Research Article

n -Tupled Fixed Points Theorem in Fuzzy Metric Spaces with Application

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We will introduce the concept of n -tupled fixed points (for positive integer n) in fuzzy metric space by mild modification of the concept of n -tupled fixed points (for even positive integer n) introduced by Imdad et al. (2013) in metric spaces. As application of the above-mentioned concept, we will establish some n -tupled fixed point theorems for contractive type mappings in fuzzy metric space which extends the result of Roldán et al. (2013). Also we have given an application to solve a kind of Lipschitzian systems for n variables and an integral system.

1. Introduction

The concept of coupled fixed point was introduced by Bhaskar and Lakshmikantham [1] and it motivated the fixed point theorists to work in the area of multidimensional fixed points; for example, see [2–9].

In recent years some of the fixed point theorists tried to establish the existence of n -tupled fixed points and common n -tupled fixed points for some contractions in metric spaces, partially ordered metric spaces and asymptotically regular metric spaces. In particular, Imdad et al. [9] introduced the concept of n -tupled coincidence points as well as n -tupled fixed point (for even positive integer) and utilize these two definitions to obtain n -tupled coincidence as well as n -tupled common fixed point theorems. Very recently, Soliman et al. [10] proved some n -tupled coincident point theorems in partially ordered complete asymptotically regular metric spaces.

The purpose of our results is to introduce n -tupled fixed points (for all positive integers) and to prove n -tupled fixed points theorems for contractive type mappings in fuzzy metric spaces. Results obtained by us in Section 2 extends the work of Roldán et al. [11]. At the end we have given an application to solve a kind of Lipschitzian systems for n

variables and an integral system. Results proved in this paper follow the lines of the proof of Roldán et al. [11] for obtaining coincidence and fixed points.

A triangular norm (also called a t -norm) is a mapping $* : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is associative, commutative, and nondecreasing in both arguments and has 1 as an identity element.

Definition 1 (see [11, 12]). For any $a \in [0, 1]$, let the sequence $\{ *^n a \}_{n=1}^{\infty}$ be defined by $*^1 a = a$ and $*^n a = (*^{n-1} a) * a$. Then a t -norm $*$ is said to be of H -type if the sequence $\{ *^n a \}_{n=1}^{\infty}$ is equicontinuous at $a = 1$, that is, for all $\epsilon \in (0, 1)$, there exists $\eta \in (0, 1)$ such that if $a \in (1 - \eta, 1]$, then $*^m a > 1 - \epsilon$ for all $m \in \mathbb{N}$.

The most important and well-known continuous t -norm of H -type is $* = \min$, which verifies $\min(a, b) \geq ab$ for all $a, b \in [0, 1]$. The following result presents a wide range of t -norms of H -type.

Lemma 2. Let $\delta \in (0, 1]$ be a real number and let $*$ be a t -norm. Define $*_{\delta}$ as $x *_{\delta} y = x * y$, if $\max(x, y) \leq 1 - \delta$ and $x *_{\delta} y = \min(x, y)$, if $\max(x, y) > 1 - \delta$. Then $*_{\delta}$ is a t -norm of H -type.

Definition 3 (see [13]). A triplet $(X, M, *)$ is said to be a fuzzy metric space (in the sense of Kramosil and Michalek; briefly, a FMS), if X is a nonempty set, $*$ is a continuous t -norm, and $M : X \times X \times [0, \infty) \rightarrow [0, 1]$ is a fuzzy set satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$:

$$(FM_1) \quad M(x, y, t) > 0,$$

$$(FM_2) \quad M(x, y, t) = 1 \text{ if and only if } x = y,$$

$$(FM_3) \quad M(x, y, t) = M(y, x, t),$$

$$(FM_4) \quad M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1] \text{ is left continuous,}$$

$$(FM_5) \quad M(x, y, t) * M(y, z, s) \leq M(x, z, t + s),$$

in this case, we also say that (X, M) is a FMS under $*$;

in the sequel, we will only consider FMS verifying

$$(FM_6) \quad \lim_{t \rightarrow \infty} M(x, y, t) = 1 \text{ for all } x, y \in X.$$

Lemma 4. $M(x, y, \cdot)$ is a nondecreasing function on $[0, \infty)$.

Definition 5 (see [14]). Let $(X, M, *)$ be a fuzzy metric space; then

- (1) a sequence $\{x_n\}$ in X is said to be convergent to x if

$$\lim_{n \rightarrow \infty} M(x_n, x, t) = 1, \quad (1)$$

for all $t > 0$;

- (2) a sequence $\{x_n\}$ in X is said to be a Cauchy sequence if, for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that

$$M(x_n, x_m, t) > 1 - \epsilon, \quad (2)$$

for all $t > 0$ and $n, m \geq n_0$;

- (3) a fuzzy metric space $(X, M, *)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

Given any t -norm $*$, it is easy to prove that $*$ $\leq$ min. Therefore, if (X, M) is a FMS under min, then (X, M) is a FMS under any (continuous or not) t -norm. This is the case in the following examples (in which, obviously, we only define $M(x, y, t)$ for $t > 0$ and $x \neq y$).

Example 6. From a metric space (X, d) , we can consider a FMS in different ways. For $t > 0$ and $x \neq y$, define

$$(i) \quad M_d(x, y, t) = t/(t + d(x, y));$$

$$(ii) \quad M_e(x, y, t) = e^{-(d(x, y)/t)};$$

$$(iii) \quad M_c(x, y, t) = \begin{cases} 0, & \text{if } t \leq d(x, y) \\ 1 & \text{if } t > d(x, y). \end{cases}$$

It is well known that (X, M_d) is a FMS under the product $*$ = $\cdot$, called the standard FMS on (X, d) , since it is the standard way of viewing the metric space (X, d) as a FMS. However, it is also true (though lesser known) that (X, M_d) , (X, M_e) , and (X, M_c) are FMS under min.

Furthermore, (X, d) is a complete metric space if and only if (X, M_d) (or (X, M_c) or (X, M_e)) is a complete FMS. For instance, this is the case of any nonempty and closed subset (or subinterval) of R provided with its Euclidean metric.

Definition 7. A function $g : X \rightarrow X$ on a fuzzy metric space is said to be continuous at a point $x_0 \in X$ if, for any sequence $\{x_n\}$ in X converging to x_0 , the sequence $\{gx_n\}$ converges to gx_0 . If g is continuous at each $x \in X$, then g is said to be continuous on X . As usual, if $x_0 \in X$, we will denote $g^{-1}(x_0) = \{x \in X : gx = x_0\}$.

Remark 8. If $x \in [0, 1]$ and $a, b \in (0, \infty)$, then $a \leq b$ implies that $x^a \geq x^b$. We will use this fact in the following way: $0 < a \leq b \leq 1$ implies that $[M(x, y, t)]^a \geq [M(x, y, t)]^b \geq M(x, y, t)$.

2. Main Results

Henceforth, X will denote a nonempty set and $X^n = \underbrace{X \times X \times \dots \times X}_{n\text{-times}}$.

Here, we introduce n -tupled fixed points (for positive integer n) in fuzzy metric space by slightly modified concept of n -tupled fixed points (for natural number n) introduced by Imdad et al. [9] in metric space.

Definition 9. Let $F : X^n \rightarrow X$ and $g : X \rightarrow X$ be two mappings.

- (a) One says that F and g are commuting if

$$gF(x^{(1)}, x^{(2)}, \dots, x^{(n)}) = F(gx^{(1)}, gx^{(2)}, \dots, gx^{(n)}) \quad (3)$$

for all $x^{(1)}, x^{(2)}, \dots, x^{(n)} \in X$.

- (b) An element $(x^{(1)}, x^{(2)}, x^{(3)}, \dots, x^{(n)}) \in X^n$ is called n -tupled coincidence points of the mappings F and g if

$$\begin{aligned} F(x^{(1)}, x^{(2)}, x^{(3)}, \dots, x^{(n)}) &= gx^{(1)} \\ F(x^{(2)}, x^{(3)}, \dots, x^{(n)}, x^{(1)}) &= gx^{(2)} \\ F(x^{(3)}, \dots, x^{(n)}, x^{(1)}, x^{(2)}) &= gx^{(3)} \\ &\vdots \\ F(x^{(n)}, x^{(1)}, x^{(2)}, \dots, x^{(n-1)}) &= gx^{(n)}. \end{aligned} \quad (4)$$

Definition 10. An element $(x^{(1)}, x^{(2)}, \dots, x^{(n)}) \in X^n$ is called an n -tupled fixed point of map $F : X^n \rightarrow X$ and $g : X \rightarrow X$ if

$$\begin{aligned} F(x^{(1)}, x^{(2)}, x^{(3)}, \dots, x^{(n)}) &= gx^{(1)} = x^{(1)} \\ F(x^{(2)}, x^{(3)}, \dots, x^{(n)}, x^{(1)}) &= gx^{(2)} = x^{(2)} \\ F(x^{(3)}, \dots, x^{(n)}, x^{(1)}, x^{(2)}) &= gx^{(3)} = x^{(3)} \\ &\vdots \\ F(x^{(n)}, x^{(1)}, x^{(2)}, \dots, x^{(n-1)}) &= gx^{(n)} = x^{(n)}. \end{aligned} \quad (5)$$

Now, we present our main theorem.

