Several Types of Totally Continuous Functions in Double Fuzzy Topological Spaces

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1. Introduction

The concept of fuzzy sets was introduced by Zadeh in his classical paper [1]. In 1968, Chang [2] used fuzzy sets to introduce the notion of fuzzy topological spaces. Coker [3, 4] defined the intuitionistic fuzzy topological spaces using intuitionistic fuzzy sets. Later on, Demirci and Çoker [5] defined intuitionistic fuzzy topological spaces in Kubiak-Šostak’s sense as a generalization of Chang’s fuzzy topological spaces and intuitionistic fuzzy topological spaces. Mondal and Samanta [6] succeeded to make the topology itself intuitionistic. The resulting structure is given the new name “intuitionistic gradation of openness.” The name “intuitionistic” did not continue due to some doubts that were thrown about the suitability of this term. These doubts were quickly ended in 2005 by Gutiérrez García and Rodabaugh [7]. They proved that this term is unsuitable in mathematics and applications. Therefore, they replaced the word “intuitionistic” by “double” and renamed its related topologies. The notion of intuitionistic gradation of openness is given the new name “double fuzzy topological spaces.”

The fuzzy type of the notion of topology can be studied in the fuzzy mathematics, which has many applications in different branches of mathematics and physics theory. For example, fuzzy topological spaces can be applied in the modeling of spatial objects such as rivers, roads, trees, and buildings. Since double fuzzy topology forms an extension of fuzzy topology and general topology, we think that our results can be applied in modern physics and GIS Problems.

Jain et al. introduced totally continuous, fuzzy totally continuous, and intuitionistic fuzzy totally continuous functions in topological spaces, respectively (see [8–11]).

In this paper, we introduce the notions of totally continuous functions, totally semicontinuous functions, and semitotally continuous functions in double fuzzy topological spaces. Their characterizations and the relationship with other already known kinds of functions are introduced and discussed.

2. Preliminaries

Throughout this paper, let \( X \) be a nonempty set and let \( I \) be the closed interval \([0, 1]\), \( I_0 = (0, 1] \) and \( I_1 = [0, 1) \). The set of all fuzzy subsets (resp., fuzzy points) of \( X \) is denoted by \( I^X \) (resp., \( P_I(X) \)). For \( x \in X \) and \( t \in I_0 \), a fuzzy point is defined by

\[
x_t(y) = \begin{cases} t, & \text{if } y = x, \\ 0, & \text{if } y \neq x.
\end{cases}
\]
Theorem 2 (see [12–14]). Let \((X, \tau, \tau^*)\) be a dfts. Then, for each \(r \in I_0, s \in I_1, \) and \(\lambda \in I^X,\) we define an operator \(C_{\tau, \tau^*} : I^X \times I_0 \times I_1 \rightarrow I^X\) as follows:

\[
C_{\tau, \tau^*}(\lambda, r, s) = \left\{ \mu \in I^X \mid \lambda \leq \mu, \tau(1 - \mu) \geq r, \tau^*(1 - \mu) \leq s \right\}.
\]

For \(\lambda, \mu \in I^X, r, r_1, r_2 \in I_0,\) and \(s, s_1, s_2 \in I_1,\) the operator \(C_{\tau, \tau^*}\) satisfies the following statements.

\[\begin{align*}
(C1) & \quad C_{\tau, \tau^*}(0, r, s) = 0, \\
(C2) & \quad \lambda \leq C_{\tau, \tau^*}(\lambda, r, s), \\
(C3) & \quad C_{\tau, \tau^*}(\lambda, r, s) \cup C_{\tau, \tau^*}(\mu, r, s) = C_{\tau, \tau^*}(\lambda \vee \mu, r, s), \\
(C4) & \quad C_{\tau, \tau^*}(\lambda, r_1, s_1) \subseteq C_{\tau, \tau^*}(\lambda, r_2, s_2) \text{ for } r_1 \leq r_2 \text{ and } s_1 \geq s_2, \\
(C5) & \quad C_{\tau, \tau^*}(C_{\tau, \tau^*}(\lambda, r, s), r, s) = C_{\tau, \tau^*}(\lambda, r, s).
\end{align*}\]

Theorem 3 (see [12–14]). Let \((X, \tau, \tau^*)\) be a dfts. Then, for each \(r \in I_0, s \in I_1,\) and \(\lambda \in I^X,\) we define an operator \(I_{\tau, \tau^*} : I^X \times I_0 \times I_0 \rightarrow I^X\) as follows:

\[
I_{\tau, \tau^*}(\lambda, r, s) = \left\{ \mu \in I^X \mid \mu \leq \lambda, \tau(\mu) \geq r, \tau^*(\mu) \leq s \right\}.
\]

For \(\lambda, \mu \in I^X, r, r_1, r_2 \in I_0,\) and \(s, s_1, s_2 \in I_1,\) the operator \(I_{\tau, \tau^*}\) satisfies the following statements.

\[\begin{align*}
(I1) & \quad I_{\tau, \tau^*}(1 - \lambda, r, s) = 1 - C_{\tau, \tau^*}(\lambda, r, s), \\
(I2) & \quad I_{\tau, \tau^*}(\lambda, r, s) = 1, \\
(I3) & \quad I_{\tau, \tau^*}(\lambda, r, s) \leq \lambda, \\
(I4) & \quad I_{\tau, \tau^*}(\lambda, r, s) \cap I_{\tau, \tau^*}(\mu, r, s) = I_{\tau, \tau^*}(\lambda \wedge \mu, r, s).
\end{align*}\]
(2) double fuzzy totally semicontinuous (briefly, dfstsc) if 
\[ f^{-1}(\mu) \text{ is } (r, s)\text{-fsco, for each } \mu \in I^Y, r \in I_0, \text{ and } s \in I_1 \]
such that \( r_1(1 - f^{-1}(\mu)) \geq r \) and \( r_2^*(1 - f^{-1}(\mu)) \leq s \),
(3) double fuzzy totally precontinuous (briefly, dftpc) if 
\[ f^{-1}(\mu) \text{ is } (r, s)\text{-fpsco, for each } \mu \in I^Y, r \in I_0, \text{ and } s \in I_1 \]
such that \( r_1(\mu) \geq r \) and \( r_2^*(\mu) \leq s \),
(4) double fuzzy semicontinuous (briefly, dfsc) if \( f^{-1}(\mu) \) is \( (r, s)\text{-fsoin } I^Y \), for each \( (r, s)\text{-fo set } \lambda \in I^X, r \in I_0, \text{ and } s \in I_1 \),
(5) double fuzzy semiopen if \( f(\lambda) \) is an \( (r, s)\text{-fco in } I^Y \), for each \( (r, s)\text{-fc set } \lambda \in I^X, r \in I_0, \text{ and } s \in I_1 \).

