Research Article

The Lattice Structure of L-Contact Relations

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From the point of view of graded truth approach, we define the notion of a contact relation on the collection of all **L**-sets, discuss the connection to the set of all close, reflexive, and symmetric relations on all **L**-ultrafilters on X , and investigate the algebraic structure of all **L**-contact relations.

1. Introduction

Contact relations have been studied on two different contexts, the proximity relations [1] and the theory of pointless geometry (topology) [2], since the early 1920s. Recently, it has become a powerful tool in several areas of artificial intelligence, such as qualitative spatial reasoning and ontology building; see [3–8].

On the other hand, the notion of an **L**-set was introduced in [9], as a generalization of Zadeh's (classical) notion of a fuzzy set [10]. Fuzzy relational modeling processes have been studied, such as fuzzy concept lattices [11].

In [12], Winter investigated time-dependent contact structure in Goguen Categories. It turns out that a suitable theory can be defined using an L -valued or L -fuzzy version of a contact relation. In [13–15], we introduced the notion of contact relation in fuzzy setting and discussed some properties of way-below relation, continuous lattice induced by an **L**-contact relation.

In this paper, we want to generalize the theory of contact relations in fuzzy setting. First, Section 2 surveys an overview of contact relations, **L**-sets. Then, Section 3 generalizes the notion of a contact relation in fuzzy setting and presents some examples. Section 4 recalls the notions of an **L**-filter, an **L**-relation, and an **L**-topology. Section 5 establishes the order preserving correspondence between the set of all **L**-contact relations on L^X and the set of all closed, reflexive, symmetric relations on $\text{Ult}(L^X)$. Section 6 focuses on the algebraic structure of all **L**-contact relations.

2. Preliminaries

Let us recall some main notions for each area, that is, contact relations [6–8] and **L**-sets [11].

2.1. Contact Relations. We assume familiarity with the notions of Boolean algebra and lattice [16, 17]. Suppose $(B, +, \cdot, *, 0, 1)$ is a Boolean algebra, $R \subseteq B \times B$ is called a binary relation on B , and all relations are denoted by $\text{Rel}(B)$. In [6], first, Düntsch and Winter considered the notion of a contact relation on a Boolean algebra B .

Definition 1. Suppose $C \in \text{Rel}(B)$, and consider the following properties: for all $x, y, z \in B$,

- (C_0) $0(-C)x$, that is, for $x \neq 0$, 0 and x are not C -related,
- (C_1) xCx for $x \neq 0$,
- (C_2) $xCy \Rightarrow yCx$,
- (C_3) xCy and $y \leq z \Rightarrow xCz$,
- (C_4) $xC(y + z) \Rightarrow xCy$ or xCz .

C is called a contact relation, and (B, C) is called a Boolean contact algebra, if C satisfies $(C_0)–(C_4)$. $\text{CR}(B)$ denotes the set of all contact relations on B .

Second they discussed the set $\text{Ult}(B)$ which is of all ultrafilters on B and the set $\text{Rel}^{\text{rsc}}(\text{Ult}(B))$ of all reflexive and symmetric relations on $\text{Ult}(B)$ that are closed in the product topology of $\text{Ult}(B)^2$.

Third, they investigated the relation between $\text{CR}(B)$ and $\text{Rel}^{\text{rsc}}(\text{Ult}(B))$ and obtained the representation theorem in [6].

Theorem 2. *Suppose that B is a Boolean algebra. Then, there is a bijective order preserving correspondence between the contact relations on B and the reflexive and symmetric relations on $\text{Ult}(B)$ that are closed in the product topology of $\text{Ult}(B)^2$.*

Then with the help of Theorem 2, they studied the structure of $\text{CR}(B)$ by means of the set $\text{Rel}^{\text{rsc}}(\text{Ult}(B))$.

For further information on contact relations and Boolean contact algebras, see [3–8].

2.2. L-Sets. As a generalization of Zadeh's (classical) notion of a fuzzy set [10], the notion of an $\mathbf{L}$ -set was introduced in [9]. An overview of the theory of $\mathbf{L}$ -sets and $\mathbf{L}$ -relations (i.e., fuzzy sets and relations in the framework of complete residuated lattices) can be found in [11].

Definition 3. A residuated lattice is an algebra $\mathbf{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, 0, 1 \rangle$ such that

- (1) $\langle L, \vee, \wedge, 0, 1 \rangle$ is a lattice with the smallest element 0 and the largest element 1,
- (2) $\langle L, \otimes, 1 \rangle$ is a commutative monoid; that is, $\otimes$ is associative, commutative, and holds the identity $a \otimes 1 = a$,
- (3) $\otimes, \rightarrow$ form an adjoint pair; that is, $a \otimes b \leq c$ iff $a \leq b \rightarrow c$ holds for all $a, b, c \in L$.

Residuated lattice $\mathbf{L}$ is called complete if $\langle L, \vee, \wedge \rangle$ is a complete lattice.

In this paper, we assume that $\mathbf{L}$ is a complete Heyting algebra which is a complete residuated lattice satisfying $a \otimes b = a \wedge b$.

For a universe set X , an $\mathbf{L}$ -set in X is a mapping $A : X \rightarrow L$. $A(x)$ indicates the truth degree of “ x belongs to A .” We use the symbol L^X to denote the set of all $\mathbf{L}$ -sets in X . The negation operator is defined: for $A \in L^X$, $A^*(x) = A(x) \rightarrow 0$ for every $x \in X$.