Theorem 11. Let $*$ be a t -norm of H -type such that $s * t \geq st$ for all $s, t \in [0, 1]$. Let $k \in (0, 1)$ and $a_1, a_2, \dots, a_n \in [0, 1]$ be real numbers such that $a_1 + a_2 + \dots + a_n \leq 1$, let $(X, M, *)$ be a complete FMS, and let $F : X^n \rightarrow X$ and $g : X \rightarrow X$ be two mappings such that $F(X^n) \subseteq g(X)$ and g is continuous and commuting with F . Suppose that, for all $x^{(1)}, x^{(2)}, \dots, x^{(n)}, y^{(1)}, y^{(2)}, \dots, y^{(n)} \in X$ and all $t > 0$,

$$\begin{aligned} & M(F(x^{(1)}, x^{(2)}, \dots, x^{(n)}), \\ & F(y^{(1)}, y^{(2)}, \dots, y^{(n)}), kt) \\ & \geq [M(gx^{(1)}, gy^{(1)}, t)]^{a_1} \\ & * [M(gx^{(2)}, gy^{(2)}, t)]^{a_2} * \dots \\ & * [M(gx^{(n)}, gy^{(n)}, t)]^{a_n}. \end{aligned} \quad (6)$$

Then there exists a unique $x^{(1)} \in X$ such that $x^{(1)} = gx^{(1)} = F(x^{(1)}, x^{(1)}, \dots, x^{(1)})$. In particular, F and g have, at least, one n -tupled coincidence point. Furthermore, $(x^{(1)}, x^{(1)}, \dots, x^{(1)})$ is the unique n -tupled coincidence point of F and g if we assume that $g^{-1}(x_0^{(1)}) = \{x_0^{(1)}\}$ only in the case that $F \equiv x_0^{(1)}$ is constant on X^n .

In this result, in order to avoid the indetermination 0^0 , we assume that $[M(gx^{(1)}, gx^{(2)}, t)]^0 = 1$ for all $t > 0$ and all $x^{(1)}, x^{(2)} \in X$.

Proof. Suppose that F is constant in X^n ; that is, there exists $x_0^{(1)} \in X$ such that

$$F(x^{(1)}, x^{(2)}, \dots, x^{(n)}) = x_0^{(1)} \quad \forall x^{(1)}, x^{(2)}, \dots, x^{(n)} \in X. \quad (7)$$

As F and g are commuting, we deduce that

$$\begin{aligned} gx_0^{(1)} &= gF(x^{(1)}, x^{(2)}, \dots, x^{(n)}) \\ &= F(gx^{(1)}, gx^{(2)}, \dots, gx^{(n)}) = x_0^{(1)}. \end{aligned} \quad (8)$$

Therefore, $x_0^{(1)} = gx_0^{(1)} = F(x_0^{(1)}, x_0^{(1)}, \dots, x_0^{(1)})$ and $(x_0^{(1)}, x_0^{(1)}, \dots, x_0^{(1)})$ is n -tupled coincidence point of F and g .

Now, suppose that $g^{-1}(x_0^{(1)}) = \{x_0^{(1)}\}$ and $(x^{(1)}, x^{(2)}, \dots, x^{(n)}) \in X^n$ is another n -tupled coincidence point of F and g . Then,

$$\begin{aligned} gx^{(1)} &= F(x^{(1)}, x^{(2)}, \dots, x^{(n)}) = x_0^{(1)}, \\ \text{so } x^{(1)} &\in g^{-1}(x_0^{(1)}) = \{x_0^{(1)}\}. \end{aligned} \quad (9)$$

Similarly, $x^{(1)} = x^{(2)} = \dots = x^{(n)} = x_0^{(1)}$ and $(x_0^{(1)}, x_0^{(1)}, \dots, x_0^{(1)})$ is the unique n -tupled coincidence points of F and g .

Next, suppose that F is not constant in X^n .

In this case, $(x^{(1)}, x^{(2)}, \dots, x^{(n)}) \neq (0, 0, 0)$ and the proof is divided into five steps. In the entire proof, n and p stand to be nonnegative integers and $t \in [0, \infty)$.

Step 1. First we define of the sequences $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \dots, \{x_n^{(n)}\}$.

Let $x_0^{(1)}, x_0^{(2)}, \dots, x_0^{(n)} \in X$ be arbitrary points in X .

Since $F(X^n) \subseteq g(X)$, we can choose $x_1^{(1)}, x_1^{(2)}, \dots, x_1^{(n)} \in X$ such that

$$\begin{aligned} g(x_1^{(1)}) &= F(x_0^{(1)}, x_0^{(2)}, \dots, x_0^{(n)}), \\ g(x_1^{(2)}) &= F(x_0^{(2)}, x_0^{(3)}, \dots, x_0^{(n)}, x_0^{(1)}), \\ &\vdots \\ g(x_1^{(n)}) &= F(x_0^{(n)}, x_0^{(1)}, \dots, x_0^{(n-1)}), \end{aligned} \quad (10)$$

Again from $F(X^n) \subseteq g(X)$, we can choose $x_2^{(1)}, x_2^{(2)}, \dots, x_2^{(n)} \in X$ such that

$$\begin{aligned} g(x_2^{(1)}) &= F(x_1^{(1)}, x_1^{(2)}, \dots, x_1^{(n)}), \\ g(x_2^{(2)}) &= F(x_1^{(2)}, x_1^{(3)}, \dots, x_1^{(n)}, x_1^{(1)}), \\ &\vdots \\ g(x_2^{(n)}) &= F(x_1^{(n)}, x_1^{(1)}, \dots, x_1^{(n-1)}), \end{aligned} \quad (11)$$

Continuing in this way, we can construct sequences $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \dots, \{x_n^{(n)}\}$ in X such that

$$\begin{aligned} g(x_n^{(1)}) &= F(x_{n-1}^{(1)}, x_{n-1}^{(2)}, \dots, x_{n-1}^{(n)}), \\ g(x_n^{(2)}) &= F(x_{n-1}^{(2)}, x_{n-1}^{(3)}, \dots, x_{n-1}^{(n)}, x_{n-1}^{(1)}), \\ &\vdots \\ g(x_n^{(n)}) &= F(x_{n-1}^{(n)}, x_{n-1}^{(1)}, \dots, x_{n-1}^{(n-1)}), \end{aligned} \quad (12)$$

for $n \geq 1$.

Step 2. Now, we will show that the sequences $\{gx_n^{(1)}\}, \{gx_n^{(2)}\}, \dots, \{gx_n^{(n)}\}$ are Cauchy sequences. Define, for $n \geq 1$ and all $t \geq 0$,

$$\begin{aligned} \delta_n(t) &= M(g(x_n^{(1)}), g(x_{n+1}^{(1)}), t) \\ &* M(g(x_n^{(2)}), g(x_{n+1}^{(2)}), t) \\ &* \dots * M(g(x_n^{(n)}), g(x_{n+1}^{(n)}), t). \end{aligned} \quad (13)$$

Since δ_n is a nondecreasing function and $t - kt \leq t \leq t/k$, we have that

$$\delta_n(t - kt) \leq \delta_n(t) \leq \delta_n\left(\frac{t}{k}\right), \quad \forall t > 0, n \geq 1. \quad (14)$$

From inequality (6) we deduce, for all $n \in N$ and all $t \geq 0$,

$$\begin{aligned} & M(g(x_n^{(1)}), g(x_{n+1}^{(1)}), t) \\ &= M(F(x_{n-1}^{(1)}, x_{n-1}^{(2)}, \dots, x_{n-1}^{(n)}), \\ & F(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(n)}), t) \end{aligned}$$

$$\begin{aligned}
&\geq \left[M \left(gx_{n-1}^{(1)}, gx_n^{(1)}, \frac{t}{k} \right) \right]^{a_1} \\
&\quad * \left[M \left(gx_{n-1}^{(2)}, gx_n^{(2)}, \frac{t}{k} \right) \right]^{a_2} \\
&\quad * \cdots * \left[M \left(gx_{n-1}^{(n)}, gx_n^{(n)}, \frac{t}{k} \right) \right]^{a_n} \\
&M \left(g \left(x_n^{(2)} \right), g \left(x_{n+1}^{(2)} \right), t \right) \\
&= M \left(F \left(x_{n-1}^{(2)}, x_{n-1}^{(3)}, \dots, x_{n-1}^{(1)} \right), \right. \\
&\quad \left. F \left(x_n^{(2)}, x_n^{(3)}, \dots, x_n^{(1)} \right), t \right) \\
&\geq \left[M \left(gx_{n-1}^{(2)}, gx_n^{(2)}, \frac{t}{k} \right) \right]^{a_1} \\
&\quad * \left[M \left(gx_{n-1}^{(3)}, gx_n^{(3)}, \frac{t}{k} \right) \right]^{a_2} \\
&\quad * \cdots * \left[M \left(gx_{n-1}^{(1)}, gx_n^{(1)}, \frac{t}{k} \right) \right]^{a_n} \\
&\quad \vdots \\
&M \left(g \left(x_n^{(n)} \right), g \left(x_{n+1}^{(n)} \right), t \right) \\
&= M \left(F \left(x_{n-1}^{(n)}, x_{n-1}^{(1)}, \dots, x_{n-1}^{(n-1)} \right), \right. \\
&\quad \left. F \left(x_n^{(n)}, x_n^{(1)}, \dots, x_n^{(n-1)} \right), t \right) \\
&\geq \left[M \left(gx_{n-1}^{(n)}, gx_n^{(n)}, \frac{t}{k} \right) \right]^{a_1} \\
&\quad * \left[M \left(gx_{n-1}^{(1)}, gx_n^{(1)}, \frac{t}{k} \right) \right]^{a_2} \\
&\quad * \cdots * \left[M \left(gx_n^{(n-1)}, gx_n^{(1)}, \frac{t}{k} \right) \right]^{a_n}.
\end{aligned} \tag{15}$$