Remark 10. A fuzzy set \( \lambda \) in a dfst \((X, r, r^*)\) is \( (r, s)\text{-fco if and only if it is an } (r, s)\text{-fco and } (r, s)\text{-fpsco set, where } r \in I_0 \text{ and } s \in I_1 \).

Theorem 11. Let \( f : (X, r_1, r_2) \rightarrow (Y, r_2, r_2^*) \) be a function. Then the following are equivalent:

(1) \( f \) is a dfstc function,
(2) \( f^{-1}(\mu) \) is an \( (r, s)\text{-fco set of } I^X \) for each \( \mu \in I^Y, r \in I_0, \text{ and } s \in I_1 \) such that \( r_1(1 - f^{-1}(\mu)) \geq r \) and \( r_2^*(1 - f^{-1}(\mu)) \leq s \),
(3) \( C_{r_1, r_2}(f^{-1}(\mu), r, s) \leq f^{-1}(C_{r_1, r_2}(\mu, r, s)) \) and \( r_2^*(f^{-1}(\mu)) \leq s \) for each \( \mu \in I^Y, r \in I_0, \text{ and } s \in I_1 \),
(4) \( f^{-1}(I_{r_1, r_2}(\mu, r, s)) \) is \( I_{r_1, r_2}^*(f^{-1}(\mu), r, s) \) and \( C_{r_1, r_2}(f^{-1}(I_{r_1, r_2}^*(\mu, r, s)), r, s) \) \( \leq f^{-1}(\mu) \) for each \( \mu \in I^Y, r \in I_0, \text{ and } s \in I_1 \).

Proof. (1) \( \Rightarrow \) (2): Let \( r_2(1 - \mu) \geq r \) and \( r_2^*(1 - \mu) \leq s \). By using (1), \( f^{-1}(1 - \mu) \) is an \( (r, s)\text{-fco set of } I^X \). Since
\[
\begin{align*}
\tau_1(1 - f^{-1}(\mu)) &= \tau_1(f^{-1}(1 - \mu)) \geq r, \\
\tau_2^*(1 - f^{-1}(\mu)) &= \tau_2^*(f^{-1}(1 - \mu)) \leq s, \\
\tau_1(1 - f^{-1}(\mu)) &= \tau_1(f^{-1}(1 - (1 - \mu))) = \tau_1(f^{-1}(\mu)) \geq r, \\
\tau_2^*(1 - f^{-1}(\mu)) &= \tau_2^*(f^{-1}(1 - (1 - \mu))) = \tau_2^*(f^{-1}(\mu)) \leq s,
\end{align*}
\]
thus \( f^{-1}(\mu) \) is an \( (r, s)\text{-fco set in } I^X \).

(2) \( \Rightarrow \) (3): Let \( \mu \in I^Y \). Then, \( C_{r_1, r_2}(\mu, r, s) \) is \( (r, s)\text{-fc set in } I^Y \). By (2), \( f^{-1}(C_{r_1, r_2}(\mu, r, s)) \) is \( (r, s)\text{-fc set of } I^X \). Hence,
\[
C_{r_1, r_2}(f^{-1}(\mu), r, s) \leq C_{r_1, r_2}(f^{-1}(C_{r_1, r_2}(\mu, r, s)), r, s) = f^{-1}(C_{r_1, r_2}(\mu, r, s)).
\]
Again by using (2),
\[
\begin{align*}
f^{-1}(\mu) &\leq f^{-1}(C_{r_1, r_2}(\mu, r, s)) \\
&= I_{r_1, r_2}(f^{-1}(C_{r_1, r_2}(\mu, r, s))).
\end{align*}
\]
(3) \( \Rightarrow \) (4): Let \( \mu \in I^Y \). By using (3), we have
\[
\begin{align*}
f^{-1}(C_{r_1, r_2}(1 - \mu, r, s)) &\leq C_{r_1, r_2}(f^{-1}(1 - \mu), r, s) \\
&= C_{r_1, r_2}(1 - f^{-1}(\mu), r, s).
\end{align*}
\]
Hence,
\[
\begin{align*}
f^{-1}(I_{r_1, r_2}(\mu, r, s)) &= f^{-1}(1 - f^{-1}(C_{r_1, r_2}(1 - \mu, r, s))) \\
&\leq 1 - f^{-1}(C_{r_1, r_2}(1 - \mu, r, s)) \\
&\leq 1 - (C_{r_1, r_2}(1 - f^{-1}(\mu), r, s)) \\
&= I_{r_1, r_2}(f^{-1}(\mu), r, s).
\end{align*}
\]
By using (3), we have
\[
\begin{align*}
f^{-1}(I_{r_1, r_2}(1 - \mu, r, s)) &\leq I_{r_1, r_2}(f^{-1}(C_{r_1, r_2}(1 - \mu, r, s)), r, s) \\
&= I_{r_1, r_2}(f^{-1}(1 - I_{r_1, r_2}(\mu, r, s)), r, s) \\
&= I_{r_1, r_2}(1 - f^{-1}(I_{r_1, r_2}(\mu, r, s)), r, s).
\end{align*}
\]
Hence,
\[
\begin{align*}
f^{-1}(\mu) &\geq 1 - f^{-1}(1 - \mu) \\
&\geq 1 - (I_{r_1, r_2}(f^{-1}(1 - \mu), r, s))) \\
&= C_{r_1, r_2}(f^{-1}(I_{r_1, r_2}(\mu, r, s)), r, s).
\end{align*}
\]
(4) \( \Rightarrow \) (1): Let \( \mu \in I^Y \) such that \( \mu = I_{r_1, r_2}(\mu, r, s) \). By using (4), we have
\[
\begin{align*}
f^{-1}(\mu) &= f^{-1}(I_{r_1, r_2}(\mu, r, s)) \leq I_{r_1, r_2}(f^{-1}(\mu), r, s). \\
&\leq I_{r_1, r_2}(f^{-1}(C_{r_1, r_2}(\mu, r, s)), r, s) \\
&= I_{r_1, r_2}(C_{r_1, r_2}(f^{-1}(\mu), r, s), r, s).
\end{align*}
\]
Hence, \( f^{-1}(\mu) = I_{r_1, r_2}(f^{-1}(\mu), r, s) \); that is, \( \tau_1(f^{-1}(\mu)) \geq r \) and \( \tau_2^*(f^{-1}(\mu)) \leq s \). By using (4), we have
\[
\begin{align*}
f^{-1}(\mu) &\geq C_{r_1, r_2}(f^{-1}(I_{r_1, r_2}(\mu, r, s)), r, s) \\
&\geq C_{r_1, r_2}(f^{-1}(\mu), r, s).
\end{align*}
\]
Hence, \( f^{-1}(\mu) = (C_{r_1, r_2}(f^{-1}(\mu), r, s)) \); that is, \( \tau_1(1 - f^{-1}(\mu)) \geq r \) and \( \tau_2^*(1 - f^{-1}(\mu)) \leq s \). Therefore \( f^{-1}(\mu) \) is an \( (r, s)\text{-fco set in } I^X \). Thus, \( f \) is dfstc function. \( \square \)
Definition 12. Let \((X, \tau, \tau^*)\) be a dfts. Then, it is called:

