Research Article

Multiaspect Soft Sets

Nor Hashimah Sulaiman and Daud Mohamad

*Department of Mathematics, Faculty of Computer Science and Mathematics, Universiti Teknologi MARA,
40450 Shah Alam, Selangor, Malaysia*

Correspondence should be addressed to Nor Hashimah Sulaiman; nhashima@tmsk.uitm.edu.my

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We introduce a novel concept of multiaspect soft set which is an extension of the ordinary soft set by Molodtsov. Some basic concepts, operations, and properties of the multiaspect soft sets are studied. We also define a mapping on multiaspect soft classes and investigate several properties related to the images and preimages of multiaspect soft sets.

1. Introduction

Soft set theory is an uncertainty-based theory introduced by Molodtsov [1]. The special characteristic of soft set lies in its parameterization component. A soft set can be considered as a parameterized family of subsets of the universe under consideration [2]. Basically, a soft set is composed of a set of parameters, each of which is associated with a set of parameter-approximate elements. The parameters can be expressed in the form of words, numbers, phrases, functions [3], and any other suitable representations depending on the context upon which a specific soft set is to be described. Unlike its counterpart, that is, fuzzy sets (see [4]), soft sets do not require any establishment of membership functions in representing fuzzy elements in the parameters. This is one of the aspects that make soft set a convenient means to represent information containing data with uncertainty elements. In addition, the structure of soft sets closely resembles an information system [5–7].

Apparently, soft sets as defined by Molodtsov [1] involve a single universe and a set of parameters with respect to the common universe. Interestingly though, real-life problems may involve descriptions of objects, situations, or entities based on certain characteristics or attributes which may be associated with different sets of elements of different types of universes. Alkhazaleh et al. [8, 9] have initiated novel concepts of *soft multiset* and *fuzzy soft multiset* which are

generalization of soft sets in representing if-then type of decision problems involving multisets of universes. Nevertheless, there exists other type of problem that may involve multisets of universes but does not pose an if-then relation. As an illustration, consider a situation where we want to specifically describe a house, say “attractiveness of House A” based on the set of parameters like *big*, *small*, *expensive*, *cheap*, and *close distance*. The parameters *big* and *small* describe the size of sections of the house such as the kitchen, hall, garden, and main bedroom. Parameters like *expensive* and *cheap* refer to the cost-related items like maintenance cost, utilities, and repair cost. The parameter *close distance* on the other hand is associated with types of distances such as distance to workplace and distance to the nearest town. Note that the kitchen, hall, and furniture form a set of aspect-related universe associated with the size aspect. Items like maintenance cost, repair cost, and utilities create another universe set for the cost aspect, while the two types of distances create another universe set for the distances aspect. The question now is how are we going to represent this information about “attractiveness of House A” in the soft set-like format? Existing soft set representation which is subjected to only one universe set may not be adequate in representing this sort of information. The conventional definition of soft set need to be further extended so as to allow a more detailed description of a particular object that takes into account items from different sets of universe. To cater this problem, in this

paper we propose a novel concept called the *multiaspect soft sets* (MASS). Some operations on the conventional soft set such as the union, intersection, complement, AND and OR operations, image, and inverse image will be extended so as to suit with the new structure of multiaspect soft set.

This paper is organized as follows. Section 2 presents basic definitions and operations in soft set theory. In Section 3, the basic concept of multiaspect soft set is elucidated. Related properties and operations are investigated and illustrated with some numerical examples. In Section 4, a notion of mapping on multiaspect soft sets is presented and related properties of images and preimages of multiaspect soft sets are also studied. Finally, Section 5 concludes the paper.

2. Preliminaries

In this section, we first review some basic definitions and operation related to soft sets. Given an initial universe $U = \{x_1, x_2, \dots, x_m\}$ and a universe set of parameters, $E = \{e_1, e_2, \dots, e_n\}$. Let $P(U)$ be the power set of U .

Definition 1 (see [1]). A pair (F, A) is called a soft set over U , where $A \subseteq E$ and F is a mapping defined by $F : A \rightarrow P(U)$.

Based on the above definition, a soft set over U can be considered as a parameterized family of subset of the universe U . For $e \in A$, $F(e)$ represents the set of e approximate elements of the soft set (F, A) .

In particular, a soft set can be illustrated by the following example.

Example 2. Suppose we want to classify a collection of storybooks $U = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$ into specific themes (parameters,) namely, $A = \{\text{fiction}(e_1), \text{love story}(e_2), \text{comedy}(e_3), \text{horror}(e_4)\}$. In this case, a soft set (S, A) over U can be used to describe the “characteristic of story books,” where

$$(S, A) = \{(e_1, \{s_1, s_6, s_7\}), (e_2, \{s_2, s_5\}), (e_3, \{s_8\}), (e_4, \{s_1, s_3, s_5\})\}. \quad (1)$$

Definition 3 (see [3]). Union of two soft sets (F, A) and (G, B) over a common universe U is a soft set (H, C) , denoted by $(F, A) \tilde{\cup} (G, B) = (H, C)$, where for all $e \in C = A \cup B$ as

$$H(e) = \begin{cases} F(e) & \text{if } e \in A \setminus B, \\ G(e) & \text{if } e \in B \setminus A, \\ F(e) \cup G(e) & \text{if } e \in A \cap B. \end{cases} \quad (2)$$

Definition 4 (see [10]). Let (F, A) and (G, B) be two soft sets over U such that $A \cap B \neq \emptyset$. Restricted union of the two soft sets is a soft set (H, C) , with $C = A \cap B$, and for all $e \in C$, $H(e) = F(e) \cap G(e)$. This notion can be denoted as $(F, A) \tilde{\cap} (G, B) = (H, C)$.

Definition 5 (see [10]). Let (F, A) and (G, B) be two soft sets over U such that $A \cap B \neq \emptyset$. Restricted intersection of the two soft sets is a soft set (H, C) , with $C = A \cap B$, and for all $e \in C$, $H(e) = F(e) \cap G(e)$. This notion can be denoted as $(F, A) \tilde{\cap} (G, B) = (H, C)$.

Definition 6 (see [10]). The extended intersection of two soft sets (F, A) and (G, B) over U denoted by $(F, A) \tilde{\cap} (G, B) = (H, C)$ is a soft set (H, C) , where for all $e \in C$ such that $C = A \cup B$ as

$$H(e) = \begin{cases} F(e) & \text{if } e \in A - B, \\ G(e) & \text{if } e \in B - A, \\ F(e) \cap G(e) & \text{if } e \in A \cap B. \end{cases} \quad (3)$$

Definition 7 (see [3]). The NOT set of $E = \{e_1, e_2, \dots, e_n\}$ notated as $\neg E$ is defined as $\neg E = \{\neg e_1, \neg e_2, \dots, \neg e_n\}$, where $\neg e_i = \text{not } e_i$, for all i .

Definition 8 (see [3]). The complement of a soft set (F, A) denoted by $(F, A)^c$ is a soft set defined by $(F, A)^c = (F^c, \neg A)$ where $F^c : \neg A \rightarrow P(U)$ is a mapping given by $F^c(a) = U \setminus F(\neg a)$ for all $\neg a \in A$.

Definition 9 (see [10]). The relative complement of a soft set (F, A) denoted by $(F, A)^r$ is a soft set defined by $(F, A)^r = (F^r, A)$ where $F^r : A \rightarrow P(U)$ is a mapping given by $F^r(a) = U \setminus F(a)$ for all $a \in A$.

The following theorem and definitions shall be used in constructing the soft image and soft preimage of multiaspect soft sets.

Theorem 10 (see [11]). Suppose $g : X \rightarrow Y$. Let A and B be subsets of X , and let P and Q be subsets of Y . Then the following holds:

- (a) $A \subseteq g^{-1}(g(A))$,
- (b) $g^{-1}(g(P)) \subseteq P$,
- (c) $g(A \cap B) \subseteq g(A) \cap g(B)$,
- (d) $g(A \cup B) = g(A) \cup g(B)$,
- (e) $g^{-1}(P \cap Q) = g^{-1}(P) \cap g^{-1}(Q)$,
- (f) $g^{-1}(P \cup Q) = g^{-1}(P) \cup g^{-1}(Q)$.

Definition 11 (see [12]). Let (X, E) and (Y, E') be soft classes. Let $u : X \rightarrow Y$ and $p : E \rightarrow E'$ be mappings. Then, a mapping $f : (X, E) \rightarrow (Y, E')$ can be defined as follows: for a soft set $(F, A) \in (X, E)$, its soft image denoted by $(f(F, A), B)$, $B = p(A)$ is a soft set in (Y, E') where for $\beta \in B \subseteq E'$, $f(F, A)(\beta) = u(\bigcup_{\alpha \in p^{-1}(\beta) \cap A} F(\alpha))$.