Using (15) and Remark 8, we have

$$\begin{aligned}
&M \left(g \left(x_n^{(1)} \right), g \left(x_{n+1}^{(1)} \right), t \right) \\
&\geq \left[M \left(gx_{n-1}^{(1)}, gx_n^{(1)}, \frac{t}{k} \right) \right]^{a_1} \\
&\quad * \left[M \left(gx_{n-1}^{(2)}, gx_n^{(2)}, \frac{t}{k} \right) \right]^{a_2} \\
&\quad * \cdots * \left[M \left(gx_{n-1}^{(n)}, gx_n^{(n)}, \frac{t}{k} \right) \right]^{a_n} \\
&\geq M \left(gx_{n-1}^{(1)}, gx_n^{(1)}, \frac{t}{k} \right) * M \left(gx_{n-1}^{(2)}, gx_n^{(2)}, \frac{t}{k} \right) \\
&\quad * \cdots * M \left(gx_{n-1}^{(n)}, gx_n^{(n)}, \frac{t}{k} \right) \\
&= \delta_{n-1} \left(\frac{t}{k} \right)
\end{aligned}$$

$$\begin{aligned}
&M \left(g \left(x_n^{(2)} \right), g \left(x_{n+1}^{(2)} \right), t \right) \\
&\geq \left[M \left(gx_{n-1}^{(2)}, gx_n^{(2)}, \frac{t}{k} \right) \right]^{a_1} \\
&\quad * \left[M \left(gx_{n-1}^{(3)}, gx_n^{(3)}, \frac{t}{k} \right) \right]^{a_2} \\
&\quad * \cdots * \left[M \left(gx_{n-1}^{(1)}, gx_n^{(1)}, \frac{t}{k} \right) \right]^{a_n} \\
&\geq M \left(gx_{n-1}^{(2)}, gx_n^{(2)}, \frac{t}{k} \right) \\
&\quad * M \left(gx_{n-1}^{(3)}, gx_n^{(3)}, \frac{t}{k} \right) \\
&\quad * \cdots * M \left(gx_{n-1}^{(1)}, gx_n^{(1)}, \frac{t}{k} \right) \\
&= \delta_{n-1} \left(\frac{t}{k} \right) \\
&M \left(g \left(x_n^{(n)} \right), g \left(x_{n+1}^{(n)} \right), t \right) \\
&\geq \left[M \left(gx_{n-1}^{(n)}, gx_n^{(n)}, \frac{t}{k} \right) \right]^{a_1} \\
&\quad * \left[M \left(gx_{n-1}^{(1)}, gx_n^{(1)}, \frac{t}{k} \right) \right]^{a_2} \\
&\quad * \cdots * \left[M \left(gx_{n-1}^{(n-1)}, gx_n^{(n-1)}, \frac{t}{k} \right) \right]^{a_n} \\
&\geq M \left(gx_{n-1}^{(n)}, gx_n^{(n)}, \frac{t}{k} \right) \\
&\quad * M \left(gx_{n-1}^{(1)}, gx_n^{(1)}, \frac{t}{k} \right) \\
&\quad * \cdots * M \left(gx_{n-1}^{(n-1)}, gx_n^{(n-1)}, \frac{t}{k} \right) \\
&= \delta_{n-1} \left(\frac{t}{k} \right).
\end{aligned} \tag{16}$$

This proves that, for all $t > 0$ and all $n \geq 0$,

$$\begin{aligned}
&M \left(g \left(x_n^{(1)} \right), g \left(x_{n+1}^{(1)} \right), t \right) \\
&\quad * M \left(g \left(x_n^{(2)} \right), g \left(x_{n+1}^{(2)} \right), t \right) \\
&\quad * \cdots * M \left(g \left(x_n^{(n)} \right), g \left(x_{n+1}^{(n)} \right), t \right) \\
&\geq \delta_{n-1} \left(\frac{t}{k} \right) \geq \delta_{n-1} (t).
\end{aligned} \tag{17}$$

Swapping t by $t - kt$, we deduce, for all $t > 0$ and $n \geq 0$, that

$$\begin{aligned}
&M \left(g \left(x_n^{(1)} \right), g \left(x_{n+1}^{(1)} \right), (t - kt) \right) \\
&\quad * M \left(g \left(x_n^{(2)} \right), g \left(x_{n+1}^{(2)} \right), (t - kt) \right) \\
&\quad * \cdots * M \left(g \left(x_n^{(n)} \right), g \left(x_{n+1}^{(n)} \right), (t - kt) \right) \\
&\geq \delta_{n-1} (t - kt).
\end{aligned} \tag{18}$$

Taking into account that $*$ is commutative and $* \geq \cdot$ and (15), we observe that

$$\begin{aligned}
 \delta_n(t) &= M\left(g(x_n^{(1)}), g(x_{n+1}^{(1)}), t\right) \\
 &\quad * M\left(g(x_n^{(2)}), g(x_{n+1}^{(2)}), t\right) \\
 &\quad * \dots * M\left(g(x_n^{(n)}), g(x_{n+1}^{(n)}), t\right) \\
 &\geq \left[M\left(gx_{n-1}^{(1)}, gx_n^{(1)}, \frac{t}{k}\right)\right]^{a_1} \\
 &\quad * \left[M\left(gx_{n-1}^{(2)}, gx_n^{(2)}, \frac{t}{k}\right)\right]^{a_2} \\
 &\quad * \dots * \left[M\left(gx_{n-1}^{(n)}, gx_n^{(n)}, \frac{t}{k}\right)\right]^{a_n} \\
 &\quad * \left[M\left(gx_{n-1}^{(2)}, gx_n^{(2)}, \frac{t}{k}\right)\right]^{a_1} \\
 &\quad * \left[M\left(gx_{n-1}^{(3)}, gx_n^{(3)}, \frac{t}{k}\right)\right]^{a_2} \\
 &\quad * \dots * \left[M\left(gx_{n-1}^{(1)}, gx_n^{(1)}, \frac{t}{k}\right)\right]^{a_n} \\
 &\quad * \dots * \left[M\left(gx_{n-1}^{(n)}, gx_n^{(n)}, \frac{t}{k}\right)\right]^{a_1} \\
 &\quad * \left[M\left(gx_{n-1}^{(1)}, gx_n^{(1)}, \frac{t}{k}\right)\right]^{a_2} \\
 &\quad * \dots * \left[M\left(gx_{n-1}^{(n-1)}, gx_n^{(n-1)}, \frac{t}{k}\right)\right]^{a_n} \\
 &= \left[M\left(gx_{n-1}^{(1)}, gx_n^{(1)}, \frac{t}{k}\right)\right]^{a_1} \\
 &\quad * \left[M\left(gx_{n-1}^{(1)}, gx_n^{(1)}, \frac{t}{k}\right)\right]^{a_2} \\
 &\quad * \dots * \left[M\left(gx_{n-1}^{(1)}, gx_n^{(1)}, \frac{t}{k}\right)\right]^{a_n} \\
 &\quad * \left[M\left(gx_{n-1}^{(2)}, gx_n^{(2)}, \frac{t}{k}\right)\right]^{a_1} \\
 &\quad * \left[M\left(gx_{n-1}^{(2)}, gx_n^{(2)}, \frac{t}{k}\right)\right]^{a_2} \\
 &\quad * \dots * \left[M\left(gx_{n-1}^{(2)}, gx_n^{(2)}, \frac{t}{k}\right)\right]^{a_n} \\
 &\quad * \dots * \left[M\left(gx_{n-1}^{(n)}, gx_n^{(n)}, \frac{t}{k}\right)\right]^{a_1} \\
 &\quad * \left[M\left(gx_{n-1}^{(n)}, gx_n^{(n)}, \frac{t}{k}\right)\right]^{a_2} \\
 &\quad * \dots * \left[M\left(gx_{n-1}^{(n)}, gx_n^{(n)}, \frac{t}{k}\right)\right]^{a_n} \\
 &\geq \left[M\left(gx_{n-1}^{(1)}, gx_n^{(1)}, \frac{t}{k}\right)\right]^{a_1} \\
 &\quad \cdot \left[M\left(gx_{n-1}^{(1)}, gx_n^{(1)}, \frac{t}{k}\right)\right]^{a_2} \\
 &\quad \cdot \dots \cdot \left[M\left(gx_{n-1}^{(1)}, gx_n^{(1)}, \frac{t}{k}\right)\right]^{a_n} \\
 &\quad * \left[M\left(gx_{n-1}^{(2)}, gx_n^{(2)}, \frac{t}{k}\right)\right]^{a_1} \\
 &\quad \cdot \left[M\left(gx_{n-1}^{(2)}, gx_n^{(2)}, \frac{t}{k}\right)\right]^{a_2} \\
 &\quad \cdot \dots \cdot \left[M\left(gx_{n-1}^{(2)}, gx_n^{(2)}, \frac{t}{k}\right)\right]^{a_n}
 \end{aligned}$$

$$\begin{aligned}
 &\cdot \dots \cdot \left[M\left(gx_{n-1}^{(2)}, gx_n^{(2)}, \frac{t}{k}\right)\right]^{a_n} \\
 &\quad * \dots * \left[M\left(gx_{n-1}^{(n)}, gx_n^{(n)}, \frac{t}{k}\right)\right]^{a_1} \\
 &\quad \cdot \left[M\left(gx_{n-1}^{(n)}, gx_n^{(n)}, \frac{t}{k}\right)\right]^{a_2} \\
 &\quad \cdot \dots \cdot \left[M\left(gx_{n-1}^{(n)}, gx_n^{(n)}, \frac{t}{k}\right)\right]^{a_n} \\
 &= \left[M\left(gx_{n-1}^{(1)}, gx_n^{(1)}, \frac{t}{k}\right)\right]^{a_1+a_2+\dots+a_n} \\
 &\quad * \left[M\left(gx_{n-1}^{(2)}, gx_n^{(2)}, \frac{t}{k}\right)\right]^{a_1+a_2+\dots+a_n} \\
 &\quad * \dots * \left[M\left(gx_{n-1}^{(n)}, gx_n^{(n)}, \frac{t}{k}\right)\right]^{a_1+a_2+\dots+a_n} \\
 &\geq M\left(gx_{n-1}^{(1)}, gx_n^{(1)}, \frac{t}{k}\right) \\
 &\quad * M\left(gx_{n-1}^{(2)}, gx_n^{(2)}, \frac{t}{k}\right) \\
 &\quad * \dots * M\left(gx_{n-1}^{(n)}, gx_n^{(n)}, \frac{t}{k}\right) \\
 &= \delta_{n-1}\left(\frac{t}{k}\right).
 \end{aligned} \tag{19}$$