1. double fuzzy semiregular (resp., double fuzzy clopen regular) if, for each \((r, s)\)-fsc (resp., \((r, s)\)-fco) set \(\nu\) of \(I^X\), \(r \in I_0\), \(s \in I_1\) and each fuzzy point \(x, \notin \nu\), there exist disjoint \((r, s)\)-fco (resp., \((r, s)\)-fo) sets \(\lambda\) and \(\mu\) such that \(\nu \leq \lambda\) and \(x, \notin \mu\),

2. double fuzzy s-regular (resp., double fuzzy ultraregular) if, for each \((r, s)\)-sc set \(\nu\) of \(X\), each \(x, \notin \nu\), \(r \in I_0\), \(s \in I_1\), there exist disjoint \((r, s)\)-fco (resp., \((r, s)\)-fco) sets \(\lambda\) and \(\mu\) such that \(\nu \leq \lambda\) and \(x, \notin \mu\),

3. double fuzzy s-normal if, for each pair of nonzero disjoint \((r, s)\)-fco sets can be separated by disjoint \((r, s)\)-fco sets, \(r \in I_0\) and \(s \in I_1\),

4. double fuzzy clopen normal if, for each pair of disjoint \((r, s)\)-fco sets \(\lambda\) and \(\mu\) of \(I^X\), there exist two disjoint \((r, s)\)-fo sets \(\alpha\) and \(\beta\), \(r \in I_0\) and \(s \in I_1\) such that \(\alpha \leq \beta\) and \(x, \notin \mu\).

Theorem 13. If \(f : (X, \tau_1, \tau_1^*) \to (Y, \tau_2, \tau_2^*)\) is dfst injective semiopen function from a double fuzzy clopen regular space \((X, \tau_1, \tau_1^*)\) onto a double fuzzy space \((Y, \tau_2, \tau_2^*)\), then \((Y, \tau_2, \tau_2^*)\) is double fuzzy s-normal.

Proof. Suppose \(\beta\) be an \((r, s)\)-fc set in \(I^X\), \(y \notin \beta\), and \(f = \text{dfst}\); \(f^{-1}(\beta)\) is \((r, s)\)-fc in \(I^Y\), for each \(\beta, r \in I_0\), \(s \in I_1\) such that \(\tau_2(\beta) \geq r\) and \(\tau_2^*(\beta) \leq s\). Put \(y = f(x, \notin)\) and \(\alpha = f^{-1}(\beta);\) then \(x, \notin \alpha\). Since \((X, \tau_1, \tau_1^*)\) is double fuzzy clopen regular, there exist disjoint \((r, s)\)-fo sets \(\lambda\) and \(\mu\) such that \(\alpha \leq \beta\) and \(x, \notin \mu\). This implies \(f(\alpha) \leq f(\beta)\).

Theorem 14. If \(f : (X, \tau_1, \tau_1^*) \to (Y, \tau_2, \tau_2^*)\) is dfst injective semiopen function from a double fuzzy clopen normal space \((X, \tau_1, \tau_1^*)\) onto a double fuzzy space \((Y, \tau_2, \tau_2^*)\), then \((Y, \tau_2, \tau_2^*)\) is double fuzzy s-normal.

Proof. Suppose \(\alpha\) and \(\beta\) are any twodisjoint \((r, s)\)-fc sets in \(I^Y\), \(r \in I_0\), \(s \in I_1\). Since \(f = \text{dfst}\), \(f^{-1}(\alpha)\) and \(f^{-1}(\beta)\) are \((r, s)\)-fc subsets of \(I^X\), for each \(\alpha\) and \(\beta\), \(r \in I_0\), \(s \in I_1\) such that \(\tau_2(\alpha) \geq r\) and \(\tau_2^*(\alpha) \leq s\), \(\tau_2(\beta) \geq r\), and \(\tau_2^*(\beta) \leq s\). Put \(\lambda = f^{-1}(\alpha)\) and \(\mu = f^{-1}(\beta)\) and \(f = \text{dfst}\), so

\[\lambda \wedge \mu = f^{-1}(\alpha) \wedge f^{-1}(\beta) = f^{-1}(\alpha \wedge \beta) = f^{-1}(\emptyset) = 0.\]

Theorem 15. Let \(f : (X, \tau_1, \tau_1^*) \to (Y, \tau_2, \tau_2^*)\) be dfst, closed injective function. If \((Y, \tau_2, \tau_2^*)\) is double fuzzy s-normal, then \((X, \tau_1, \tau_1^*)\) is double fuzzy ultraregular.

Proof. Suppose \(\beta\) is an \((r, s)\)-fc set not containing \(x, \notin \alpha\), \(r \in I_0\), and \(s \in I_1\) such that \(\tau_2(\beta) \geq r\), and \(\tau_2^*(\beta) \leq s\). Put \(y = f(x, \notin)\) and \(\alpha = f^{-1}(\beta)\); then \(x, \notin \alpha\). Since \((X, \tau_1, \tau_1^*)\) is double fuzzy clopen normal, there exist disjoint \((r, s)\)-fo sets \(\lambda\) and \(\mu\) such that \(\alpha \leq \beta\) and \(x, \notin \mu\).