Definition 12 (see [12]). Let $f : (X, E) \rightarrow (Y, E')$ be a mapping from a soft class (X, E) to another soft class (Y, E') . Let $u : X \rightarrow Y$ and $p : E \rightarrow E'$ be mappings. Suppose $(G, C) \in (Y, E')$ where $C \subseteq E'$. Then, the soft inverse image of (G, C) denoted by $(f^{-1}(G, C), D)$, $D = p^{-1}(C)$ is a soft set in (X, E) given as $f^{-1}(G, C)(\alpha) = u^{-1}(G(p(\alpha)))$ for $\alpha \in D \subseteq E$.

3. Multiaspect Soft Sets

In this section, we introduce a novel concept of *multiaspect soft sets* (MASS) generalized from the theory of soft set

by Molodtsov [1]. Some existing definitions, mathematical operations, and properties of soft sets [1, 3, 9] can be extended to this new concept.

Given a collection of n nonempty finite aspect-related universe sets $\mathcal{U} = \{U^j : j \in I\}$, where for any distinct p and q for which $U^p, U^q \in \mathcal{U}$, we have $U^p \neq U^q$. Let $E = \cup_{j \in I} E^{U^j}$ be a set of parameters such that $E^{U^j} = \{e_i^j : i = 1, 2, \dots, |E^{U^j}|\}$ is the set of parameters associated with U^j . Suppose $P_{\mathcal{U}} = \cup_{j \in I} P(U^j)$ where $P(U^j)$ is the power set of U^j , and given $A \subseteq E$ where $A = \cup_{j \in I} A^{U^j}$ with $A^{U^j} \subseteq E^{U^j}$, $A^{U^j} \neq \emptyset$ for $j \in I$. We define the multispect soft set as follows.

Definition 13. Let $F : A \rightarrow P_{\mathcal{U}}$ and $F_j : A^{U^j} \rightarrow P(U^j)$ be mappings. A pair (F, A) is called a multispect soft set over $\mathcal{U}$ defined as

$$(F, A) = \left\{ \left(e_i^j, F(e_i^j) \right) : F(e_i^j) = F_j(e_i^j), e_i^j \in A \right\}. \quad (4)$$

Remark 14. A soft set is a special case of a MASS. Given a single (common) universe set say U^1 and the corresponding parameter set $E = E^{U^1}$. The soft set (F, A) over U^1 is defined by the mapping $F : A \rightarrow P(U^1)$ which gives the conventional definition of soft set by Molodtsov [1].

The notion of multispect soft set can be illustrated by the following example.

Example 15. Consider the “Attractiveness of House A ” problem as mentioned earlier in the introduction. Given a collection of aspect-related universes, $\mathcal{U} = \{U^j : j = 1, 2, 3\}$ that is, the aspect-related universe set of sections in House $A(U^1)$, set of costs (U^2), and set of distance types (U^3). The individual aspect-related universes are given as follows.

$$\begin{aligned} U^1 &= \{\text{kitchen } (a), \text{ family hall } (b), \\ &\quad \text{main bedroom } (c), \text{ garden } (d)\} \\ U^2 &= \{\text{maintenance cost } (g), \text{ repair cost } (h), \\ &\quad \text{utilities } (k)\} \\ U^3 &= \{\text{distance to workplace } (s), \\ &\quad \text{distance to nearest town } (t)\} \end{aligned} \quad (5)$$

and the set of parameters that describes the specific characteristic of the house as

$$\begin{aligned} E &= \{\text{big } (e_1^1), \text{ small } (e_2^1), \text{ expensive } (e_2^2), \\ &\quad \text{cheap } (e_2^2), \text{ close distance } (e_1^3)\} \end{aligned} \quad (6)$$

Given $P_{\mathcal{U}} = \cup_{j=1}^3 P(U^j)$ and $E = \cup_{j=1}^3 E^{U^j}$ where $E^{U^1} = \{e_1^1, e_2^1\}$, $E^{U^2} = \{e_1^2, e_2^2\}$ and $E^{U^3} = \{e_1^3\}$, the MASS representation of House A can be described as

$$(H, E) = \left\{ (e_1^1, \{a, b, c\}), (e_2^1, \{d\}), (e_1^2, \{g, h\}), \right. \\ \left. (e_2^2, \{k\}), (e_1^3, \{s, t\}) \right\}. \quad (7)$$

In this example, a MASS can be used to highlight specific element(s) or component(s) associated with the house that is (are) best described by appropriate parameter(s) in describing the characteristics of an apartment under study.

Definition 16. A multispect soft class is a collection of all multispect soft sets over $\mathcal{U}$ with attributes from E and is denoted by $\text{MASS}(\mathcal{U}_E)$.

Definition 17. A null multispect soft set (with respect to parameter set A), denoted by (Φ, A) , is defined as $(\Phi, A) = \{(e_i^j, \Phi_A(e_i^j)) : \Phi_A(e_i^j) = \emptyset, \forall e_i^j \in A\}$.

Definition 18. An absolute multispect soft set (with respect to parameter set A), denoted by (Ω, A) is defined as $(\Omega, A) = \{(e_i^j, \Omega_A(e_i^j)) : \Omega_A(e_i^j) = U^j, \forall e_i^j \in A\}$.

Example 19. Given $\mathcal{U} = \{U^j : j = 1, 2, 3\}$ a collection of aspect-related universe sets where $U^1 = \{a, b, c\}$, $U^2 = \{p, q, r\}$, $U^3 = \{s, t, w\}$, and a collection of sets of parameters $E^{U^1} = \{e_1^1, e_2^1, e_3^1\}$, $E^{U^2} = \{e_1^2, e_2^2\}$, and $E^{U^3} = \{e_1^3\}$. Let $E = \cup_{j=1}^3 E^{U^j}$, that is, $E = \{e_1^1, e_2^1, e_3^1, e_1^2, e_2^2, e_1^3\}$, and $A = \{e_1^1, e_1^2, e_2^2, e_1^3\} \subseteq E$, then we have

(a) A null MASS with respect to A ,

$$(\Phi, A) = \{(e_1^1, \emptyset), (e_1^2, \emptyset), (e_2^2, \emptyset), (e_1^3, \emptyset)\}, \quad (8)$$

(b) an absolute null MASS with respect to A ,

$$(\Omega, A) = \{(e_1^1, U^1), (e_1^2, U^2), (e_2^2, U^2), (e_1^3, U^3)\}. \quad (9)$$

Definition 20. Let $(F, A), (G, B) \in \text{MASS}(\mathcal{U}_E)$. Then, (F, A) is a soft subset of (G, B) denoted by $(F, A) \subseteq (G, B)$, if

- (i) $A \subseteq B$ with $A^{U^j} \subseteq B^{U^j}$, for all j ,
- (ii) $F(e_i^j) \subseteq G(e_i^j)$, for all $e_i^j \in A$.

Definition 21. Two multispect soft sets $(F, A), (G, B) \in \text{MASS}(\mathcal{U}_E)$ are said to be soft equal, that is, $(F, A) = (G, B)$, if and only if $A = B$ and $F(e_i^j) = G(e_i^j)$, for all $e_i^j \in A$.

Proposition 22. Let $(F, A), (G, B), (H, C), (\Omega, E), (\Phi, A) \in \text{MASS}(\mathcal{U}_E)$, and $A, B, C \subseteq E$, then

- (1) $(F, A) \subseteq (\Omega, E)$,
- (2) $(\Phi, A) \subseteq (F, A)$,
- (3) if $(F, A) \subseteq (G, B)$ and $(G, B) \subseteq (H, C)$, then $(F, A) \subseteq (H, C)$.

Proof. Trivially follow Definition 20. $\square$

Proposition 23. Let $(F, A), (G, B), (H, C) \in \text{MASS}(\mathcal{U}_E)$ and $A, B, C \subseteq E$ then

- (1) if $(F, A) = (G, B)$ and $(G, B) = (H, C)$, then $(F, A) = (H, C)$,
- (2) if $(F, A) \subseteq (G, B)$ and $(G, B) \subseteq (F, A)$, then $(F, A) = (G, B)$.

Proof. Proofs are straight forward. $\square$

Definition 24. The NOT set of E notated as $\neg E$ is defined as $\neg E = \{\neg e_i^j\}$ where $\neg e_i^j = \text{not } e_i^j$, for all $e_i^j \in E$.

Example 25. Given a set of parameter $E = \{e_1^1, e_2^1, e_3^1, e_1^2, e_2^2, e_3^2\}$. The NOT set of E is given as $\neg E = \{\neg e_1^1, \neg e_2^1, \neg e_3^1, \neg e_1^2, \neg e_2^2, \neg e_3^2\}$.

Definition 26. The complement of (F, A) is a multiaspect soft set defined as $(F, A)^c = (F^c, \neg A)$ where $F^c : \neg A \rightarrow P_{\mathcal{U}}$ is a mapping such that $\neg A = \bigcup_{j \in I} (\neg A^{U^j})$ and for all $\neg e_i^j \in \neg A^{U^j}$, $F^c(\neg e_i^j) = U^j \setminus F(e_i^j)$.