If we join this property to (14),

$$\delta_n(t) \geq \delta_{n-1}\left(\frac{t}{k}\right) \geq \delta_{n-1}(t) \geq \delta_{n-1}(t - kt), \tag{20}$$

$\forall t > 0, n \geq 1.$

Repeatedly applying the first inequality, we deduce that

$$\delta_n(t) \geq \delta_{n-1}\left(\frac{t}{k}\right) \geq \delta_{n-2}\left(\frac{t}{k^2}\right) \geq \dots \geq \delta_0\left(\frac{t}{k^n}\right) \tag{21}$$

for all $t > 0$ and $n \geq 1$. This means that, for all $t > 0$,

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \delta_n(t) &\geq \lim_{n \rightarrow \infty} \delta_0\left(\frac{t}{k^n}\right) = 1 \\
 &\implies \lim_{n \rightarrow \infty} \delta_n(t) = 1.
 \end{aligned} \tag{22}$$

Properties (17) and (20) imply that

$$\begin{aligned}
 &M\left(g(x_n^{(1)}), g(x_{n+1}^{(1)}), t\right) \\
 &\quad * M\left(g(x_n^{(2)}), g(x_{n+1}^{(2)}), t\right) \\
 &\quad * \dots * M\left(g(x_n^{(n)}), g(x_{n+1}^{(n)}), t\right) \\
 &\geq \delta_n(t) \geq \delta_{n-1}(t - kt).
 \end{aligned} \tag{23}$$

Next, we claim that

$$\begin{aligned}
 &M\left(g(x_n^{(1)}), g(x_{n+p}^{(1)}), t\right) \\
 &\quad * M\left(g(x_n^{(2)}), g(x_{n+p}^{(2)}), t\right) \\
 &\quad * \dots * M\left(g(x_n^{(n)}), g(x_{n+p}^{(n)}), t\right) \\
 &\geq *^p \delta_{n-1}(t - kt)
 \end{aligned} \tag{24}$$

for all $t > 0$ and $n, p \geq 1$. We prove it by using mathematical induction on $p \geq 1$. If $p = 1$, (24) is true for all $n \geq 1$ and all $t > 0$ by (23). Suppose that (24) is true for all $n \geq 1$ and all $t > 0$ for some p , and we are going to prove it is also true for $p + 1$. Applying (6), the induction hypothesis, and that $* \geq \cdot$,

$$\begin{aligned}
 & M(g(x_{n+1}^{(1)}), g(x_{n+p+1}^{(1)}), kt) \\
 &= M(F(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(n)}), F(x_{n+p}^{(1)}, x_{n+p}^{(2)}, \dots, x_{n+p}^{(n)}), kt) \\
 &\geq [M(gx_n^{(1)}, gx_{n+p}^{(1)}, t)]^{a_1} \\
 &\quad * [M(gx_n^{(2)}, gx_{n+p}^{(2)}, t)]^{a_2} \\
 &\quad * \dots * [M(gx_n^{(n)}, gx_{n+p}^{(n)}, t)]^{a_n} \\
 &\geq (*^p \delta_{n-1}(t - kt))^{a_1} \\
 &\quad * (*^p \delta_{n-1}(t - kt))^{a_2} \\
 &\quad * \dots * (*^p \delta_{n-1}(t - kt))^{a_n} \\
 &\geq (*^p \delta_{n-1}(t - kt))^{a_1} \\
 &\quad \cdot (*^p \delta_{n-1}(t - kt))^{a_2} \\
 &\quad \cdot \dots \cdot (*^p \delta_{n-1}(t - kt))^{a_n} \\
 &\geq (*^p \delta_{n-1}(t - kt))^{a_1 + a_2 + \dots + a_n} \\
 &\geq *^p \delta_{n-1}(t - kt).
 \end{aligned} \tag{25}$$

Arguing in the same way, we arrive at the following:

$$\begin{aligned}
 & M(g(x_{n+1}^{(2)}), g(x_{n+p+1}^{(2)}), kt), \\
 & M(g(x_{n+1}^{(3)}), g(x_{n+p+1}^{(3)}), kt), \dots, \\
 & M(g(x_{n+1}^{(n)}), g(x_{n+p+1}^{(n)}), kt) \\
 &\geq *^p \delta_{n-1}(t - kt).
 \end{aligned} \tag{26}$$

Applying the axiom (FM₅) of a FMS, (18), and the induction hypothesis,

$$\begin{aligned}
 & M(g(x_n^{(1)}), g(x_{n+p+1}^{(1)}), t) \\
 &= M(g(x_n^{(1)}), g(x_{n+p+1}^{(1)}), (t + kt - kt)) \\
 &\geq M(g(x_n^{(1)}), g(x_{n+1}^{(1)}), (t - kt)) \\
 &\quad * M(g(x_{n+1}^{(1)}), g(x_{n+p+1}^{(1)}), kt) \\
 &\geq \delta_{n-1}(t - kt) * (*^p \delta_{n-1}(t - kt)) \\
 &= *^{p+1} \delta_{n-1}(t - kt).
 \end{aligned} \tag{27}$$

Arguing in the same manner, we have

$$\begin{aligned}
 & M(g(x_n^{(2)}), g(x_{n+p+1}^{(2)}), t), \\
 & M(g(x_n^{(3)}), g(x_{n+p+1}^{(3)}), t), \dots, \\
 & M(g(x_n^{(n)}), g(x_{n+p+1}^{(n)}), t).
 \end{aligned} \tag{28}$$

Therefore, (24) is true.

This permits us to show that $\{gx_n^{(1)}\}$ is Cauchy. Suppose that $t > 0$ for a given $\epsilon \in (0, 1)$.

By the hypothesis, as $*$ is a t -norm of H -type, there exists $0 < \eta < 1$ such that $*^p a > 1 - \epsilon$ for all $a \in (1 - \eta, 1]$ and for all $p \geq 1$.

By (22), $\lim_{n \rightarrow \infty} \delta_n(t) = 1$, so there exists $n_0 \in \mathbb{N}$ such that $\delta_n(t - kt) > 1 - \eta$ for all $n \geq n_0$.

Hence (24), we get

$$\begin{aligned}
 & M(g(x_n^{(1)}), g(x_{n+p+1}^{(1)}), t), \\
 & M(g(x_n^{(2)}), g(x_{n+p+1}^{(2)}), t), \dots, \\
 & M(g(x_n^{(n)}), g(x_{n+p+1}^{(n)}), t) > 1 - \epsilon
 \end{aligned} \tag{29}$$

for all $n \geq n_0$ and $p \geq 1$.

Therefore, $\{gx_n^{(1)}\}$ is a Cauchy sequence. Similarly, we can show that $\{gx_n^{(2)}\}, \{gx_n^{(3)}\}, \dots, \{gx_n^{(n)}\}$ are also Cauchy sequences.

Step 3. Now, we claim that g and F have a tripled coincidence point. Since X is complete, there exists $x^{(1)}, x^{(2)}, \dots, x^{(n)} \in X$ such that

$$\begin{aligned}
 \lim_{n \rightarrow \infty} gx_n^{(1)} &= x^{(1)}, \lim_{n \rightarrow \infty} gx_n^{(2)} = x^{(2)}, \dots, \\
 \lim_{n \rightarrow \infty} gx_n^{(n)} &= x^{(n)}.
 \end{aligned} \tag{30}$$

As g is continuous, we have that

$$\begin{aligned}
 \lim_{n \rightarrow \infty} ggx_n^{(1)} &= gx^{(1)}, \lim_{n \rightarrow \infty} ggx_n^{(2)} = gx^{(2)}, \dots, \\
 \lim_{n \rightarrow \infty} ggx_n^{(n)} &= gx^{(n)}.
 \end{aligned} \tag{31}$$

The commutativity of F with g implies that

$$\begin{aligned}
 ggx_{n+1}^{(1)} &= gF(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(n)}) \\
 &= F(gx_n^{(1)}, gx_n^{(2)}, \dots, gx_n^{(n)}).
 \end{aligned} \tag{32}$$

By (6) and Remark 8, we get

$$\begin{aligned}
 & M(ggx_{n+1}^{(1)}, F(x^{(1)}, x^{(2)}, \dots, x^{(n)}), kt) \\
 &= M(F(gx_n^{(1)}, gx_n^{(2)}, \dots, gx_n^{(n)}), \\
 &\quad F(x^{(1)}, x^{(2)}, \dots, x^{(n)}), kt) \\
 &\geq [M(ggx_n^{(1)}, gx_n^{(1)}, t)]^{a_1}
 \end{aligned}$$

$$\begin{aligned}
& * [M(ggx_n^{(2)}, gx^{(2)}, t)]^{a_2} \\
& * \dots * [M(ggx_n^{(n)}, gx^{(n)}, t)]^{a_n} \\
& \geq M(ggx_n^{(1)}, gx^{(1)}, t) * M(ggx_n^{(2)}, gx^{(2)}, t) \\
& * \dots * M(ggx_n^{(n)}, gx^{(n)}, t).
\end{aligned} \tag{33}$$

Letting $n \rightarrow \infty$, we deduce that

$$\lim_{n \rightarrow \infty} gg x_n^{(1)} = F(x^{(1)}, x^{(2)}, \dots, x^{(n)}). \tag{34}$$

Hence, $F(x^{(1)}, x^{(2)}, \dots, x^{(n)}) = g(x^{(1)})$.