Theorem 16. Let \(f : (X, \tau_1, \tau_1^*) \to (Y, \tau_2, \tau_2^*)\) be a function. Then the following are equivalent:

1. \(f\) is a dfst function,
2. \(f^{-1}(\mu)\) is an \((r, s)\)-fc set of \(I^X\) for each \(\mu, r \in I_0\), \(s \in I_1\) such that \(\tau_2(1 - \mu) \geq r\) and \(\tau_2^*(1 - \mu) \leq s\),
3. \(C_{\tau_1, \tau_1^*}(f^{-1}(\mu), r, s) \leq f^{-1}(C_{\tau_2, \tau_2^*}(\mu, r, s))\) and \(f^{-1}(\mu) \leq I_{\tau_1, \tau_1^*}(f^{-1}(C_{\tau_2, \tau_2^*}(\mu, r, s)), r, s)\) for each \(\mu, r \in I_0\), \(s \in I_1\),
4. \(f^{-1}(I_{\tau_1, \tau_1^*}(\mu, r, s)) \leq I_{\tau_1, \tau_1^*}(f^{-1}(\mu), r, s)\) and \(C_{\tau_1, \tau_1^*}(f^{-1}(I_{\tau_1, \tau_1^*}(\mu, r, s)), r, s) \leq f^{-1}(\mu)\) for each \(\mu, r \in I_0\), \(s \in I_1\).

Proof. This proof is similar to the proof of Theorem 11.

Theorem 17. Let \((X, \tau_1, \tau_1^*)\) and \((Y, \tau_2, \tau_2^*)\) be dfsts. A function \(f : (X, \tau_1, \tau_1^*) \to (Y, \tau_2, \tau_2^*)\) is dfst if and only if \(f\) is dfst and dfstpc.

Proof. Let \(\mu \in I^Y\), \(r \in I_0\), and \(s \in I_1\) such that \(\tau_2(\mu) \geq r\) and \(\tau_2^*(\mu) \leq s\); then \(f^{-1}(\mu)\) is \((r, s)\)-fco set and \((r, s)\)-fco. From Remark 10, \(f^{-1}(\mu)\) is \((r, s)\)-fco set in \(I^X\). Therefore, \(f\) is dfst.

The completion of the proof is straightforward.
Theorem 18. Let \( f : (X, \tau_1, \tau'_1) \to (Y, \tau_2, \tau'_2) \) be a function. Then one has the following:

1. if \( f \) is dftc function, then \( f \) is dftsc;
2. if \( f \) is dftsc function, then \( f \) is dftc.

Proof. (1) Let \( f \) be a dftc function, \( \mu \in I^Y, r \in I_0, \) and \( s \in I_0 \) such that \( \tau_2(\mu) \geq r \) and \( \tau'_2(\mu) \leq s. \) By hypothesis, \( f^{-1}(\mu) \) is \( (r, s) \)-fsco set in \( I^X. \) Therefore \( f \) is a dftc function.

(2) Let \( \mu \in I^Y, r \in I_0, \) and \( s \in I_1 \) such that \( \tau_2(\mu) \geq r \) and \( \tau'_2(\mu) \leq s. \) By the hypothesis, \( f^{-1}(\mu) \) is \( (r, s) \)-fsco set in \( I^X. \) Hence \( f \) is a dftsc function.

The converse of the above theorem need not be true in general as shown by the following example.

Example 19. (1) Let \( X = \{a, b, c\} \) and \( Y = \{x, y, z\}. \) Define fuzzy sets \( \lambda_1, \lambda_2, \) and \( \mu_1 \) as follows:

\[
\lambda_1(a) = 0.4, \quad \lambda_1(b) = 0.3, \quad \lambda_1(c) = 0.2,
\lambda_2(a) = \mu_1(x) = 0.6, \quad \lambda_2(b) = \mu_1(y) = 0.7,
\lambda_2(c) = \mu_1(z) = 0.8.
\]

(18)

Let \( (\tau_1, \tau'_1) \) and \( (\tau_2, \tau'_2) \) be defined as follows:

\[
\tau_1(\lambda) = \begin{cases} 
1, & \text{if } \lambda = 0 \text{ or } 1; \\
1/2, & \text{if } \lambda = \lambda_1; \\
0, & \text{otherwise},
\end{cases}
\tau'_1(\lambda) = \begin{cases} 
0, & \text{if } \lambda = 0 \text{ or } 1; \\
1/2, & \text{if } \lambda = \lambda_1; \\
1, & \text{otherwise},
\end{cases}
\]

\[
\tau_2(\mu) = \begin{cases} 
1, & \text{if } \mu = 0 \text{ or } 1; \\
1/2, & \text{if } \mu = \mu_1; \\
0, & \text{otherwise},
\end{cases}
\tau'_2(\mu) = \begin{cases} 
0, & \text{if } \mu = 0 \text{ or } 1; \\
1/2, & \text{if } \mu = \mu_1; \\
1, & \text{otherwise},
\end{cases}
\]

Then the function \( f : (X, \tau_1, \tau'_1) \to (Y, \tau_2, \tau'_2) \) is defined by

\[
f(a) = x, \quad f(b) = y, \quad f(c) = z.
\]

(20)

Since \( \mu_1 \) is an \((1/2, 1/2)\)-fo set and \( f^{-1}(\mu_1) = \lambda_2 \) is an \((1/2, 1/2)\)-fo set if \((1/2, 1/2)\)-fo set, then \( f \) is dftsc but not dftc.

(2) in (1) define \( \lambda_1 \) and \( \mu_1 \) as follows:

\[
\lambda_1(a) = 0.2, \quad \lambda_1(b) = 0.3, \quad \lambda_1(c) = 0.2,
\mu_1(x) = 0.2, \quad \mu_1(y) = 0.4, \quad \mu_1(z) = 0.2.
\]

(21)

So \( f^{-1}(\mu_1) = \lambda_1 \) is \((1/2, 1/2)\)-fo set in \((X, \tau_1, \tau'_1)\) and not \((1/2, 1/2)\)-fo set set; that is, \( f \) is not dftsc.