Definition 27. The relative complement of a multiaspect soft set $(F, A) \in \text{MASS}(\mathcal{U}_E)$ denoted by $(F, A)^r$ is a multiaspect over $\mathcal{U}$ defined as $(F, A)^r = (F^r, A)$ where $F^r : A \rightarrow P_{\mathcal{U}}$ is a mapping such that $A = \bigcup_{j \in I} (A^{U^j})$ and for all $e_i^j \in A^{U^j}$, $F^r(e_i^j) = (F(e_i^j))^r = U^j \setminus F(e_i^j)$.

Example 28. Let $(F, A) = \{(e_1^1, \{a\}), (e_1^2, \{p, q\}), (e_2^2, \{p, q, r\}), (e_3^3, \{s, w\})\}$ be a multiaspect soft set in $\text{MASS}(\mathcal{U}_E)$ where $\mathcal{U}$ and E are defined as in Example 19. The complement and relative complement of (F, A) are respectively obtained as

$$\begin{aligned} (F, A)^c &= (F^c, \neg A) \\ &= \{(\neg e_1^1, \{b, c\}), (\neg e_1^2, \{r\}), (\neg e_2^2, \emptyset), (\neg e_3^3, \{t\})\}, \\ (F, A)^r &= (F^r, A) \\ &= \{(e_1^1, \{b, c\}), (e_1^2, \{r\}), (e_2^2, \emptyset), (e_3^3, \{t\})\}. \end{aligned} \quad (10)$$

Proposition 29. Let $(F, A), (G, A) \in \text{MASS}(\mathcal{U}_E)$ and $A \subseteq E$, then the following hold:

- (1) $((F, A)^c)^c = ((F, A)^r)^r = (F, A)$,
- (2) $(\Omega, A)^c = (\Phi, \neg A)$,
- (3) $(\Omega, A)^r = (\Phi, A)$.

Proof.

- (1) Proof is straight forward from Definitions 26 and 27.
- (2) We have $(\Omega, A)^c = (\Omega^c, \neg A)$, where for all $\neg e_i^j \in \neg A$, $\Omega_{\neg A}^c(\neg e_i^j) = U^j \setminus \Omega_A(e_i^j) = U^j \setminus U^j = \emptyset = \Phi_{\neg A}(\neg e_i^j)$. Hence, $(\Omega, A)^c = (\Phi, \neg A)$.

- (3) Using Definitions 18 and 27, $(\Omega, A)^r = (\Omega^r, A)$, where for all $e_i^j \in A$, $\Omega_A^r(e_i^j) = U^j \setminus \Omega(e_i^j) = U^j \setminus U^j = \Phi_A(e_i^j)$. Hence, $(\Omega, A)^r = (\Phi, A)$.

In the following definitions, let (F, A) and (G, B) be two multiaspect soft sets in $\text{MASS}(\mathcal{U}_E)$. $\square$

Definition 30. The union of (F, A) and (G, B) is a multiaspect soft set (H, C) over $\mathcal{U}$ denoted by $(F, A) \tilde{\cup} (G, B) = (H, C)$, where for all $e_i^j \in C = A \cup B$,

$$H(e_i^j) = \begin{cases} F(e_i^j) & \text{if } e_i^j \in A \setminus B, \\ G(e_i^j) & \text{if } e_i^j \in B \setminus A, \\ F(e_i^j) \cup G(e_i^j) & \text{if } e_i^j \in A \cap B. \end{cases} \quad (11)$$

Definition 31. The restricted union of (F, A) and (G, B) where $A \cap B \neq \emptyset$ with $A^{U^j} \cap B^{U^j} \neq \emptyset$, for $j \in I$ is a multiaspect soft set (H, C) over $\mathcal{U}$ denoted by $(F, A) \tilde{\cup}_{\mathcal{R}} (G, B) = (H, C)$ such that for all $e_i^j \in C = A \cap B$,

$$H(e_i^j) = F(e_i^j) \cup G(e_i^j). \quad (12)$$

Definition 32. The restricted intersection of (F, A) and (G, B) where $A \cap B \neq \emptyset$ with $A^{U^j} \cap B^{U^j} \neq \emptyset$, for $j \in I$ is a multiaspect soft set (H, C) over $\mathcal{U}$ denoted by $(F, A) \tilde{\cap}_{\mathcal{R}} (G, B) = (H, C)$ such that for all $e_i^j \in C = A \cap B$,

$$H(e_i^j) = F(e_i^j) \cap G(e_i^j). \quad (13)$$

Definition 33. The extended intersection of (F, A) and (G, B) is a multiaspect soft set over $\mathcal{U}$ denoted by $(F, A) \tilde{\cap} (G, B) = (H, C)$, where for all $e_i^j \in C = A \cup B$, as

$$H(e_i^j) = \begin{cases} F(e_i^j) & \text{if } e_i^j \in A \setminus B, \\ G(e_i^j) & \text{if } e_i^j \in B \setminus A, \\ F(e_i^j) \cap G(e_i^j) & \text{if } e_i^j \in A \cap B. \end{cases} \quad (14)$$

Remark 34. If $A = B$, then $(F, A) \tilde{\cap} (G, B) = (F, A) \tilde{\cap}_{\mathcal{R}} (G, B)$.

Proposition 35. Let $(F, A), (\Omega, A), (\Omega, E), (\Phi, A) \in \text{MASS}(\mathcal{U}_E)$, then

- (1) $(F, A) \tilde{\cup} (\Omega, A) = (F, A) \tilde{\cup}_{\mathcal{R}} (\Omega, A) = (\Omega, A)$,
- (2) $(F, A) \tilde{\cup} (\Omega, E) = (\Omega, E)$,
- (3) $(F, A) \tilde{\cup}_{\mathcal{R}} (\Omega, E) = (\Omega, A)$,
- (4) $(F, A) \tilde{\cup} (\Phi, A) = (F, A) \tilde{\cup}_{\mathcal{R}} (\Phi, A) = (F, A)$,
- (5) $(F, A) \tilde{\cup} (F, A)^r = (F, A) \tilde{\cup}_{\mathcal{R}} (F, A)^r = (\Omega, A)$.

Proof. Straightforward using Definitions 30 and 31. $\square$

Proposition 36. Let $(F, A), (\Omega, A), (\Phi, B) \in \text{MASS}(\mathcal{U}_E)$, then

- (1) $(F, A) \tilde{\cap} (\Omega, A) = (F, A) \tilde{\cap}_{\mathcal{R}} (\Omega, A) = (F, A)$,
- (2) $(F, A) \tilde{\cap} (\Phi, A) = (F, A) \tilde{\cap}_{\mathcal{R}} (\Phi, A) = (\Phi, A)$,
- (3) $(F, A) \tilde{\cap} (F, A)^r = (F, A) \tilde{\cap}_{\mathcal{R}} (F, A)^r = (\Phi, A)$,
- (4) $(F, A) \tilde{\cap}_{\mathcal{R}} (\Phi, B) = (\Phi, A \cap B)$,
- (5) $(F, A) \tilde{\cap}_{\mathcal{R}} (\Omega, E) = (F, A)$.

Proof. Proofs are straightforward using Definitions 32 and 33. $\square$

Example 37. Let $\mathcal{U}$ and E be given as in Example 19, and $(F, A), (G, B), (L, D) \in \text{MASS}(\mathcal{U}_E)$ with $A, B, D \subseteq E$ where

$$(F, A) = \{(e_1^1, \{a, b\}), (e_2^1, \{a, c\}), (e_2^2, \{p, q, r\}), (e_1^3, \{w\})\} \quad (15)$$

$$(G, B) = \{(e_1^1, \{a, b\}), (e_3^1, \{a, b, c\}), (e_1^2, \{p\}), (e_2^2, \{p, q, r\}), (e_1^3, \{s, t, w\})\},$$

$$(L, D) = \{(e_1^1, \{a\}), (e_1^2, \{p, q, r\}), (e_2^2, \{p, q\}), (e_1^3, \{t\})\}. \quad (16)$$

The operations of union, restricted union, extended intersection, and restricted intersection of multispect soft sets are illustrated as follows.