Similarly, we can show that

$$\begin{aligned}
F(x^{(2)}, x^{(3)}, \dots, x^{(1)}) &= g(x^{(2)}), \\
F(x^{(3)}, x^{(4)}, \dots, x^{(2)}) &= g(x^{(3)}), \\
&\vdots \\
F(x^{(n)}, x^{(1)}, \dots, x^{(n-1)}) &= g(x^{(n)}).
\end{aligned} \tag{35}$$

So $(x^{(1)}, x^{(2)}, \dots, x^{(n)})$ is n -tupled coincidence point of the mappings F and g :

$$\begin{aligned}
F(x^{(1)}, x^{(2)}, \dots, x^{(n)}) &= g(x^{(1)}) \\
F(x^{(2)}, x^{(3)}, \dots, x^{(1)}) &= g(x^{(2)}) \\
&\vdots \\
F(x^{(n)}, x^{(1)}, \dots, x^{(n-1)}) &= g(x^{(n)}).
\end{aligned} \tag{36}$$

Step 4. Now, we claim that

$$\begin{aligned}
F(x^{(1)}, x^{(2)}, \dots, x^{(n)}) &= x^{(2)} \\
F(x^{(2)}, x^{(3)}, \dots, x^{(1)}) &= x^{(3)} \\
&\vdots \\
F(x^{(n)}, x^{(1)}, \dots, x^{(n-1)}) &= x^{(1)}.
\end{aligned} \tag{37}$$

We note that, by condition (6),

$$\begin{aligned}
& M(gx^{(1)}, gx_n^{(2)}, kt) \\
&= M(F(x^{(1)}, x^{(2)}, \dots, x^{(n)}), \\
&\quad F(x_{n-1}^{(2)}, x_{n-1}^{(3)}, \dots, x_{n-1}^{(1)}), kt)
\end{aligned}$$

$$\begin{aligned}
& \geq [M(gx^{(1)}, gx_{n-1}^{(2)}, t)]^{a_1} \\
& * [M(gx^{(2)}, gx_{n-1}^{(3)}, t)]^{a_2} * \dots \\
& * [M(gx^{(n)}, gx_{n-1}^{(1)}, t)]^{a_n} \\
& M(gx^{(2)}, gx_n^{(3)}, kt) \\
&= M(F(x^{(2)}, x^{(3)}, \dots, x^{(1)}), \\
&\quad F(x_{n-1}^{(3)}, x_{n-1}^{(4)}, \dots, x_{n-1}^{(2)}), kt) \\
& \geq [M(gx^{(2)}, gx_{n-1}^{(3)}, t)]^{a_1} \\
& * [M(gx^{(3)}, gx_{n-1}^{(4)}, t)]^{a_2} * \dots \\
& * [M(gx^{(1)}, gx_{n-1}^{(2)}, t)]^{a_n} \\
& \vdots \\
& M(gx^{(n)}, gx_n^{(1)}, kt) \\
&= M(F(x^{(n)}, x^{(1)}, \dots, x^{(n-1)}), \\
&\quad F(x_{n-1}^{(1)}, x_{n-1}^{(2)}, \dots, x_{n-1}^{(n)}), kt) \\
& \geq [M(gx^{(n)}, gx_{n-1}^{(1)}, t)]^{a_1} \\
& * [M(gx^{(1)}, gx_{n-1}^{(2)}, t)]^{a_2} * \dots \\
& * [M(gx^{(n-1)}, gx_{n-1}^{(n)}, t)]^{a_n}.
\end{aligned} \tag{38}$$

Let $\beta_n(t) = M(gx^{(1)}, gx_n^{(2)}, t) * M(gx^{(2)}, gx_n^{(3)}, t) * \dots * M(gx^{(n)}, gx_n^{(1)}, t)$ for all $t > 0$ and $n \geq 0$.

By (38),

$$\begin{aligned}
\beta_{n+1}(kt) &= M(gx^{(1)}, gx_{n+1}^{(2)}, kt) \\
& * M(gx^{(2)}, gx_{n+1}^{(3)}, kt) \\
& * \dots * M(gx^{(n)}, gx_{n+1}^{(1)}, kt) \\
& \geq ([M(gx^{(1)}, gx_n^{(2)}, t)]^{a_1} \\
& * [M(gx^{(2)}, gx_n^{(3)}, t)]^{a_2} \\
& * \dots * [M(gx^{(n)}, gx_n^{(1)}, t)]^{a_n}) \\
& * ([M(gx^{(2)}, gx_n^{(3)}, t)]^{a_1} \\
& * [M(gx^{(3)}, gx_n^{(4)}, t)]^{a_2} \\
& * \dots * [M(gx^{(1)}, gx_n^{(2)}, t)]^{a_n}) \\
& * ([M(gx^{(n)}, gx_n^{(1)}, t)]^{a_1} \\
& * [M(gx^{(1)}, gx_n^{(2)}, t)]^{a_2} \\
& * \dots * [M(gx^{(n-1)}, gx_n^{(n)}, t)]^{a_n}) \\
& * \dots *
\end{aligned}$$

$$\begin{aligned}
&= \left(\left[M \left(gx^{(1)}, gx_n^{(2)}, t \right) \right]^{a_1} \right. \\
&\quad * \left[M \left(gx^{(1)}, gx_n^{(2)}, t \right) \right]^{a_2} \\
&\quad * \cdots * \left[M \left(gx^{(1)}, gx_n^{(2)}, t \right) \right]^{a_n} \\
&\quad * \left(\left[M \left(gx^{(2)}, gx_n^{(3)}, t \right) \right]^{a_1} \right. \\
&\quad * \left[M \left(gx^{(2)}, gx_n^{(3)}, t \right) \right]^{a_2} \\
&\quad * \cdots * \left[M \left(gx^{(2)}, gx_n^{(3)}, t \right) \right]^{a_n} \\
&\quad * \cdots \\
&\quad * \left(\left[M \left(gx^{(n)}, gx_n^{(1)}, t \right) \right]^{a_1} \right. \\
&\quad * \left[M \left(gx^{(n)}, gx_n^{(1)}, t \right) \right]^{a_2} \\
&\quad * \cdots * \left[M \left(gx^{(n)}, gx_n^{(1)}, t \right) \right]^{a_n} \\
&\geq \left(\left[M \left(gx^{(1)}, gx_n^{(2)}, t \right) \right]^{a_1} \right. \\
&\quad \cdot \left[M \left(gx^{(1)}, gx_n^{(2)}, t \right) \right]^{a_2} \\
&\quad \cdot \cdots \cdot \left[M \left(gx^{(1)}, gx_n^{(2)}, t \right) \right]^{a_n} \\
&\quad * \left(\left[M \left(gx^{(2)}, gx_n^{(3)}, t \right) \right]^{a_1} \right. \\
&\quad \cdot \left[M \left(gx^{(2)}, gx_n^{(3)}, t \right) \right]^{a_2} \\
&\quad \cdot \cdots \cdot \left[M \left(gx^{(2)}, gx_n^{(3)}, t \right) \right]^{a_n} \\
&\quad * \cdots \\
&\quad * \left(\left[M \left(gx^{(n)}, gx_n^{(1)}, t \right) \right]^{a_1} \right. \\
&\quad \cdot \left[M \left(gx^{(n)}, gx_n^{(1)}, t \right) \right]^{a_2} \\
&\quad \cdot \cdots \cdot \left[M \left(gx^{(n)}, gx_n^{(1)}, t \right) \right]^{a_n} \\
&= \left[M \left(gx^{(1)}, gx_n^{(2)}, t \right) \right]^{a_1+a_2+\cdots+a_n} \\
&\quad * \left[M \left(gx^{(1)}, gx_n^{(2)}, t \right) \right]^{a_1+a_2+\cdots+a_n} \\
&\quad * \cdots * \left[M \left(gx^{(1)}, gx_n^{(2)}, t \right) \right]^{a_1+a_2+\cdots+a_n} \\
&\geq M \left(gx^{(1)}, gx_n^{(2)}, t \right) * M \left(gx^{(2)}, gx_n^{(3)}, t \right) \\
&\quad * M \left(gx^{(n)}, gx_n^{(1)}, t \right) \\
&= \beta_n(t).
\end{aligned} \tag{39}$$

This proves that $\beta_{n+1}(kt) \geq \beta_n(t)$ for all $t > 0$ and $n \geq 0$. Repeating this process,

$$\beta_n(t) \geq \beta_{n-1}\left(\frac{t}{k}\right) \geq \beta_{n-2}\left(\frac{t}{k^2}\right) \geq \cdots \geq \beta_0\left(\frac{t}{k^n}\right) \tag{40}$$

for all $t > 0$ and $n \geq 0$.