Definition 20. Let \( f : (X, \tau_1, \tau'_1) \to (Y, \tau_2, \tau'_2) \) be a function between dfts’s \((X, \tau_1, \tau'_1)\) and \((Y, \tau_2, \tau'_2)\). Then \( f \) is called:

1. double fuzzy irresolute (dfir, for short) if \( f^{-1}(\mu) \) is \((r, s)\)-fsoset, for each \((r, s)\)-fo set \( \mu \in I^Y, r \in I_0, \) and \( s \in I_1. \)
2. double fuzzy semi- irresolute (dfisr, for short) if \( f^{-1}(\mu) \) is \((r, s)\)-fsco, for each \((r, s)\)-fo set \( \mu \in I^Y, r \in I_0, \) and \( s \in I_1, \)

Theorem 21. If a function \( f : (X, \tau_1, \tau'_1) \to (Y, \tau_2, \tau'_2) \) is a dfir function and \( g : (Y, \tau_2, \tau'_2) \to (Z, \tau_3, \tau'_3) \) is dfir (dfisr) function, then \( g \circ f : (X, \tau_1, \tau'_1) \to (Z, \tau_3, \tau'_3) \) is dfir (dfisr) function.

Proof. Let \( \mu \in I^Z, r \in I_0, \) and \( s \in I_1 \) such that \( \tau_3(\mu) \geq r \) and \( \tau'_3(\mu) \leq s. \) Since \( g \) is a dftsc (dftc, resp.) function, \( g^{-1}(\mu) \) is \((r, s)\)-fsco, \((r, s)\)-fsyo, (resp.) set in \( I^Y. \) Also, since \( f \) is a dftc function, \( f^{-1}(g^{-1}(\mu)) \) is \((r, s)\)-fo set in \( I^X. \) Since \( (g \circ f)^{-1}(\mu) = f^{-1}(g^{-1}(\mu)), (g \circ f)^{-1}(\mu) \) is \((r, s)\)-fo set in \( I^X. \) Therefore, \( g \circ f \) is a dftc function.

4. Semitotally Continuous Functions in Double Fuzzy Topological Spaces

Now, we introduce the concept of semitotally continuous function which is stronger than totally continuous function in double fuzzy topological spaces, and then we investigate some characteristic properties.

Definition 22. Let \( f : (X, \tau_1, \tau'_1) \to (Y, \tau_2, \tau'_2) \) be a function between dfts’s \((X, \tau_1, \tau'_1)\) and \((Y, \tau_2, \tau'_2)\); then \( f \) is called double fuzzy semitotally continuous function (briefly, dfstc) if \( f^{-1}(\mu) \) is \((r, s)\)-fo, for each \((r, s)\)-fo set \( \mu \in I^Y, r \in I_0, \) and \( s \in I_1. \)

Theorem 23. Let \( f : (X, \tau_1, \tau'_1) \to (Y, \tau_2, \tau'_2) \) be a function between dfts’s \((X, \tau_1, \tau'_1)\) and \((Y, \tau_2, \tau'_2)\). Then the following are equivalent:

1. \( f \) is a dfstc function,
2. for each \( x_i \in I^X \) and each \((r, s)\)-fo set \( \mu \in I^Y, r \in I_0, \) and \( s \in I_1, \) and \( f(x_i) \leq \mu, \) there exists an \((r, s)\)-fo set \( \lambda \in I^X \) such that \( x_i \in \lambda \) and \( f(\lambda) \leq \mu. \)

Proof. (1) \( \Rightarrow \) (2): Suppose \( f : (X, \tau_1, \tau'_1) \to (Y, \tau_2, \tau'_2) \) is dfstc and \( \mu \) is any \((r, s)\)-fo set in \( I^Y \) containing \( f(x_i), r \in I_0, \) and \( \lambda \in I^X \) so that \( x_i \in f^{-1}(\mu). \) Take \( \lambda = f^{-1}(\mu); \) then \( \lambda \) is an \((r, s)\)-fo set in \( I^X \) and \( x_i \in \lambda, \) since \( f \) is dfstc and \( f^{-1}(\mu) \) is \((r, s)\)-fo set in \( I^X. \) Therefore, \( f(\lambda) = f\left(f^{-1}(\mu)\right) \leq \mu. \)

This implies \( f(\lambda) \leq \mu. \)

(2) \( \Rightarrow \) (1): Suppose \( \mu \) is an \((r, s)\)-fo set in \( I^Y, r \in I_0, \) and \( s \in I_1 \) and let \( x_i \in f^{-1}(\mu) \) be any fuzzy point; then \( f(x_i) \leq \mu. \) Therefore, by hypothesis, there is an \((r, s)\)-fo set \( f(\lambda x_i) \) containing \( x_i \) such that \( f(x_i) \leq \mu, \) so

\[
x_i \in \lambda x_i \leq f^{-1}(\mu)
\]

(23)
which implies that \( f^{-1}(\mu) \) is an \((r, s)\)-fco of \( x_t \) and \( r(f^{-1}(\mu)) \geq r \) and \( r^*\left(f^{-1}(\mu)\right) \leq s \) and \( x_t \) of \( f^{-1}(\mu) \). Hence it is an \((r, s)\)-fco in \( I^X \). Therefore, \( f \) is dfstc function.

**Theorem 24.** Let \( f : (X, \tau_1, \tau_1^*) \to (Y, \tau_2, \tau_2^*) \) be a function. Then the following are equivalent:

1. \( f \) is a dfstc function,
2. \( f^{-1}(\mu) \) is an \((r, s)\)-fco of \( x_t \) and \( \tau(\mu) \geq r \) and \( \tau^*(\mu) \leq s \) and \( x_t \) of \( \mu \).

(4) \( \Rightarrow \) (1): Let \( \mu \in I^Y \) be an \((r, s)\)-fsc set, \( r \in I_0, s \in I_1 \); then
\[
 f^{-1}(\mu) \geq C_{\tau_1, \tau_1^*} \left( f^{-1} \left( C_{\tau_2, \tau_2^*} \left( \left( \left( \left( \left( \left( \mu, r, s \right), r, s \right), r, s \right), r, s \right), r, s \right), r, s \right), r, s \right), r, s \right), r, s \right).
\]

The converse of the theorem need not be true in general as shown by the following example.