(a) Union of multispect soft sets:

$$\begin{aligned} (F, A) \tilde{\cup} (G, B) &= \{(e_1^1, \{a, b\}), (e_2^1, \{a, c\}), (e_3^1, \{a, b, c\}), (e_1^2, \{p\}), (e_2^2, \{p, q, r\}), (e_1^3, \{s, t, w\})\}, \\ (F, A) \tilde{\cup} (L, D) &= \{(e_1^1, \{a, b\}), (e_2^1, \{a, c\}), (e_1^2, \{p, q, r\}), (e_2^2, \{p, q, r\}), (e_1^3, \{t, w\})\}. \end{aligned} \quad (17)$$

(b) Extended intersection of multispect soft sets:

$$\begin{aligned} (F, A) \tilde{\cap} (G, B) &= \{(e_1^1, \{a, b\}), (e_2^1, \{a, c\}), (e_3^1, \{a, b, c\}), (e_1^2, \{p\}), (e_2^2, \{p, q, r\}), (e_1^3, \{w\})\}, \\ (F, A) \tilde{\cap} (L, D) &= \{(e_1^1, \{a\}), (e_2^1, \{a, c\}), (e_1^2, \{p, q, r\}), (e_2^2, \{p, q\}), (e_1^3, \{\phi\})\}. \end{aligned} \quad (18)$$

(c) Restricted union of multispect soft sets:

$$(F, A) \tilde{\cup}_{\mathcal{R}} (G, B) = \{(e_1^1, \{a, b\}), (e_2^2, \{p, q, r\}), (e_1^3, \{s, t, w\})\}, \quad (19)$$

$$(F, A) \tilde{\cup}_{\mathcal{R}} (L, D) = \{(e_1^1, \{a, b\}), (e_2^2, \{p, q, r\}), (e_1^3, \{t, w\})\}. \quad (20)$$

(d) Restricted intersection of multispect soft sets:

$$\begin{aligned} (F, A) \tilde{\cap}_{\mathcal{R}} (G, B) &= \{(e_1^1, \{a, b\}), (e_2^2, \{p, q, r\}), (e_1^3, \{w\})\}, \\ (F, A) \tilde{\cap}_{\mathcal{R}} (L, D) &= \{(e_1^1, \{a\}), (e_2^2, \{p, q\}), (e_1^3, \{\phi\})\}. \end{aligned} \quad (21)$$

Proposition 38. Let $(F, A), (G, B), (H, C) \in \text{MASS}(\mathcal{U}_E)$, then the following hold.

(1) Identity property:

$$(F, A) \tilde{\cup} (F, A) = (F, A) \tilde{\cap} (F, A) = (F, A). \quad (22)$$

(2) Associative property:

- (i) $(F, A) \tilde{\cup} [(G, B) \tilde{\cup} (H, C)] = [(F, A) \tilde{\cup} (G, B)] \tilde{\cup} (H, C)$,
- (ii) $(F, A) \tilde{\cap} [(G, B) \tilde{\cap} (H, C)] = [(F, A) \tilde{\cap} (G, B)] \tilde{\cap} (H, C)$,
- (iii) $(F, A) \tilde{\cup}_{\mathcal{R}} [(G, B) \tilde{\cup}_{\mathcal{R}} (H, C)] = [(F, A) \tilde{\cup}_{\mathcal{R}} (G, B)] \tilde{\cup}_{\mathcal{R}} (H, C)$,
- (iv) $(F, A) \tilde{\cap}_{\mathcal{R}} [(G, B) \tilde{\cap}_{\mathcal{R}} (H, C)] = [(F, A) \tilde{\cap}_{\mathcal{R}} (G, B)] \tilde{\cap}_{\mathcal{R}} (H, C)$.

(3) Distributive property:

- (i) $(F, A) \tilde{\cup} [(G, B) \tilde{\cap} (H, C)] = [(F, A) \tilde{\cup} (G, B)] \tilde{\cap} [(F, A) \tilde{\cup} (H, C)]$,
- (ii) $(F, A) \tilde{\cap} [(G, B) \tilde{\cup} (H, C)] = [(F, A) \tilde{\cap} (G, B)] \tilde{\cup} [(F, A) \tilde{\cap} (H, C)]$,
- (iii) $(F, A) \tilde{\cup}_{\mathcal{R}} [(G, B) \tilde{\cap}_{\mathcal{R}} (H, C)] = [(F, A) \tilde{\cup}_{\mathcal{R}} (G, B)] \tilde{\cap}_{\mathcal{R}} [(F, A) \tilde{\cup}_{\mathcal{R}} (H, C)]$,
- (iv) $(F, A) \tilde{\cap}_{\mathcal{R}} [(G, B) \tilde{\cup}_{\mathcal{R}} (H, C)] = [(F, A) \tilde{\cap}_{\mathcal{R}} (G, B)] \tilde{\cup}_{\mathcal{R}} [(F, A) \tilde{\cap}_{\mathcal{R}} (H, C)]$.

Proof. Straightforward from Definitions 30 to 33. $\square$

Definition 39. Let $(F, A), (G, B) \in \text{MASS}(\mathcal{U}_E)$. The OR operation of (F, A) and (G, B) denoted by $(F, A) \underline{\vee} (G, B)$ is defined as $(F, A) \underline{\vee} (G, B) = (M, C)$ where $C = \cup_{j \in I} C^{U^j}$ with $C^{U^j} = A^{U^j} \times B^{U^j}$ such that for all $(\alpha, \beta) \in C^{U^j}$, $M(\alpha, \beta) = F(\alpha) \cup G(\beta)$.

Definition 40. Let $(F, A), (G, B) \in \text{MASS}(\mathcal{U}_E)$. The AND operation of (F, A) and (G, B) denoted by $(F, A) \underline{\wedge} (G, B)$ is defined as $(F, A) \underline{\wedge} (G, B) = (M, C)$ where $C = \cup_{j \in I} C^{U^j}$ with $C^{U^j} = A^{U^j} \times B^{U^j}$ such that for all $(\alpha, \beta) \in C^{U^j}$, $M(\alpha, \beta) = F(\alpha) \cap G(\beta)$.

Example 41. Let (F, A) and (G, B) be given as in Example 37. We have

$$\begin{aligned} C &= \bigcup_{j=1}^3 (A^{U^j} \times B^{U^j}) \\ &= \{(e_1^1, e_1^1), (e_1^1, e_3^1), (e_2^1, e_1^1), (e_2^1, e_3^1), (e_2^2, e_1^2), (e_2^2, e_2^2), (e_1^3, e_1^3)\}. \end{aligned} \quad (23)$$

The “OR” and “AND” operation between (F, A) and (G, B) are obtained as follows:

$$\begin{aligned}
 (F, A) \sqcup (G, B) &= \{((e_1^1, e_1^1), \{a, b\}), \\
 &\quad ((e_1^1, e_3^1), \{a, b, c\}), ((e_2^1, e_1^1), \{a, b, c\}), \\
 &\quad ((e_2^1, e_3^1), \{a, b, c\}), ((e_2^2, e_1^2), \{p, q, r\}), \\
 &\quad ((e_2^2, e_2^2), \{p, q, r\}), ((e_1^3, e_1^3), \{s, t, w\})\}, \\
 (F, A) \sqcap (G, B) &= \{((e_1^1, e_1^1), \{a, b\}), \\
 &\quad ((e_1^1, e_3^1), \{a, b\}), ((e_2^1, e_1^1), \{a\}), \\
 &\quad ((e_2^1, e_3^1), \{a\}), ((e_2^2, e_1^2), \{p\}), \\
 &\quad ((e_2^2, e_2^2), \{p, q, r\}), ((e_1^3, e_1^3), \{w\})\}.
 \end{aligned} \tag{24}$$

Proposition 42. Let $(F, A), (G, B) \in \text{MASS}(\mathcal{U}_E)$. Then,

- (1) $[(F, A) \tilde{\cup}_{\mathcal{A}} (G, B)]^r = (F, A)^r \tilde{\cap}_{\mathcal{A}} (G, B)^r$,
- (2) $[(F, A) \tilde{\cap}_{\mathcal{A}} (G, B)]^r = (F, A)^r \tilde{\cup}_{\mathcal{A}} (G, B)^r$,
- (3) $[(F, A) \tilde{\cup} (G, B)]^c = (F, A)^c \tilde{\cap} (G, B)^c$,
- (4) $[(F, A) \tilde{\cap} (G, B)]^c = (F, A)^c \tilde{\cup} (G, B)^c$.

Proof. We only prove (1) as (2), (3), and (4) can be proven in similar manner.

Let $[(F, A) \tilde{\cup}_{\mathcal{A}} (G, B)]^r = (H, C)^r = (H^r, C)$ and $(F, A)^r \tilde{\cap}_{\mathcal{A}} (G, B)^r = (F^r, A) \tilde{\cap}_{\mathcal{A}} (G^r, B) = (L, C)$. By definition, for all $e_i^j \in C$, we have

$$\begin{aligned}
 H^r(e_i^j) &= U^j \setminus H(e_i^j) = U^j \setminus [F(e_i^j) \cup G(e_i^j)] \\
 &= (U^j \setminus F(e_i^j)) \cap (U^j \setminus G(e_i^j)) \\
 &= F^r(e_i^j) \cap G^r(e_i^j) = L(e_i^j).
 \end{aligned} \tag{25}$$

Note that $H^r(e_i^j) = L(e_i^j)$. Hence, $[(F, A) \tilde{\cup}_{\mathcal{A}} (G, B)]^r = (F, A)^r \tilde{\cap}_{\mathcal{A}} (G, B)^r$. $\square$

Proposition 43. Let $(F, A), (G, B) \in \text{MASS}(\mathcal{U}_E)$. Then,

- (1) $[(F, A) \sqcup (G, B)]^r = (F, A)^r \sqcap (G, B)^r$,
- (2) $[(F, A) \sqcap (G, B)]^r = (F, A)^r \sqcup (G, B)^r$,
- (3) $[(F, A) \sqcup (G, B)]^c = (F, A)^c \sqcap (G, B)^c$,
- (4) $[(F, A) \sqcap (G, B)]^c = (F, A)^c \sqcup (G, B)^c$.