Now, by (40) and (38),

$$\begin{aligned}
M \left(gx^{(1)}, gx_{n+1}^{(2)}, kt \right) &\geq \left[M \left(gx^{(1)}, gx_n^{(2)}, t \right) \right]^{a_1} \\
&\quad * \left[M \left(gx^{(2)}, gx_n^{(3)}, t \right) \right]^{a_2} \\
&\quad * \cdots * \left[M \left(gx^{(n)}, gx_n^{(1)}, t \right) \right]^{a_n} \\
&\geq \beta_n(t) \geq \beta_0\left(\frac{t}{k^n}\right) \\
M \left(gx^{(2)}, gx_{n+1}^{(3)}, kt \right) &\geq \left[M \left(gx^{(2)}, gx_n^{(3)}, t \right) \right]^{a_1} \\
&\quad * \left[M \left(gx^{(3)}, gx_n^{(4)}, t \right) \right]^{a_2} \\
&\quad * \cdots * \left[M \left(gx^{(1)}, gx_n^{(2)}, t \right) \right]^{a_n} \\
&\geq \beta_n(t) \geq \beta_0\left(\frac{t}{k^n}\right) \\
&\vdots \\
M \left(gx^{(n)}, gx_{n+1}^{(1)}, kt \right) &\geq \left[M \left(gx^{(n)}, gx_n^{(1)}, t \right) \right]^{a_1} \\
&\quad * \left[M \left(gx^{(1)}, gx_n^{(2)}, t \right) \right]^{a_2} \\
&\quad * \cdots * \left[M \left(gx^{(n-1)}, gx_n^{(n)}, t \right) \right]^{a_n} \\
&\geq \beta_n(t) \geq \beta_0\left(\frac{t}{k^n}\right).
\end{aligned} \tag{41}$$

Therefore,

$$\begin{aligned}
&M \left(gx^{(1)}, gx_{n+1}^{(2)}, kt \right), M \left(gx^{(2)}, gx_{n+1}^{(3)}, kt \right), \dots, \\
&M \left(gx^{(n)}, gx_{n+1}^{(1)}, kt \right) \geq \beta_0\left(\frac{t}{k^n}\right)
\end{aligned} \tag{42}$$

for all $t > 0$ and $n \geq 1$.

Since $\lim_{n \rightarrow \infty} \beta_0(t/k^n) = 1$ for all $t > 0$, we have, taking limit in (41), that

$$\begin{aligned}
\lim_{n \rightarrow \infty} gx_{n+1}^{(2)} &= gx^{(1)}, \lim_{n \rightarrow \infty} gx_{n+1}^{(3)} = gx^{(2)}, \dots, \\
\lim_{n \rightarrow \infty} gx_{n+1}^{(1)} &= gx^{(n)}.
\end{aligned} \tag{43}$$

This shows, using (36), that

$$\begin{aligned}
F \left(x^{(1)}, x^{(2)}, \dots, x^{(n)} \right) &= g \left(x^{(1)} \right) = \lim_{n \rightarrow \infty} gx_n^{(2)} = x^{(2)}, \\
F \left(x^{(2)}, x^{(3)}, \dots, x^{(1)} \right) &= g \left(x^{(2)} \right) = \lim_{n \rightarrow \infty} gx_n^{(3)} = x^{(3)}, \\
&\vdots \\
F \left(x^{(n-1)}, x^{(n)}, \dots, x^{(n-2)} \right) &= g \left(x^{(n-1)} \right) \\
&= \lim_{n \rightarrow \infty} gx_n^{(n)} = x^{(n)}.
\end{aligned} \tag{44}$$

Step 5. Now, we will prove that $x^{(1)} = x^{(2)} = \cdots = x^{(n)}$.

Let

$$\begin{aligned}
\theta(t) &= M \left(x^{(1)}, x^{(2)}, t \right) * M \left(x^{(2)}, x^{(3)}, t \right) \\
&\quad * \cdots * M \left(x^{(n)}, x^{(1)}, t \right)
\end{aligned} \tag{45}$$

for all $t > 0$.

Using condition (6), we have

$$\begin{aligned}
 & M(x^{(1)}, x^{(2)}, kt) \\
 &= M(F(x^{(n)}, x^{(1)}, \dots, x^{(n-1)}), \\
 &\quad F(x^{(1)}, x^{(2)}, \dots, x^{(n)}), kt) \\
 &\geq [M(gx^{(n)}, gx^{(1)}, t)]^{a_1} \\
 &\quad * [M(gx^{(1)}, gx^{(2)}, t)]^{a_2} \\
 &\quad * \dots * [M(gx^{(n-1)}, gx^{(n)}, t)]^{a_n} \\
 &= [M(x^{(1)}, x^{(2)}, t)]^{a_1} \\
 &\quad * [M(x^{(2)}, x^{(3)}, t)]^{a_2} \\
 &\quad * \dots * [M(x^{(n)}, x^{(1)}, t)]^{a_n} \\
 &M(x^{(2)}, x^{(3)}, kt) \\
 &= M(F(x^{(1)}, x^{(2)}, \dots, x^{(n)}), \\
 &\quad F(x^{(2)}, x^{(3)}, \dots, x^{(1)}), kt) \\
 &\geq [M(gx^{(1)}, gx^{(2)}, t)]^{a_1} * [M(gx^{(2)}, gx^{(3)}, t)]^{a_2} \\
 &\quad * \dots * [M(gx^{(n)}, gx^{(1)}, t)]^{a_n} \\
 &= [M(x^{(2)}, x^{(3)}, t)]^{a_1} * [M(x^{(3)}, x^{(4)}, t)]^{a_2} \\
 &\quad * \dots * [M(x^{(1)}, x^{(2)}, t)]^{a_n} \\
 &\quad \vdots \\
 &M(x^{(n)}, x^{(1)}, kt) \\
 &= M(F(x^{(n-1)}, x^{(n)}, \dots, x^{(n-2)}), \\
 &\quad F(x^{(n)}, x^{(1)}, \dots, x^{(n-1)}), kt) \\
 &\geq [M(gx^{(n-1)}, gx^{(n)}, t)]^{a_1} * [M(gx^{(n)}, gx^{(1)}, t)]^{a_2} \\
 &\quad * \dots * [M(gx^{(n-2)}, gx^{(n-1)}, t)]^{a_n} \\
 &= [M(x^{(n)}, x^{(1)}, t)]^{a_1} * [M(x^{(1)}, x^{(2)}, t)]^{a_2} \\
 &\quad * \dots * [M(x^{(n-1)}, x^{(n)}, t)]^{a_n}.
 \end{aligned} \tag{46}$$

Combining all the inequalities, we will have the following:

$$\begin{aligned}
 & \theta(kt) \\
 &= M(x^{(1)}, x^{(2)}, kt) * M(x^{(2)}, x^{(3)}, kt) \\
 &\quad * \dots * M(x^{(n)}, x^{(1)}, kt) \\
 &\geq ([M(x^{(1)}, x^{(2)}, t)]^{a_1} * [M(x^{(2)}, x^{(3)}, t)]^{a_2} \\
 &\quad * \dots * [M(x^{(n)}, x^{(1)}, t)]^{a_n}) \\
 &\quad * ([M(x^{(2)}, x^{(3)}, t)]^{a_1} * [M(x^{(3)}, x^{(4)}, t)]^{a_2}
 \end{aligned}$$

$$\begin{aligned}
 & * \dots * [M(x^{(1)}, x^{(2)}, t)]^{a_n}) \\
 & * \dots \\
 & * ([M(x^{(n)}, x^{(1)}, t)]^{a_1} * [M(x^{(1)}, x^{(2)}, t)]^{a_2} \\
 &\quad * \dots * [M(x^{(n-1)}, x^{(n)}, t)]^{a_n}) \\
 &= ([M(x^{(1)}, x^{(2)}, t)]^{a_1} * [M(x^{(1)}, x^{(2)}, t)]^{a_2} \\
 &\quad * \dots * [M(x^{(1)}, x^{(2)}, t)]^{a_n}) \\
 &\quad * ([M(x^{(2)}, x^{(3)}, t)]^{a_1} * [M(x^{(2)}, x^{(3)}, t)]^{a_2} \\
 &\quad * \dots * [M(x^{(2)}, x^{(3)}, t)]^{a_n}) \\
 & * \dots \\
 &\quad * ([M(x^{(n)}, x^{(1)}, t)]^{a_1} * [M(x^{(n)}, x^{(1)}, t)]^{a_2} \\
 &\quad * \dots * [M(x^{(n)}, x^{(1)}, t)]^{a_n}) \\
 &\geq ([M(x^{(1)}, x^{(2)}, t)]^{a_1} * [M(x^{(1)}, x^{(2)}, t)]^{a_2} \\
 &\quad \dots * [M(x^{(1)}, x^{(2)}, t)]^{a_n}) \\
 &\quad * ([M(x^{(2)}, x^{(3)}, t)]^{a_1} * [M(x^{(2)}, x^{(3)}, t)]^{a_2} \\
 &\quad \dots * [M(x^{(2)}, x^{(3)}, t)]^{a_n}) \\
 & * \dots \\
 &\quad * ([M(x^{(n)}, x^{(1)}, t)]^{a_1} * [M(x^{(n)}, x^{(1)}, t)]^{a_2} \\
 &\quad \dots * [M(x^{(n)}, x^{(1)}, t)]^{a_n}) \\
 &= ([M(x^{(1)}, x^{(2)}, t)]^{a_1+a_2+\dots+a_n}) \\
 &\quad * ([M(x^{(2)}, x^{(3)}, t)]^{a_1+a_2+\dots+a_n}) \\
 &\quad * \dots * ([M(x^{(n)}, x^{(1)}, t)]^{a_1+a_2+\dots+a_n}) \\
 &\geq M(x^{(1)}, x^{(2)}, t) * M(x^{(2)}, x^{(3)}, t) \\
 &\quad * \dots * M(x^{(n)}, x^{(1)}, t) = \theta(t).
 \end{aligned} \tag{47}$$