**Example 26.** See Example 19 (1) with \( \mu_1 \) and take \( \lambda_1(\alpha) = \lambda_1(b) = \lambda_1(c) = 0.5 \). Clearly \( f \) is dfstc, but not dfstc.

**Theorem 25.** Every dfstc function is a dfstc function.

**Proof.** Suppose \( f : (X, \tau_1, \tau_1^*) \to (Y, \tau_2, \tau_2^*) \) is a dfstc function and \( \mu \in I^Y \), \( r \in I_0 \), and \( s \in I_1 \) such that \( \tau_2(\mu) \geq r \) and \( \tau_2^*(\mu) \leq s \). Since \( \mu \) is \((r, s)\)-fso set in \( I^Y \) and \( f \) is dfstc function, it follows that \( f^{-1}(\mu) \) is \((r, s)\)-fco set in \( I^X \). Therefore \( f \) is dfstc function.

The converse of the above need not be true as shown by the following example.

**Example 28.** Let \( X = \{a, b, c\} \) and \( Y = \{x, y, z\} \) define \( \lambda_1 \) and \( \mu_1 \) as follows:
\[
\begin{align*}
\lambda_1(a) &= 0.4, & \lambda_1(b) &= 0.3, & \lambda_1(c) &= 0.2, \\
\mu_1(a) &= 0.3, & \mu_1(y) &= 0.4, & \mu_1(z) &= 0.5,
\end{align*}
\]

Define \((\tau_1, \tau_1^*)\) and \((\tau_2, \tau_2^*)\) on \( X \) as follows:
\[
\begin{align*}
\tau_1(\lambda) &= \begin{cases} 
1, & \text{if } \lambda = 0 \text{ or } 1; \\
\frac{1}{2}, & \text{if } \lambda = \lambda_1; \\
0, & \text{otherwise,}
\end{cases} \\
\tau_1^*(\lambda) &= \begin{cases} 
0, & \text{if } \lambda = 0 \text{ or } 1; \\
\frac{1}{2}, & \text{if } \lambda = \lambda_1; \\
1, & \text{otherwise,}
\end{cases}
\end{align*}
\]
We defined the function $f : (X, r_1, r_1^*) \rightarrow (Y, r_2, r_2^*)$ by

$$f(a) = x, \quad f(b) = y, \quad f(c) = z. \quad (33)$$

So $f^{-1}(\mu_1)$ is $(1/2, 1/2)$-fsc set but not $(1/2, 1/2)$-fco set; that is, $f$ is not dfstc function.

**Theorem 29.** Let $(X, r_1, r_1^*)$, $(Y, r_2, r_2^*)$, and $(Z, r_3, r_3^*)$ be dfts, and let $f : (X, r_1, r_1^*) \rightarrow (Y, r_2, r_2^*)$, $g : (Y, r_2, r_2^*) \rightarrow (Z, r_3, r_3^*)$, and $g \circ f : (X, r_1, r_1^*) \rightarrow (Z, r_3, r_3^*)$ be functions. Then one has the following:

1. If $f$ is dfstc and $g$ is dfstc, then $g \circ f$ is dfstc;
2. If $f$ is dfstc and $g$ is dfir, then $g \circ f$ is dfstc;
3. If $f$ is dfstc and $g$ is dfsc, then $g \circ f$ is dfstc;
4. If $f$ is dfstc and $g$ is any function, then $g \circ f$ is dfstc if and only if $g$ is dfir.

**Proof.** (1) It is clear.

(2) Let $\mu$ be an $(r, s)$-fso set in $I^X$, $r \in I_0$, and $s \in I_1$ and $g$ be dfir, so $g^{-1}(\mu)$ is $(r, s)$-fso set in $I^Y$. Also since $f$ is dfstc function, then $f^{-1}(g^{-1}(\mu)) = (g \circ f)^{-1}(\mu)$ is $(r, s)$-fco set in $I^X$. Hence $g \circ f$ is dfstc function.

(3) It is similar to the proof of (2).

(4) The proof follows from (2).

Conversely, let $g \circ f$ be dfstc and let $\mu$ be an $(r, s)$-fso set in $I^Y$, $r \in I_0$, and $s \in I_1$. Now, by hypothesis $g \circ f$ is dfstc;

$$(g \circ f)^{-1}(\mu) = f^{-1}(g^{-1}(\mu)) \quad (r, s)\text{-fco set in } I^X.$$ But $f$ is dfstc; then $g^{-1}(\mu)$ is $(r, s)$-fco in $I^Y$. Hence $g \circ f$ is dfstc.

**Definition 30.** Let $(X, r, r^*)$ be a dfts. Then it is called:

1. double fuzzy semi-$T_0$ if, for each pair of distinct fuzzy points in $I^X$, there exists an $(r, s)$-fso set containing one fuzzy point but not the other, $r \in I_0$ and $s \in I_1$.
2. double fuzzy semi-$T_1$ (resp., double fuzzy clopen $T_1$) if, for each pair of distinct fuzzy points $x_n$ and $y_m$ of $I^X$, $r \in I_0$, and $s \in I_1$, there exist $(r, s)$-fso (resp., $(r, s)$-fco) sets $\lambda$ and $\mu$ containing $x_n$ and $y_m$, respectively, such that $x_n \in \lambda, y_m \notin \lambda$ and $x_n \notin \mu, y_m \in \mu$.
3. double fuzzy semi-$T_2$ (resp., double fuzzy ultra-$T_2$) if, for each disjoint points $x_n$ and $y_m$ of $I^X$ can be separated by disjoint $(r, s)$-fso (resp., $(r, s)$-fco) sets, $r \in I_0$, and $s \in I_1$.
4. double fuzzy ultranormal if, for each pair of nonzero disjoint $(r, s)$-fco sets can be separated by disjoint $(r, s)$-fso sets, $r \in I_0$, and $s \in I_1$.
5. double fuzzy seminormal if, for each pair of disjoint $(r, s)$-fso sets $\alpha$ and $\beta$, there exist two disjoint $(r, s)$-fco sets $\lambda$ and $\mu$, $r \in I_0$, and $s \in I_1$ such that $\lambda \leq \alpha$ and $\mu \leq \beta$.