Proof. We only prove (1) as (2), (3), and (4) can be proven in similar manner.

- (1) Let $[(F, A) \sqcup (G, B)]^r = (M, C)^r = (M^r, C)$ and $(F, A)^r \sqcap (G, B)^r = (F^r, A) \sqcap (G^r, B) = (H, C)$ where

$C = \cup_{j \in I} C^{U^j} = \cup_{j \in I} (A^{U^j} \times B^{U^j})$. By definition, for all $(\alpha, \beta) \in C^{U^j}$, we have

$$\begin{aligned}
 M^r(\alpha, \beta) &= U^j \setminus M(\alpha, \beta) = U^j \setminus (F(\alpha) \cup G(\beta)) \\
 &= (U^j \setminus F(\alpha)) \cap (U^j \setminus G(\beta)) \\
 &= F^r(\alpha) \cap G^r(\beta) = H(\alpha, \beta).
 \end{aligned} \tag{26}$$

Hence, $[(F, A) \sqcup (G, B)]^r = (F, A)^r \sqcap (G, B)^r$. $\square$

4. Mapping on Multiaspect Soft Classes

Studies on mapping involving soft sets were initiated among others by [12, 13]. In this section, we extend the concept of mapping on soft classes [12] into a mapping on multiaspect soft classes. Several properties related to the soft image and soft preimage of multiaspect soft sets are investigated and illustrated with numerical examples.

Let $\text{MASS}(\mathcal{U}_E)$ and $\text{MASS}(\mathcal{V}_{E'})$ be two multiaspect soft classes where $\mathcal{U} = \{U^j : j \in I\}$ and $\mathcal{V} = \{V^j : j \in I\}$. Let the corresponding sets of parameters $E = \cup_{j \in I} E^{U^j}$ and $E' = \cup_{j \in I} E^{V^j}$ such that $E^{U^j} = \{e_i^j : i = 1, 2, \dots, |E^{U^j}|\}$ and $E^{V^j} = \{e_k^j : k = 1, 2, \dots, |E^{V^j}|\}$. Suppose $P(U^j)$ and $P(V^j)$ are the power set of U^j and V^j , respectively, $A = \cup_{j \in I} A^{U^j} \subseteq E$ with $A^{U^j} \subseteq E^{U^j}$ and $D' = \cup_{j \in I} D'^{V^j} \subseteq E'$ with $D'^{V^j} \subseteq E^{V^j}$. We define the soft image and soft preimage of a multiaspect soft set as follows.

Definition 44. Let $\rho_j : P(U^j) \rightarrow P(V^j)$, $\lambda : E \rightarrow E'$, and $\lambda_j : E^{U^j} \rightarrow E^{V^j}$ for $j \in I$ be mappings. Suppose $\psi : \text{MASS}(\mathcal{U}_E) \rightarrow \text{MASS}(\mathcal{V}_{E'})$ is a mapping from a multiaspect soft class $\text{MASS}(\mathcal{U}_E)$ to another multiaspect soft class $\text{MASS}(\mathcal{V}_{E'})$. For a multiaspect soft set (F, A) in $\text{MASS}(\mathcal{U}_E)$, the soft image of (F, A) under ψ denoted by $\psi(F, A)$ is a multiaspect soft set in $\text{MASS}(\mathcal{V}_{E'})$ defined as follows: for $\varepsilon_k^j \in \lambda(A) \subseteq E'$ with $\lambda(A) = \cup_{j \in I} \lambda_j(A^{U^j})$,

$$\psi(F, A)(\varepsilon_k^j) = \rho_j \left(\bigcup_{e_i^j \in \lambda_j^{-1}(\varepsilon_k^j) \cap A} F(e_i^j) \right). \tag{27}$$

Definition 45. Let $\rho_j : P(U^j) \rightarrow P(V^j)$, $\lambda : E \rightarrow E'$ and $\lambda_j : E^{U^j} \rightarrow E^{V^j}$ for $j \in I$ be mappings. Suppose $\psi : \text{MASS}(\mathcal{U}_E) \rightarrow \text{MASS}(\mathcal{V}_{E'})$ is a mapping and (M', D') is a multiaspect soft set in $\text{MASS}(\mathcal{V}_{E'})$ where $D' \subseteq E'$. Then, the soft preimage of (M', D') is a multiaspect soft set in $\text{MASS}(\mathcal{U}_E)$ denoted by $\psi^{-1}(M', D')$ where for $e_i^j \in \lambda^{-1}(D') \subseteq E$ with $\lambda^{-1}(D') = \cup_{j \in I} \lambda_j^{-1}(D'^{V^j})$,

$$\psi^{-1}(M', D')(e_i^j) = \rho_j^{-1}(M'(\lambda_j(e_i^j))). \tag{28}$$

Example 46. Consider two multispect soft classes $\text{MASS}(\mathcal{U}_E)$ and $\text{MASS}(\mathcal{V}_{E'})$ where $\mathcal{U} = \{U^1, U^2\}$, and $\mathcal{V} = \{V^1, V^2\}$ such that $U^1 = \{a, b, c\}$, $U^2 = \{p, q\}$, $V^1 = \{r, s, t\}$, and $V^2 = \{w, z\}$. Let $E^{U^1} = \{e_1^1, e_2^1\}$, $E^{U^2} = \{e_1^2, e_2^2\}$, $E^{V^1} = \{e_1^1\}$, and $E^{V^2} = \{e_1^2, e_2^2, e_3^2\}$. We have $E = \{e_1^1, e_2^1, e_1^2, e_2^2\}$ and $E' = \{e_1^1, e_2^1, e_2^2, e_3^2\}$. Suppose $\rho_j : P(U^j) \rightarrow P(V^j)$ and $\lambda_j : E^{U^j} \rightarrow E^{V^j}$ for $j = 1, 2$ such that

$$\begin{aligned} \rho_1(\{a\}) &= \{s\}, & \rho_1(\{b\}) &= \{t\}, \\ \rho_1(\{c\}) &= \{r\}, & \rho_1(\{a, b\}) &= \{s, t\}, \\ \rho_1(\{b, c\}) &= \{r, t\}, & \rho_1(\{a, b, c\}) &= \{r, s, t\}, \\ \rho_2(\{q\}) &= \{w\}, & \rho_2(\{p, q\}) &= \{w, z\}, \\ \rho_2(\{p\}) &= \{z\}, & \rho_1(\phi) &= \rho_2(\phi) = \phi, \\ \lambda_1(e_1^1) &= e_1^1, & \lambda_1(e_2^1) &= e_1^1, \\ \lambda_2(e_1^2) &= e_1^2, & \lambda_2(e_2^2) &= e_2^2. \end{aligned} \quad (29)$$

Given $(F, A) \in \text{MASS}(\mathcal{U}_E)$ where $(F, A) = \{(e_1^1, \{a, b\}), (e_2^1, \{b\}), (e_1^2, \{p, q\})\}$. Using Definition 44, the soft image of (F, A) under ψ , that is, $\psi(F, A)$, with parameters $\varepsilon_k^j \in \lambda(A) = \{e_1^1, e_1^2\}$ is obtained as follows.

$$\begin{aligned} \text{(i) For } \varepsilon_k^j &= e_1^1, e_i^j \in \lambda_1^{-1}(e_1^1) \cap A = \{e_1^1, e_2^1\}: \\ \psi(F, A)(\varepsilon_1^1) &= \rho_1(F(e_1^1) \cup F(e_2^1)) \\ &= \rho_1(\{a, b\}) = \{s, t\}. \end{aligned} \quad (30)$$

$$\begin{aligned} \text{(ii) For } \varepsilon_k^j &= e_1^2, e_i^j \in \lambda_2^{-1}(e_1^2) \cap A = \{e_1^2\}: \\ \psi(F, A)(\varepsilon_1^2) &= \rho_2(F(e_1^2)) \\ &= \rho_2(\{p, q\}) = \{w, z\}. \end{aligned} \quad (31)$$

Hence, $\psi(F, A) = \{(e_1^1, \{s, t\}), (e_1^2, \{w, z\})\}$.

Next, suppose we want to find the soft preimage of a multispect soft set (M', D') in $\text{MASS}(\mathcal{V}_{E'})$ where $(M', D') = \{(e_1^1, \{s\}), (e_2^2, \{w\})\}$. By Definition 45, we have $\lambda^{-1}(D') = \lambda_1^{-1}(\{e_1^1\}) \cup \lambda_2^{-1}(\{e_2^2\}) = \{e_1^1, e_2^1, e_2^2\}$. Thus,

$$\begin{aligned} \psi^{-1}(M', D')(e_1^1) &= \rho_1^{-1}(M'(\lambda_1(e_1^1))) \\ &= \rho_1^{-1}(M'(e_1^1)) = \rho_1^{-1}(\{s\}) = \{a\}, \\ \psi^{-1}(M', D')(e_2^1) &= \rho_1^{-1}(M'(\lambda_1(e_2^1))) \\ &= \rho_1^{-1}(M'(e_1^1)) = \rho_1^{-1}(\{s\}) = \{a\}, \\ \psi^{-1}(M', D')(e_2^2) &= \rho_2^{-1}(M'(\lambda_2(e_2^2))) \\ &= \rho_2^{-1}(M'(e_2^2)) = \rho_2^{-1}(\{w\}) = \{q\}. \end{aligned} \quad (32)$$

Hence, $\psi^{-1}(M', D') = \{(e_1^1, \{a\}), (e_2^1, \{a\}), (e_2^2, \{q\})\}$.