We conclude that $\theta(kt) \geq \theta(t)$ implies that $\theta(t) \geq \theta(t/k) \geq \theta(t/k^2) \geq \dots \geq \theta(t/k^n)$ for all $t > 0$ and $n \geq 1$. From (46),

$$\begin{aligned}
 M(x^{(1)}, x^{(2)}, kt) &\geq [M(x^{(1)}, x^{(2)}, t)]^{a_1} \\
 &\quad * [M(x^{(2)}, x^{(3)}, t)]^{a_2} \\
 &\quad * \dots * [M(x^{(n)}, x^{(1)}, t)]^{a_n} \\
 &\geq M(x^{(1)}, x^{(2)}, t) * M(x^{(2)}, x^{(3)}, t)
 \end{aligned}$$

$$\begin{aligned}
& * \cdots * M(x^{(n)}, x^{(1)}, t) \\
& = \theta(t) \geq \theta\left(\frac{t}{k^n}\right) \\
M(x^{(2)}, x^{(3)}, kt) & \geq [M(x^{(2)}, x^{(3)}, t)]^{a_1} \\
& * [M(x^{(3)}, x^{(4)}, t)]^{a_2} \\
& * \cdots * [M(x^{(1)}, x^{(2)}, t)]^{a_n} \\
& \geq M(x^{(2)}, x^{(3)}, t) * M(x^{(3)}, x^{(4)}, t) \\
& * \cdots * M(x^{(1)}, x^{(2)}, t) \\
& = \theta(t) \geq \theta\left(\frac{t}{k^n}\right) \\
& \vdots \\
M(x^{(n)}, x^{(1)}, kt) & \geq [M(x^{(n)}, x^{(1)}, t)]^{a_1} \\
& * [M(x^{(1)}, x^{(2)}, t)]^{a_2} \\
& * \cdots * [M(x^{(n-1)}, x^{(n)}, t)]^{a_n} \\
& \geq M(x^{(n)}, x^{(1)}, t) * M(x^{(1)}, x^{(2)}, t) \\
& * \cdots * M(x^{(n-1)}, x^{(n)}, t) \\
& = \theta(t) \geq \theta\left(\frac{t}{k^n}\right).
\end{aligned} \tag{48}$$

Letting $n \rightarrow \infty$, we have $\lim_{n \rightarrow \infty} \theta(t/k^n) = 1$ for all $t > 0$, and it means that $M(x^{(1)}, x^{(2)}, kt) = M(x^{(2)}, x^{(3)}, kt) = \cdots = M(x^{(n)}, x^{(1)}, kt) = 1$ for all $t > 0$; that is,

$$x^{(1)} = x^{(2)} = \cdots = x^{(n)}. \tag{49}$$

The uniqueness of $x^{(1)}$ follows directly from (6). $\square$

Remark 12. The uniqueness of the coincidence point of F and g is not always true. For instance, if $F \equiv x_0$ is constant and $g \equiv x_0$ is also constant, then every $(x^{(1)}, x^{(2)}, \dots, x^{(n)}) \in X^n$ is a coincidence point of F and g .

Consequences. In the proof of the next result, the view of (X, d) as the crisp FMS $(X, M_c, \min)$ is used (see Example 6). Here we deduce results for n -tupled coincidence point in metric spaces which are generalization of the results of Roldán et al. [11] which are not in a partially ordered set.

Theorem 13. Let (X, d) be a complete metric space and let $F : X^n \rightarrow X$ and $g : X \rightarrow X$ be two mappings such that $F(X^n) \subseteq g(X)$ and g is continuous and commuting with F . Suppose that F and g satisfy some of the following conditions for all $x^{(1)}, x^{(2)}, \dots, x^{(n)}, y^{(1)}, y^{(2)}, \dots, y^{(n)} \in X$;

$$\begin{aligned}
(1) \quad & d(F(x^{(1)}, x^{(2)}, \dots, x^{(n)}), F(y^{(1)}, y^{(2)}, \dots, y^{(n)})) \\
& \leq k \max \{d(gx^{(1)}, gy^{(1)}), \\
& \quad d(gx^{(2)}, gy^{(2)}), \dots, d(gx^{(n)}, gy^{(n)})\}
\end{aligned} \tag{50}$$

for some $k \in (0, 1)$,

$$\begin{aligned}
(2) \quad & d(F(x^{(1)}, x^{(2)}, \dots, x^{(n)}), F(y^{(1)}, y^{(2)}, \dots, y^{(n)})) \\
& \leq k \max \{a_1 d(gx^{(1)}, gy^{(1)}) \\
& \quad + a_2 d(gx^{(2)}, gy^{(2)}) \\
& \quad + \cdots + a_n d(gx^{(n)}, gy^{(n)})\}
\end{aligned} \tag{51}$$

for some $k \in (0, 1)$ and some $a_1, a_2, \dots, a_n \in [0, 1/n]$,

$$\begin{aligned}
(3) \quad & d(F(x^{(1)}, x^{(2)}, \dots, x^{(n)}), F(y^{(1)}, y^{(2)}, \dots, y^{(n)})) \\
& \leq a_1 d(gx^{(1)}, gy^{(1)}) \\
& \quad + a_2 d(gx^{(2)}, gy^{(2)}) \\
& \quad + \cdots + a_n d(gx^{(n)}, gy^{(n)})
\end{aligned} \tag{52}$$

for some $a_1, a_2, \dots, a_n \in [0, 1]$, such that $\sum_{i=1}^n a_i < 1$.

Then there exists a unique $x \in X$ such that $x = gx = F(x, x, \dots, x)$.

Proof. (1) Consider M_c defined as in Example 6. As (X, d) is complete, then $(X, M_c, \min)$ is a complete FMS. Fix $x^{(1)}, x^{(2)}, \dots, x^{(n)}, y^{(1)}, y^{(2)}, \dots, y^{(n)} \in X$ and $t > 0$.

Now we are ready to prove (6).

Put $a^{(1)} = a^{(2)} = \cdots = a^{(n)} = 1/n$ and $* = \min$. If $M_c(g(x^{(1)}), gy^{(1)}, t) = 0$ or $M_c(g(x^{(2)}), gy^{(2)}, t) = 0$ or $\cdots$ or $M_c(g(x^{(n)}), gy^{(n)}, t) = 0$, then (6) is obvious.

Suppose that

$$\begin{aligned}
& M_c(g(x^{(1)}), gy^{(1)}, t) = 1, \\
& M_c(g(x^{(2)}), gy^{(2)}, t) = 1, \dots, \\
& M_c(g(x^{(n)}), gy^{(n)}, t) = 1.
\end{aligned} \tag{53}$$

This means that

$$\begin{aligned}
& d(gx^{(1)}, gy^{(1)}) < t, d(gx^{(2)}, gy^{(2)}) < t, \dots, \\
& d(gx^{(n)}, gy^{(n)}) < t.
\end{aligned} \tag{54}$$

Therefore,

$$\begin{aligned}
 t &> \max \{d(g(x^{(1)}), gy^{(1)}), d(g(x^{(2)}), gy^{(2)}), \dots, \\
 &\quad d(g(x^{(n)}), gy^{(n)})\}, \\
 kt &> k \max \{d(g(x^{(1)}), gy^{(1)}), d(gx^{(2)}, gy^{(2)}), \dots, \\
 &\quad d(gx^{(n)}, gy^{(n)})\} \\
 &\geq d(F(x^{(1)}, x^{(2)}, \dots, x^{(n)}), F(y^{(1)}, y^{(2)}, \dots, y^{(n)})).
 \end{aligned} \tag{55}$$

Hence, $M_c(F(x^{(1)}, x^{(2)}, \dots, x^{(n)}), F(y^{(1)}, y^{(2)}, \dots, y^{(n)}), kt) = 1$ and (6) is also true.