6. double fuzzy s-connected if $(X, r, r^*)$ is not the union of two disjoint nonzero $(r, s)$-fso subsets of $I^X$, $r \in I_0$, and $s \in I_1$.

**Theorem 31.** Let $(X, r_1, r_1^*)$ and $(Y, r_2, r_2^*)$ be dfts’s and $f : (X, r_1, r_1^*) \rightarrow (Y, r_2, r_2^*)$ be a function. Then one has the following:

1. If $f$ is dfstc injection and $(Y, r_2, r_2^*)$ is double fuzzy semi-$T_1$, then $(X, r_1, r_1^*)$ is double fuzzy clopen-$T_1$.
2. If $f$ is dfstc injection and $(Y, r_2, r_2^*)$ is double fuzzy semi-$T_0$, then $(X, r_1, r_1^*)$ is double fuzzy ultra-$T_2$.
3. If $f$ is dfstc injection and $(Y, r_2, r_2^*)$ is double fuzzy semi-$T_2$, then $(X, r_1, r_1^*)$ is double fuzzy ultra-$T_2$.
4. If $f$ is dfstc injective double fuzzy semiopen function from a double fuzzy clopen regular space $(X, r_1, r_1^*)$ onto $(Y, r_2, r_2^*)$, then $(Y, r_2, r_2^*)$ is double fuzzy semiregular.
5. If $f$ is dfstc injective double fuzzy semiopen function from a double fuzzy clopen normal space $(X, r_1, r_1^*)$ onto $(Y, r_2, r_2^*)$, then $(Y, r_2, r_2^*)$ is double fuzzy seminormal.

**Proof.** (1) Suppose fuzzy points $x_n$ and $y_m$ are in $I^X$ such that $x_n \neq y_m$. Since $f$ is injective, then $f(x_n) \neq f(y_m)$ in $I^Y$. Also $(Y, r_2, r_2^*)$ is double fuzzy semi-$T_0$ so there exist $\lambda$ and $\mu$ which are $(r, s)$-fso sets in $I^Y$, $r \in I_0$, and $s \in I_1$ such that $f(x_n) \leq \lambda, f(y_m) \notin \lambda$, $f(y_m) \leq \mu$, and $f(y_m) \notin \lambda$; that is, $x_n \in f^{-1}(\lambda), y_m \notin f^{-1}(\mu), y_m \notin f^{-1}(\lambda)$, and $x_n \notin f^{-1}(\mu)$. Since $f$ is dfstc, then $f^{-1}(\lambda)$ and $f^{-1}(\mu)$ are $(r, s)$-fco subsets of $I^X$. That is, $(X, r_1, r_1^*)$ is double fuzzy clopen-$T_1$.

(2) Suppose fuzzy points $x_n$ and $y_m$ in $I^X$ such that $x_n \neq y_m$. Since $f$ is injective, then $f(x_n) \neq f(y_m)$ in $I^Y$. Also $(Y, r_2, r_2^*)$ is double fuzzy semi-$T_0$ so there exist $\lambda$ and $\mu$ which are $(r, s)$-fso sets in $I^Y$, $r \in I_0$, and $s \in I_1$ such that $f(x_n) \leq \lambda, f(y_m) \notin \lambda$; that is, $x_n \in f^{-1}(\lambda), y_m \notin f^{-1}(\lambda)$. But $f$ is dfstc; $f^{-1}(\lambda)$ and $f^{-1}(\mu)$ are $(r, s)$-fco subsets in $I^X$. This implies that every pair of distinct points of $I^X$ can be separated by disjoint $(r, s)$-fco subsets in $I^X$. Therefore $(X, r_1, r_1^*)$ is double fuzzy ultra-$T_2$.

(3) Suppose fuzzy points $x_n$ and $y_m$ in $I^X$ such that $x_n \neq y_m$. Since $f$ is injective, then $f(x_n) \neq f(y_m)$ in $I^Y$. But $(Y, r_2, r_2^*)$ is double fuzzy semi-$T_2$ so there exist $\lambda$ and $\mu$ which are $(r, s)$-fso sets in $I^Y$, $r \in I_0$, and $s \in I_1$ such that $f(x_n) \leq \lambda, f(y_m) \leq \mu$, and $\lambda \wedge \mu = \emptyset$; that is, $x_n \in f^{-1}(\lambda)$ and $y_m \in f^{-1}(\mu)$. But $f$ is dfstc; $f^{-1}(\lambda)$ and $f^{-1}(\mu)$ are $(r, s)$-fco sets in $I^X$ such that $f^{-1}(\lambda) \wedge f^{-1}(\mu) = f^{-1}(\lambda \wedge \mu) = \emptyset$. Therefore $(X, r_1, r_1^*)$ is double fuzzy ultra-$T_2$.

(4) Suppose $\beta$ is an $(r, s)$-fso set in $I^Y$ and $y_n \notin \beta$. Assume $y_n = f(x_k)$ and $\alpha = f^{-1}(\beta)$. Since $f$ is dfstc function, then $f^{-1}(\beta)$ is an $(r, s)$-fso set in $I^X$ such that $x_k \notin \alpha$. But $(X, r_1, r_1^*)$
is double fuzzy clopen regular; then there exist disjoint \((r, s)-\)fo sets \(\lambda\) and \(\mu\) such that \(\alpha \leq \lambda\) and \(x_t \in \mu\), which implies
\[
\beta = f(\alpha) \leq f(\lambda),
\]
\[
y_n = f(x_t) \leq f(\mu).
\]
Also, by hypothesis \(f\) is injective double fuzzy semiopen; then \(f(\lambda)\) and \(f(\mu)\) are \((r, s)-\)fo sets such that
\[
f(\lambda) \wedge f(\mu) = f(\lambda \wedge \mu) = f(\emptyset) = 0.
\]
Therefore \((Y, \tau_2, \tau_2')\) is double fuzzy semirregular.