Definition 47. Let $\psi : \text{MASS}(\mathcal{U}_E) \rightarrow \text{MASS}(\mathcal{V}_{E'})$, $\rho_j : P(U^j) \rightarrow P(V^j)$, $\lambda : E \rightarrow E'$, and $\lambda_j : E^{U^j} \rightarrow E^{V^j}$ for $j \in I$ be mappings, and given $(F, A), (G, B) \in \text{MASS}(\mathcal{U}_E)$. Then, for $\varepsilon_k^j \in \lambda(A) \cup \lambda(B) \subseteq E'$, the union of two multispect soft images $\psi(F, A)$ and $\psi(G, B)$ is a multispect soft sets in $\text{MASS}(\mathcal{V}_{E'})$ defined as

$$(\psi(F, A) \cup \psi(G, B))(\varepsilon_k^j) = \psi(F, A)(\varepsilon_k^j) \cup \psi(G, B)(\varepsilon_k^j). \quad (33)$$

Definition 48. Let $\psi : \text{MASS}(\mathcal{U}_E) \rightarrow \text{MASS}(\mathcal{V}_{E'})$, $\rho_j : P(U^j) \rightarrow P(V^j)$, $\lambda : E \rightarrow E'$, and $\lambda_j : E^{U^j} \rightarrow E^{V^j}$ for $j \in I$ be mappings, and given $(F, A), (G, B) \in \text{MASS}(\mathcal{U}_E)$. Then, for $\varepsilon_k^j \in \lambda(A) \cap \lambda(B) \subseteq E'$ with $\lambda(A) \cap \lambda(B) \neq \emptyset$, the restricted intersection of two multispect soft images $\psi(F, A)$ and $\psi(G, B)$ is a multispect soft sets in $\text{MASS}(\mathcal{V}_{E'})$ defined as

$$\begin{aligned} (\psi(F, A) \tilde{\cap} \psi(G, B))(\varepsilon_k^j) \\ = \psi(F, A)(\varepsilon_k^j) \tilde{\cap} \psi(G, B)(\varepsilon_k^j). \end{aligned} \quad (34)$$

Definition 49. Let $\psi : \text{MASS}(\mathcal{U}_E) \rightarrow \text{MASS}(\mathcal{V}_{E'})$, $\rho_j : P(U^j) \rightarrow P(V^j)$, $\lambda : E \rightarrow E'$, and $\lambda_j : E^{U^j} \rightarrow E^{V^j}$ for $j \in I$ be mappings, and given $(F', A'), (G', B') \in \text{MASS}(\mathcal{V}_{E'})$. Then, for $e_i^j \in \lambda^{-1}(A') \cup \lambda^{-1}(B') \subseteq E$, the union of two multispect soft preimages $\psi^{-1}(F', A')$ and $\psi^{-1}(G', B')$ is a multispect soft sets in $\text{MASS}(\mathcal{U}_E)$ defined as

$$\begin{aligned} (\psi^{-1}(F', A') \cup \psi^{-1}(G', B'))(e_i^j) \\ = \psi^{-1}(F', A')(e_i^j) \cup \psi^{-1}(G', B')(e_i^j). \end{aligned} \quad (35)$$

Definition 50. Let $\psi : \text{MASS}(\mathcal{U}_E) \rightarrow \text{MASS}(\mathcal{V}_{E'})$, $\rho_j : P(U^j) \rightarrow P(V^j)$, $\lambda : E \rightarrow E'$, and $\lambda_j : E^{U^j} \rightarrow E^{V^j}$ for $j \in I$ be mappings, and given $(F', A'), (G', B') \in \text{MASS}(\mathcal{V}_{E'})$. Then, for $e_i^j \in C \subseteq E$ with $C = \lambda^{-1}(A') \cap \lambda^{-1}(B') \neq \emptyset$, the restricted intersection of two multispect soft preimages $\psi^{-1}(F', A')$ and $\psi^{-1}(G', B')$ is a multispect soft sets in $\text{MASS}(\mathcal{U}_E)$ defined as

$$\begin{aligned} (\psi^{-1}(F', A') \tilde{\cap} \psi^{-1}(G', B'))(e_i^j) \\ = \psi^{-1}(F', A')(e_i^j) \tilde{\cap} \psi^{-1}(G', B')(e_i^j). \end{aligned} \quad (36)$$

Proposition 51. Let $\psi : \text{MASS}(\mathcal{U}_E) \rightarrow \text{MASS}(\mathcal{V}_{E'})$, $\rho_j : P(U^j) \rightarrow P(V^j)$, $\lambda : E \rightarrow E'$, and $\lambda_j : E^{U^j} \rightarrow E^{V^j}$ for $j \in I$ be mappings. Then, for $(F, A), (G, B), (\Omega, E) \in \text{MASS}(\mathcal{U}_E)$, the following hold:

- (1) $(F, A) \subseteq \psi^{-1}(\psi(F, A))$,
- (2) $(\psi(F, A))^r \subseteq \psi((F, A)^r)$ if ρ_j is surjective for all j ,

- (3) $\psi((F, A) \tilde{\cup} (G, B)) = \psi(F, A) \tilde{\cup} \psi(G, B)$,
 (4) $\psi((F, A) \tilde{\cap}_{\mathcal{R}} (G, B)) \tilde{\subseteq} \psi(F, A) \tilde{\cap}_{\mathcal{R}} \psi(G, B)$,
 (5) If $(F, A) \tilde{\subseteq} (G, B)$, then $\psi(F, A) \tilde{\subseteq} \psi(G, B)$.

Proof.

- (1) Let $\psi(F, A) = (R', D') \in \text{MASS}(\mathcal{V}_{E'})$ where for $\varepsilon_k^j \in D' \subseteq E'$, we have

$$R'(\varepsilon_k^j) = \rho_j \left(\bigcup_{e_i^j \in \lambda_j^{-1}(\varepsilon_k^j) \cap A} F(e_i^j) \right). \quad (37)$$

By definition, we have $\psi^{-1}(\psi(F, A)) = \psi^{-1}(R', D')$ where for $e_i^j \in \lambda^{-1}(D') \subseteq A$, we have

$$\begin{aligned} \psi^{-1}(R', D')(\varepsilon_k^j) &= \rho_j^{-1}(R'(\lambda_j(e_i^j))), \quad \lambda_j(e_i^j) \in D' \\ &= \rho_j^{-1}(R'(\varepsilon_k^j)) \quad \text{where } \varepsilon_k^j = \lambda_j(e_i^j) \\ &= \rho_j^{-1} \left(\rho_j \left(\bigcup_{e_i^j \in \lambda_j^{-1}(\varepsilon_k^j) \cap A} F(e_i^j) \right) \right) \\ &\supseteq \bigcup_{e_i^j \in \lambda_j^{-1}(\varepsilon_k^j) \cap A} F(e_i^j) \\ &\supseteq F(e_i^j). \end{aligned} \quad (38)$$

Thus, we have $\psi^{-1}(R', D') = \psi^{-1}(\psi(F, A)) \supseteq (F, A)$.

- (2) Suppose ρ_j is surjective for all j . For $\varepsilon_k^j \in \lambda(A) \subseteq E'$, we have

$$\psi(F, A)(\varepsilon_k^j) = \rho_j \left(\bigcup_{e_i^j \in \lambda_j^{-1}(\varepsilon_k^j) \cap A} F(e_i^j) \right). \quad (39)$$

Thus, by the definition of relative complement, we have

$$(\psi(F, A))^r(\varepsilon_k^j) = \left(\rho_j \left(\bigcup_{e_i^j \in \lambda_j^{-1}(\varepsilon_k^j) \cap A} F(e_i^j) \right) \right)^r$$

$$\begin{aligned} &\subseteq \rho_j \left(\left(\bigcup_{e_i^j \in \lambda_j^{-1}(\varepsilon_k^j) \cap A} F(e_i^j) \right)^r \right) \\ &= \rho_j \left(\bigcap_{e_i^j \in \lambda_j^{-1}(\varepsilon_k^j) \cap A} F^r(e_i^j) \right) \\ &\subseteq \rho_j \left(\bigcup_{e_i^j \in \lambda_j^{-1}(\varepsilon_k^j) \cap A} F^r(e_i^j) \right) \\ &= \psi(F^r, A)(\varepsilon_k^j) = \psi((F, A)^r)(\varepsilon_k^j). \end{aligned} \quad (40)$$

Hence, $(\psi(F, A))^r \tilde{\subseteq} \psi((F, A)^r)$ as required.