(2) In this case,

$$\begin{aligned}
 &d(F(x^{(1)}, x^{(2)}, \dots, x^{(n)}), F(y^{(1)}, y^{(2)}, \dots, y^{(n)})) \\
 &\leq k \{a_1 d(gx^{(1)}, gy^{(1)}) + a_2 d(gx^{(2)}, gy^{(2)}) \\
 &\quad + \dots + a_n d(gx^{(n)}, gy^{(n)})\} \\
 &\leq k \left\{ \frac{1}{n} d(gx^{(1)}, gy^{(1)}) + \frac{1}{n} d(gx^{(2)}, gy^{(2)}) \right. \\
 &\quad \left. + \dots + \frac{1}{n} d(gx^{(n)}, gy^{(n)}) \right\} \\
 &= \frac{k}{n} \{d(gx^{(1)}, gy^{(1)}) + d(gx^{(2)}, gy^{(2)}) \\
 &\quad + \dots + d(gx^{(n)}, gy^{(n)})\} \\
 &\leq \frac{k}{n} \max \{d(gx^{(1)}, gy^{(1)}), d(gx^{(2)}, gy^{(2)}), \dots, \\
 &\quad d(gx^{(n)}, gy^{(n)})\} \\
 &= k \max \{d(gx^{(1)}, gy^{(1)}), d(gx^{(2)}, gy^{(2)}), \dots, \\
 &\quad d(gx^{(n)}, gy^{(n)})\}.
 \end{aligned} \tag{56}$$

(3) If $k = \sum_{i=1}^n a_i < 1$.

$$\begin{aligned}
 &d(F(x^{(1)}, x^{(2)}, \dots, x^{(n)}), F(y^{(1)}, y^{(2)}, \dots, y^{(n)})) \\
 &\leq a_1 d(gx^{(1)}, gy^{(1)}) + a_2 d(gx^{(2)}, gy^{(2)}) \\
 &\quad + \dots + a_n d(gx^{(n)}, gy^{(n)}) \\
 &\leq a_1 \max \{d(gx^{(1)}, gy^{(1)}), d(gx^{(2)}, gy^{(2)}), \dots, \\
 &\quad d(gx^{(n)}, gy^{(n)})\} \\
 &\quad + a_2 \max \{d(gx^{(1)}, gy^{(1)}), d(gx^{(2)}, gy^{(2)}), \dots, \\
 &\quad d(gx^{(n)}, gy^{(n)})\} \\
 &\quad + \dots \\
 &\quad + a_n \max \{d(gx^{(1)}, gy^{(1)}), d(gx^{(2)}, gy^{(2)}), \dots,
 \end{aligned}$$

$$\begin{aligned}
 &d(gx^{(n)}, gy^{(n)})\} \\
 &= (a_1 + a_2 + \dots + a_n) \\
 &\quad \cdot \max \{d(gx^{(1)}, gy^{(1)}), d(gx^{(2)}, gy^{(2)}), \dots, \\
 &\quad d(gx^{(n)}, gy^{(n)})\} \\
 &= k \max \{d(gx^{(1)}, gy^{(1)}), d(gx^{(2)}, gy^{(2)}), \dots, \\
 &\quad d(gx^{(n)}, gy^{(n)})\}.
 \end{aligned} \tag{57}$$

□

Remark 14. We can obtain Theorems 11 and 15 due to Roldán et al. [11] as corollaries, if we establish our main result for the mappings $F : X \times X \times X \rightarrow X$ and $g : X \rightarrow X$.

Applications. Now we are ready to establish application.

Lipschitzian Systems. Let $f_1, f_2, \dots, f_n : R \rightarrow R$ be Lipschitzian mappings and let $\beta_1, \beta_2, \dots, \beta_n \in R$ be real numbers. Define $h : R \rightarrow R$ as $h(x) = \beta_1 f_1(x) + \beta_2 f_2(x) + \dots + \beta_n f_n(x)$ for all $x \in R$. Then h is another Lipschitzian mapping and $k_h \leq |\beta_1|k_{f_1} + |\beta_2|k_{f_2} + \dots + |\beta_n|k_{f_n}$. Obviously, if $K = |\beta_1|k_{f_1} + |\beta_2|k_{f_2} + \dots + |\beta_n|k_{f_n} < 1$, then h is a contraction, so there exists a unique $x^{(1)} \in R$ such that $h_{x^{(1)}} = x^{(1)}$.

Next, define $F : R^n \rightarrow R$ as

$$\begin{aligned}
 &F(x^{(1)}, x^{(2)}, \dots, x^{(n)}) \\
 &= \beta_1 f_1(x^{(1)}) + \beta_2 f_2(x^{(2)}) + \dots + \beta_n f_n(x^{(n)})
 \end{aligned} \tag{58}$$

for all $x^{(1)}, x^{(2)}, \dots, x^{(n)} \in R$. It is clear that $F(x^{(1)}, x^{(1)}, \dots, x^{(1)}) = hx^{(1)}$ for all $x^{(1)} \in R$. Furthermore,

$$\begin{aligned}
 &d(F(x^{(1)}, x^{(2)}, \dots, x^{(n)}), F(y^{(1)}, y^{(2)}, \dots, y^{(n)})) \\
 &\leq \sum_{i=1}^n |\beta_i| |f_i(x^{(i)}) - f_i(y^{(i)})| \\
 &\leq \sum_{i=1}^n |\beta_i| k_{f_i} |x^{(i)} - y^{(i)}| \\
 &\leq K \max_{1 \leq j \leq n} d(x^{(j)}, y^{(j)}).
 \end{aligned} \tag{59}$$

If $K < 1$, then F verifies (6) with $g(x^{(1)}) = x^{(1)}$ for all $x^{(1)} \in R$.

Corollary 15. Let $f_1, f_2, \dots, f_n : R \rightarrow R$ be Lipschitzian mappings on R (provided with the Euclidean metric) and let $\beta_1, \beta_2, \dots, \beta_n \in R$ such that $|\beta_1|k_{f_1} + |\beta_2|k_{f_2} + \dots + |\beta_n|k_{f_n} < 1$. Then the system

$$\begin{aligned}
 &\beta_1 f_1(x^{(1)}) + \beta_2 f_2(x^{(2)}) + \dots + \beta_n f_n(x^{(n)}) = (x^{(1)}), \\
 &\beta_1 f_1(x^{(2)}) + \beta_2 f_2(x^{(3)}) + \dots + \beta_n f_n(x^{(1)}) = (x^{(2)}) \\
 &\quad \vdots \\
 &\beta_1 f_1(x^{(n)}) + \beta_2 f_2(x^{(1)}) + \dots + \beta_n f_n(x^{(n-1)}) = (x^{(n)})
 \end{aligned} \tag{60}$$

has a unique solution, which is $(x^{(1)}, x^{(1)}, \dots, x^{(1)})$, where $x^{(1)}$ is the only real solution of $\beta_1 f_1(x^{(1)}) + \beta_2 f_2(x^{(2)}) + \dots + \beta_n f_n(x^{(n)}) = (x^{(1)})$.

An Integral System. Let $a, b \in R$ with $a < b$ and let $I = [a, b]$. Consider $X = L^1(I)$ with the distance $d_1 \cdot (f, g) = \int_I |f(t) - g(t)| dt$, where $\int$ represents the Lebesgue integral. It is well known that $(L(I), d_1)$ is a complete metric space. Let $k, \beta_1, \beta_2, \dots, \beta_n \in R$, be real numbers and let $G : R^n \rightarrow R$ be a mapping verifying $G(0, 0, \dots, 0) = 0$, and

$$\left| G_{x^{(1)}, x^{(2)}, \dots, x^{(n)}} - G_{y^{(1)}, y^{(2)}, \dots, y^{(n)}} \right| \leq k \sum_{i=1}^n \beta_i |x^{(i)} - y^{(i)}| \quad (61)$$

for all $(x^{(1)}, x^{(2)}, \dots, x^{(n)}), (y^{(1)}, y^{(2)}, \dots, y^{(n)}) \in R^n$.

If $A \in R$, we want to find functions $f_1, f_2, \dots, f_n \in L^1(I)$ such that

$$f_i(x) = A + \int_{[a,x]} G(f_1(t), f_{i+1}(t), \dots, f_{i+n-n}(t)) dt \quad (62)$$

holds for all $x \in I$, $i = 1, 2, \dots, n$.

For all $f_1, f_2, \dots, f_n \in L^1(I)$ and all $x \in I$, define

$$\begin{aligned} F_{f_1, f_2, \dots, f_n}(x) \\ = A + \int_{[a,x]} G(f_1(t), f_2(t) \cdots f_n(t)) dt. \end{aligned} \quad (63)$$

On the one hand, it is not difficult to prove that $F_{f_1, f_2, \dots, f_n} \in L^1(I)$; hence $F : L^1(I)^n \rightarrow L^1(I)$ is well defined. On the other hand,

$$\begin{aligned} d_1(F_{f_1, f_2, \dots, f_n}, F_{g_1, g_2, \dots, g_n}) \\ = \int_I |F_{f_1, f_2, \dots, f_n}(x) - F_{g_1, g_2, \dots, g_n}(x)| dx \\ \leq \int_I \left(\int_{[a,x]} |G(f_1(t), f_2(t) \cdots f_n(t)) \right. \\ \left. - G(g_1(t), g_2(t) \cdots g_n(t))| dt \right) dx \\ \leq \int_I \left(\int_{[a,x]} k \sum_{i=1}^n \beta_i |f_i(t) - g_i(t)| dt \right) dx \\ \leq k \sum_{i=1}^n \beta_i \int_I \left(\int_I |f_i(t) - g_i(t)| dt \right) dx \\ = k \sum_{i=1}^n \beta_i \int_I d_1(f_i, g_i) dx \\ = k(b-a) \sum_{i=1}^n \beta_i d_1(f_i, g_i). \end{aligned} \quad (64)$$

If we suppose that $K = k(b-a)(\beta_1, \beta_2, \dots, \beta_n) < 1$, then F verifies (6) with $g(f) = f$ for all $f \in L^1(I)$. Then system (62) has a unique solution, which is of the form $(f_0, f_0, \dots, f_0)$, where $f_0 \in L^1(I)$ is the only solution of the equation

$$f_0(x) = A + \int_{[a,x]} G(f_0(t), f_0(t), \dots, f_0(t)) dt \quad (65)$$

for all $x \in I$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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