(5) Assume that \(\beta_1\) and \(\beta_2\) are any two disjoint \((r, s)-\)fsc sets in \(I^X\). Take \(\lambda = f^{-1}(\beta_1)\) and \(\mu = f^{-1}(\beta_2)\). Since \(f\) is dfstc injective, then \(f^{-1}(\beta_1)\) and \(f^{-1}(\beta_2)\) are \((r, s)-\)fo subsets of \(I^X\) such that
\[
\lambda \wedge \mu = f^{-1}(\beta_1) \wedge f^{-1}(\beta_2) = f^{-1}(\beta_1 \wedge \beta_2) = f^{-1}(\emptyset) = 0.
\]
But \((X, \tau_1, \tau_1')\) is double fuzzy clopen normal; then there exist disjoint \((r, s)-\)fo sets \(\delta_1\) and \(\delta_2\) such that \(\lambda \leq \delta_1\) and \(\mu \leq \delta_2\) which implies
\[
\beta_1 = f(\lambda) \leq f(\delta_1),
\]
\[
\beta_2 = f(\mu) \leq f(\delta_2).
\]
Also, since \(f\) is injective double fuzzy semiopen, \(f(\delta_1)\) and \(f(\delta_2)\) are disjoint \((r, s)-\)fo sets. Therefore \((Y, \tau_2, \tau_2')\) is double fuzzy seminormal.

**Theorem 32.** Let \((X, \tau_1, \tau_1')\) and \((Y, \tau_2, \tau_2')\) be dfstcs and let \(f : (X, \tau_1, \tau_1') \rightarrow (Y, \tau_2, \tau_2')\) be a function. Then one has the following.

1. If \(f\) is dfstc, double fuzzy closed injection and \((Y, \tau_2, \tau_2')\) is double fuzzy s-normal, then \((X, \tau_1, \tau_1')\) is double fuzzy ultranormal.
2. If \(f\) is dfstc, double fuzzy semilocal closed injection and \((Y, \tau_2, \tau_2')\) is double fuzzy s-regular, then \((X, \tau_1, \tau_1')\) is double fuzzy ultralinear.

**Proof.** (1) Suppose \(\mu_1, \mu_2\) are disjoint \((r, s)-\)fc subsets in \(I^X, r \in I_0, s \in I_1\). Since \(f\) is double fuzzy closed and injective, then \(\tau_2(1 - f(\mu_1)) \geq f(1 - \mu_1), \tau_2'(1 - f(\mu_1)) \leq \tau_2'(1 - \mu_1), \tau_3(1 - f(\mu_2)) \geq f(1 - \mu_2), \tau_3'(1 - f(\mu_2)) \leq \tau_3'(1 - \mu_2),\) for each \(\mu_1, \mu_2 \in I^X, r \in I_0, s \in I_1\) such that \(f(\mu_1)\) and \(f(\mu_2)\) are disjoint \((r, s)-\)fc subsets in \(I^Y, f(\mu_1)\) and \(f(\mu_2)\) are separated by disjoint \((r, s)-\)fo sets \(\lambda_1, \lambda_2,\) respectively, because \((Y, \tau_2, \tau_2')\) is double fuzzy s-normal space; then \(\mu_1 \leq f^{-1}(\lambda_1)\) and \(\mu_2 \leq f^{-1}(\lambda_2).\) Also \(f\) is dfstc, and \(f^{-1}(\lambda_1)\) and \(f^{-1}(\lambda_2)\) are \((r, s)-\)fc sets in \(I^X\) such that
\[
f^{-1}(\lambda_1) \wedge f^{-1}(\lambda_2) = f^{-1}(\lambda_1 \wedge \lambda_2) = 0.
\]
Therefore \((X, \tau_1, \tau_1')\) is double fuzzy ultralinear.

(2) Assume that an \((r, s)-\)fc set \(\alpha\) does not contain fuzzy point \(x_t\), and let \(f\) be a double fuzzy semiclosed function; then \(f(\alpha)\) is an \((r, s)-\)fsc set in \(I^Y\) such that \(f(\alpha)\) does not contain \(f(x_t)\). Since \((Y, \tau_2, \tau_2')\) is double fuzzy semirregular, then there exist disjoint \((r, s)-\)fo sets \(\lambda\) and \(\mu\) such that \(f(x_t) \leq \lambda\) and \(f(\alpha) \leq \mu\); that is \(x_t \in f^{-1}(\lambda)\) and \(\alpha \leq f^{-1}(\mu).\) Since \(f\) is dfstc function, then \(f^{-1}(\lambda)\) and \(f^{-1}(\mu)\) are \((r, s)-\)fo sets in \(I^X\). But \(f\) is injective; then
\[
f^{-1}(\lambda) \wedge f^{-1}(\mu) = f^{-1}(\lambda \wedge \mu) = f^{-1}(\emptyset) = 0.
\]
Therefore, \((X, \tau_1, \tau_1')\) is double fuzzy ultralinear.

**Theorem 33.** If \(f : (X, \tau_1, \tau_1') \rightarrow (Y, \tau_2, \tau_2')\) is dfstc surjection and \((X, \tau_1, \tau_1')\) is double fuzzy connected, then \((Y, \tau_2, \tau_2')\) is double fuzzy s-connected.

**Proof.** Assume \((Y, \tau_2, \tau_2')\) is not double fuzzy s-connected and \(\lambda\) and \(\mu\) are \((r, s)-\)fo sets in \(I^Y, r \in I_0, s \in I_1\) such that \(\lambda \wedge \mu = 1\) and \(\lambda \wedge \mu = 0\). Since \(f\) is dfstc,
\[
f^{-1}(\lambda \wedge \mu) = f^{-1}(\lambda) \wedge f^{-1}(\mu) = f^{-1}(1) = 1.
\]
Also \(f^{-1}(\lambda)\) and \(f^{-1}(\mu)\) are nonzero \((r, s)-\)fc sets in \(I^X\) such that
\[
f^{-1}(\lambda) \wedge f^{-1}(\mu) = f^{-1}(\lambda \wedge \mu) = 0.
\]
Then \((X, \tau_1, \tau_1')\) is not double fuzzy connected, so it is a contradiction. Hence \((Y, \tau_2, \tau_2')\) is double fuzzy s-connected.

5. Conclusion

In this paper, we introduced and characterized the notions of totally continuous functions, totally semicontinuous functions, and semitotally continuous functions in double fuzzy topological spaces. The relationship with other kinds of functions is studied. We could know that double fuzzy topological spaces are a generalization of some other kinds of topological spaces; therefore, our results can be considered as a generalization of the same results in other kinds of topological spaces. Also, it is possible to study this topic for a completely distributive DeMorgan algebra.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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