- (3) We want to show that $\psi((F, A) \tilde{\cup} (G, B))(\varepsilon_k^j) = (\psi(F, A) \tilde{\cup} \psi(G, B))(\varepsilon_k^j)$. Let $(F, A) \cup (G, B) = (H, A \cup B)$. Then, for all $\varepsilon_k^j \in \lambda(A \cup B) \subseteq E'$, we have

$$\begin{aligned} \psi((F, A) \cup (G, B))(\varepsilon_k^j) &= \psi(H, A \cup B)(\varepsilon_k^j) \\ &= \rho_j \left(\bigcup_{e_i^j \in \lambda_j^{-1}(\varepsilon_k^j) \cap (A \cup B)} H(e_i^j) \right) \end{aligned} \quad (41)$$

such that

$$H(e_i^j) = \begin{cases} F(e_i^j) & \text{if } e_i^j \in A \setminus B, \\ G(e_i^j) & \text{if } e_i^j \in B \setminus A, \\ F(e_i^j) \cup G(e_i^j) & \text{if } e_i^j \in A \cap B. \end{cases} \quad (42)$$

This means

$$\begin{aligned} &\psi((F, A) \tilde{\cup} (G, B))(\varepsilon_k^j) \\ &= \rho_j \left(\bigcup \begin{cases} F(e_i^j) & \text{if } e_i^j \in (A \setminus B) \cap \lambda_j^{-1}(\varepsilon_k^j), \\ G(e_i^j) & \text{if } e_i^j \in (B \setminus A) \cap \lambda_j^{-1}(\varepsilon_k^j), \\ F(e_i^j) \cup G(e_i^j) & \text{if } e_i^j \in (A \cap B) \cap \lambda_j^{-1}(\varepsilon_k^j) \end{cases} \right). \end{aligned} \quad (43)$$

On the right hand side, for $\varepsilon_k^j \in \lambda(A) \cup \lambda(B) \subseteq E'$, we have

$$\begin{aligned}
 & (\psi(F, A) \tilde{\cup} \psi(G, B))(\varepsilon_k^j) \\
 &= \psi(F, A)(\varepsilon_k^j) \cup \psi(G, B)(\varepsilon_k^j) \\
 &= \rho_j \left(\bigcup_{e_i^j \in \lambda_j^{-1}(\varepsilon_k^j) \cap A} F(e_i^j) \right) \cup \rho_j \left(\bigcup_{e_i^j \in \lambda_j^{-1}(\varepsilon_k^j) \cap B} G(e_i^j) \right) \\
 &= \rho_j \left(\left(\bigcup_{e_i^j \in \lambda_j^{-1}(\varepsilon_k^j) \cap A} F(e_i^j) \right) \cup \left(\bigcup_{e_i^j \in \lambda_j^{-1}(\varepsilon_k^j) \cap B} G(e_i^j) \right) \right) \\
 &= \rho_j \left(\bigcup \left\{ \begin{array}{ll} F(e_i^j) & \text{if } e_i^j \in \lambda_j^{-1}(\varepsilon_k^j) \cap (A \setminus B), \\ G(e_i^j) & \text{if } e_i^j \in \lambda_j^{-1}(\varepsilon_k^j) \cap (B \setminus A), \\ F(e_i^j) \cup G(e_i^j) & \text{if } e_i^j \in \lambda_j^{-1}(\varepsilon_k^j) \cap (A \cap B) \end{array} \right. \right). \quad (44)
 \end{aligned}$$

Note that $\varepsilon_k^j \in \lambda(A \cup B) = \lambda(A) \cup \lambda(B)$ and for all $\varepsilon_k^j \in \lambda(A \cup B)$, $\psi((F, A) \tilde{\cup} (G, B))(\varepsilon_k^j) = (\psi(F, A) \tilde{\cup} \psi(G, B))(\varepsilon_k^j)$. Hence, we have (3).

(4) Note that $\lambda(A \cap B) \subseteq \lambda(A) \cap \lambda(B)$. For $\varepsilon_k^j \in \lambda(A \cap B) \subseteq E'$, we want to show that $\psi((F, A) \tilde{\cap} (G, B))(\varepsilon_k^j) \subseteq (\psi(F, A) \tilde{\cap} \psi(G, B))(\varepsilon_k^j)$. For $\varepsilon_k^j \in \lambda(A \cap B) \subseteq E'$, we have

$$\begin{aligned}
 & \psi((F, A) \tilde{\cap} (G, B))(\varepsilon_k^j) \\
 &= \psi(F(e_i^j) \cap G(e_i^j), A \cap B)(\varepsilon_k^j) \\
 &= \rho_j \left(\bigcup_{e_i^j \in \lambda_j^{-1}(\varepsilon_k^j) \cap (A \cap B)} F(e_i^j) \cap G(e_i^j) \right) \\
 &= \rho_j \left(\left(\bigcup_{e_i^j \in \lambda_j^{-1}(\varepsilon_k^j) \cap (A \cap B)} F(e_i^j) \right) \cap \left(\bigcup_{e_i^j \in \lambda_j^{-1}(\varepsilon_k^j) \cap (A \cap B)} G(e_i^j) \right) \right) \\
 &\subseteq \left(\rho_j \left(\bigcup_{e_i^j \in \lambda_j^{-1}(\varepsilon_k^j) \cap (A \cap B)} F(e_i^j) \right) \right) \\
 &\cap \left(\rho_j \left(\bigcup_{e_i^j \in \lambda_j^{-1}(\varepsilon_k^j) \cap (A \cap B)} G(e_i^j) \right) \right)
 \end{aligned}$$

$$\begin{aligned}
 &\subseteq \left(\rho_j \left(\bigcup_{e_i^j \in \lambda_j^{-1}(\varepsilon_k^j) \cap A} F(e_i^j) \right) \right) \\
 &\cap \left(\rho_j \left(\bigcup_{e_i^j \in \lambda_j^{-1}(\varepsilon_k^j) \cap B} G(e_i^j) \right) \right) \\
 &= \psi(F, A)(\varepsilon_k^j) \cap \psi(G, B)(\varepsilon_k^j) \\
 &= (\psi(F, A) \cap \psi(G, B))(\varepsilon_k^j). \quad (45)
 \end{aligned}$$

Hence, we have (4).

(5) We have $F(e_i^j) \subseteq G(e_i^j)$, $e_i^j \in A \subseteq B$. Thus, for $\varepsilon_k^j \in \lambda(A)$, we want to show that $\psi(F, A)(\varepsilon_k^j) \subseteq \psi(G, B)(\varepsilon_k^j)$. Note that

$$\begin{aligned}
 \psi(F, A)(\varepsilon_k^j) &= \rho_j \left(\bigcup_{e_i^j \in \lambda_j^{-1}(\varepsilon_k^j) \cap A} F(e_i^j) \right) \\
 &\subseteq \rho_j \left(\bigcup_{e_i^j \in \lambda_j^{-1}(\varepsilon_k^j) \cap B} G(e_i^j) \right) \quad (46) \\
 &= \psi(G, B)(\varepsilon_k^j).
 \end{aligned}$$

Therefore, we have (5). $\square$

Example 52. Consider $(F, A) = \{(e_1^1, \{a, b\}), (e_2^1, \{b\}), (e_1^2, \{p, q\})\}$ and its image under ψ , $\psi(F, A) = \{(\varepsilon_1^1, \{s, t\}), (\varepsilon_1^2, \{w, z\})\}$ as in Example 46. We have

$$\begin{aligned}
 & \psi^{-1}(\psi(F, A)) \\
 &= \{(e_1^1, \{a, b\}), (e_2^1, \{a, b\}), (e_2^2, \{p, q\})\} \tilde{\mathcal{C}}(F, A), \\
 & \psi(F, A)^r \\
 &= \{(\varepsilon_1^1, \{r, s, t\}), (\varepsilon_1^2, \phi)\} \tilde{\mathcal{C}}(\psi(F, A))^r \\
 &= \{(\varepsilon_1^1, \{r\}), (\varepsilon_1^2, \phi)\}. \quad (47)
 \end{aligned}$$

This indicates that reversals of inequality (1) and the reversal of inequality (2) in Proposition 51 do not hold.

Proposition 53. Let $\psi : \text{MASS}(\mathcal{U}_E) \rightarrow \text{MASS}(\mathcal{V}_{E'})$ be a mapping and let $\{(F_k, A_k)\}_{k=1,2,\dots,n}$ be a family of multiaspect soft sets in $\text{MASS}(\mathcal{U}_E)$. Then, the following hold.

- (1) $\psi(\tilde{\cup}_{k=1}^n (F_k, A_k)) = \tilde{\cup}_{k=1}^n (\psi(F_k, A_k))$,
- (2) $\psi(\tilde{\cap}_{k=1}^n (F_k, A_k)) \subseteq \tilde{\cap}_{k=1}^n (\psi(F_k, A_k))$.

Proof. We only show (1) as (2) can be shown in similar manner.

If $k = \{1, 2\}$, then $\psi((F_1, A_1) \tilde{\cup} (F_2, A_2)) = \psi(F_1, A_1) \tilde{\cup} \psi(F_2, A_2)$.

Suppose $\psi(\tilde{\cup}_{k=1}^n (F_k, A_k)) = \tilde{\cup}_{k=1}^n (\psi(F_k, A_k))$ holds for $n \geq 1$. Then,

$$\begin{aligned} \psi(\tilde{\cup}_{k=1}^{n+1} (F_k, A_k)) &= \psi((\tilde{\cup}_{k=1}^n (F_k, A_k)) \tilde{\cup} (F_{n+1}, A_{n+1})) \\ &= \psi(\tilde{\cup}_{k=1}^n (F_k, A_k)) \tilde{\cup} \psi(F_{n+1}, A_{n+1}) \\ &= (\tilde{\cup}_{k=1}^n \psi(F_k, A_k)) \tilde{\cup} \psi(F_{n+1}, A_{n+1}) \\ &= \tilde{\cup}_{k=1}^{n+1} \psi(F_k, A_k). \end{aligned} \quad (48)$$

Thus, (1) holds for $n+1$. This shows that (1) holds for all n . $\square$

Proposition 54. Let $\psi : \text{MASS}(\mathcal{U}_E) \rightarrow \text{MASS}(\mathcal{V}_{E'})$, $\rho_j : P(U^j) \rightarrow P(V^j)$, $\lambda : E \rightarrow E'$, and $\lambda_j : E^{U^j} \rightarrow E'^{V^j}$ for $j \in I$ be mappings and (F', A') , $(G', B') \in \text{MASS}(\mathcal{V}_{E'})$. Then, the following hold:

- (1) $\psi(\psi^{-1}(F', A')) \subseteq (F', A')$,
- (2) $\psi(\psi^{-1}(F', A')) = (F', A')$, if ρ_j, λ_j are surjective for all j ,
- (3) $(\psi^{-1}(F', A'))^r = \psi^{-1}(F', A')^r$,
- (4) $\psi^{-1}((F', A') \tilde{\cup} (G, B)) = \psi^{-1}(F', A') \tilde{\cup} \psi^{-1}(G', B')$,
- (5) $\psi^{-1}((F', A') \tilde{\cap}_{\mathcal{R}} (G', B')) = \psi^{-1}(F', A') \tilde{\cap}_{\mathcal{R}} \psi^{-1}(G', B')$,
- (6) if $(F', A') \tilde{\subseteq} (G', B')$, then $\psi^{-1}(F', A') \tilde{\subseteq} \psi^{-1}(G', B')$.

Proof. We only give proofs of (1), (3), and (5) as the rest can be shown in similar fashion.

- (1) Let the soft preimage of (F', A') be $\psi^{-1}(F', A') = (W, A) \in \text{MASS}(\mathcal{U}_E)$ where for all $e_i^j \in A$, $W(e_i^j) = \psi^{-1}(F', A')(e_i^j) = \rho_j^{-1}(F'(\lambda_j(e_i^j)))$.

Then, $\psi(\psi^{-1}(F', A')) = \psi(W, A)$ where for all $\varepsilon_k^j \in \lambda(A) \subseteq A'$, we have

$$\begin{aligned} \psi(\psi^{-1}(F', A'))(\varepsilon_k^j) &= \psi(W, A)(\varepsilon_k^j) \\ &= \rho_j \left(\bigcup_{e_i^j \in \lambda_j^{-1}(\varepsilon_k^j) \cap A} W(e_i^j) \right) \end{aligned}$$

$$\begin{aligned} &= \rho_j \left(\bigcup_{e_i^j \in \lambda_j^{-1}(\varepsilon_k^j) \cap A} \rho_j^{-1}(F'(\lambda_j(e_i^j))) \right) \\ &= \rho_j \left(\rho_j^{-1} \left(\bigcup_{e_i^j \in \lambda_j^{-1}(\varepsilon_k^j) \cap A} (F'(\lambda_j(e_i^j))) \right) \right) \\ &\subseteq \bigcup_{e_i^j \in \lambda_j^{-1}(\varepsilon_k^j) \cap A} (F'(\lambda_j(e_i^j))) = F'(\varepsilon_k^j). \end{aligned} \quad (49)$$

Thus, $\psi(\psi^{-1}(F', A')) \tilde{\subseteq} (F', A')$.

- (3) By Definition 45, for $e_i^j \in \lambda^{-1}(A')$, $\psi^{-1}(F', A')(e_i^j) = \rho_j^{-1}(F'(\lambda_j(e_i^j)))$. Thus,

$$\begin{aligned} (\psi^{-1}(F', A'))^r(e_i^j) &= (\rho_j^{-1}(F'(\lambda_j(e_i^j))))^r \\ &= \rho_j^{-1}(F'(\lambda_j(e_i^j)))^r \\ &= \rho_j^{-1}((F')^r(\lambda_j(e_i^j))) \\ &= \psi^{-1}((F')^r, A')(e_i^j) \\ &= \psi^{-1}(F', A')^r(e_i^j). \end{aligned} \quad (50)$$

Hence, $(\psi^{-1}(F', A'))^r = \psi^{-1}((F', A')^r)$.

- (5) We want to show that for all $e_i^j \in \lambda^{-1}(A' \cap B')$ = $\lambda^{-1}(A') \cap \lambda^{-1}(B')$, we have

$$\begin{aligned} \psi^{-1}((F', A') \tilde{\cap}_{\mathcal{R}} (G', B'))(e_i^j) &= (\psi^{-1}(F', A') \tilde{\cap}_{\mathcal{R}} \psi^{-1}(G', B'))(e_i^j). \end{aligned} \quad (51)$$

Note that

$$\begin{aligned} \psi^{-1}((F', A') \tilde{\cap}_{\mathcal{R}} (G', B'))(e_i^j) &= \psi^{-1}(H', A' \cap B')(e_i^j) \\ &= \rho_j^{-1}(H'(\lambda_j(e_i^j))), \quad \text{where } \lambda_j(e_i^j) \in A' \cap B' \\ &= \rho_j^{-1}(F'(\varepsilon_k^j) \cap G'(\varepsilon_k^j)), \quad \text{where } \lambda_j(e_i^j) = \varepsilon_k^j \\ &= \rho_j^{-1}(F'(\lambda_j(e_i^j))) \cap \rho_j^{-1}(G'(\lambda_j(e_i^j))) \\ &= \psi^{-1}(F', A')(e_i^j) \cap \psi^{-1}(G', B')(e_i^j) \\ &= (\psi^{-1}(F', A') \tilde{\cap}_{\mathcal{R}} \psi^{-1}(G', B'))(e_i^j). \end{aligned} \quad (52)$$

Hence, we have (5). $\square$

Proposition 55. Let $\psi : \text{MASS}(\mathcal{U}_E) \rightarrow \text{MASS}(\mathcal{V}_{E'})$ be a mapping and $\{(F'_k, A'_k)\}_{k=\{1,2,\dots,n\}}$ be a family of multiaspect soft sets in $\text{MASS}(\mathcal{V}_{E'})$. Then, the following hold:

- (1) $\psi^{-1}(\bar{\cup}_{k=1}^n(F'_k, A'_k)) = \bar{\cup}_{k=1}^n(\psi^{-1}(F'_k, A'_k)),$
- (2) $\psi^{-1}(\bar{\cap}_{k=1}^n(F'_k, A'_k)) = \bar{\cap}_{k=1}^n(\psi^{-1}(F'_k, A'_k)).$

Proof. Similar to the proofs of Proposition 53. $\square$

Example 56. Let $\psi : \text{MASS}(\mathcal{U}_E) \rightarrow \text{MASS}(\mathcal{V}_{E'})$, $\rho_j : P(U^j) \rightarrow P(V^j)$, $\lambda : E \rightarrow E'$, and $\lambda_j : E^{U^j} \rightarrow E^{V^j}$ for $j = 1, 2$ be mappings defined as in Example 46. Suppose we choose a multiaspect soft set $(K', N') \in \text{MASS}(\mathcal{V}_{E'})$ where $(K', N') = \{(\epsilon_1^1, \{s, t\}), (\epsilon_1^2, \{w\}), (\epsilon_2^2, \{z\}), (\epsilon_3^2, \{w, z\})\}$. By Definition 45, the soft preimage of (K', N') is obtained as $\psi^{-1}(K', N') = \{(e_1^1, \{a, b\}), (e_2^1, \{a, b\}), (e_1^2, \{q\}), (e_2^2, \{p, q\})\}$. Using Definition 44, we have $\psi(\psi^{-1}(K', N')) = \{(\epsilon_1^1, \{s, t\}), (\epsilon_1^2, \{w\}), (\epsilon_2^2, \{z\})\} \subseteq (K', N')$.

5. Conclusion

This paper presented a new concept of multiaspect soft sets which can be used to represent more detailed descriptions of objects or situations involving a collection of aspect-related universes. Several operations on multiaspect soft set have been established and their important properties are studied. This concept provides a new dimension of information representation in soft set-based structure.

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