Definition 4. (1) Suppose $\{A_i \mid i \in I\} \subseteq L^X$ is a system of $\mathbf{L}$ -sets, and $\bigvee_{i \in I} A_i$ and $\bigwedge_{i \in I} A_i$ are two $\mathbf{L}$ -sets defined as follows, for every $x \in X$:

$$\begin{aligned} \left(\bigvee_{i \in I} A_i \right)(x) &= \bigvee_{i \in I} A_i(x), \\ \left(\bigwedge_{i \in I} A_i \right)(x) &= \bigwedge_{i \in I} A_i(x). \end{aligned} \quad (1)$$

(2) Suppose $\mathcal{U}$ is an $\mathbf{L}$ -set in L^X ; that is, $\mathcal{U} : L^X \rightarrow L$ is a mapping. For every $A \in L^X$, $\mathcal{U}(A)$ is called the degree of membership of A in $\mathcal{U}$. Then $\mathcal{U}$ is a generalization of a system of subsets in the classical case.

Two $\mathbf{L}$ -sets $\bigcup \mathcal{U}, \bigcap \mathcal{U}$ in X are defined: for every $x \in X$,

$$\begin{aligned} \bigcup \mathcal{U}(x) &= \bigvee_{A \in L^X} \mathcal{U}(A) \wedge A(x), \\ \bigcap \mathcal{U}(x) &= \bigwedge_{A \in L^X} \mathcal{U}(A) \longrightarrow A(x). \end{aligned} \quad (2)$$

Clearly, $\bigcup \mathcal{U}$ and $\bigcap \mathcal{U}$ are generalizations of the union and the intersection of a system of sets in the classical case (see [11]).

If $\mathcal{U}$ is a system of $\mathbf{L}$ -sets in L^X , that is, $\mathcal{U} = \{A_i \mid i \in I\} \subseteq L^X$, we have $\bigcup \mathcal{U} = \bigvee \mathcal{U}$, and $\bigcap \mathcal{U} = \bigwedge \mathcal{U}$.

Note 1. In ordinary set theory, suppose X is a set and $\{A_i \mid i \in I\}$ is a system of subsets of X , if $A_i \subseteq X$. Clearly $\{A_i \mid i \in I\} \subseteq 2^X$, where 2^X is the power set of X .

The classical order $\leq$ and equality $=$ are generalized in fuzzy setting, that is, $\mathbf{L}$ -relation and $\mathbf{L}$ -equality.

$I \in L^{X \times X}$ is called an $\mathbf{L}$ -binary relation. The truth degree to which elements x and y are related by an $\mathbf{L}$ -relation I is denoted by $I(x, y)$ or (xIy) .

A binary $\mathbf{L}$ -relation I on X is an $\mathbf{L}$ -equivalence if it satisfies the following: $\forall x, y, z \in X$, $I(x, x) = 1$ (reflexivity), $I(x, y) = I(y, x)$ (symmetry), and $I(x, y) \otimes I(y, z) \leq I(x, z)$ (transitivity). An $\mathbf{L}$ -equivalence is an $\mathbf{L}$ -equality if it satisfies the following: $I(x, y) = 1$ implies $x = y$.

An $\mathbf{L}$ -order on X with an $\mathbf{L}$ -equality relation $\approx$ is a binary $\mathbf{L}$ -relation $\leq$ which is compatible with respect to $\approx$ and satisfies the following: $\forall x, y, z \in X$, $(x \leq x) = 1$ (reflexivity), $(x \leq y) \wedge (y \leq x) \leq (x \approx y)$ (antisymmetry), and $(x \leq y) \otimes (y \leq z) \leq (x \leq z)$ (transitivity). A set X equipped with an $\mathbf{L}$ -order $\leq$ and an $\mathbf{L}$ -equality $\approx$ is called an $\mathbf{L}$ -ordered set $\langle \langle X, \approx \rangle, \leq \rangle$.

The subethood degree $S(A, B)$ is defined as follows: for $A, B \in L^X$,

$$\begin{aligned} S(A, B) &= \bigwedge_{x \in X} A(x) \longrightarrow B(x), \\ (A \leq B) &= S(A, B), \\ (A \approx B) &= S(A, B) \wedge S(B, A). \end{aligned} \quad (3)$$

We write $A \leq B$, if $S(A, B) = 1$.

Example 5. For $\emptyset \neq M \subseteq L^X$, we obtain that $\langle \langle M, \approx \rangle, S \rangle$ is an $\mathbf{L}$ -ordered set. In fact, reflexivity and antisymmetry are trivial, and we have to prove transitivity and compatibility. Transitivity is described as follows: $S(A, B) \otimes S(B, C) \leq S(A, C)$ holds if and only if $S(A, B) \otimes S(B, C) \leq A(x) \rightarrow C(x)$; that is, $\forall x \in X$, $A(x) \otimes S(A, B) \otimes S(B, C) \leq C(x)$, and it is true since $A(x) \otimes S(A, B) \otimes S(B, C) \leq A(x) \otimes (A(x) \rightarrow B(x)) \otimes (B(x) \rightarrow C(x)) \leq C(x)$. In a similar way, we also prove compatibility: $S(A, B) \otimes (A \approx A') \otimes (B \approx B') \leq S(A', B')$.

By Example 5, we know that $\langle \langle L^X, \approx \rangle, \leq \rangle$ is an $\mathbf{L}$ -ordered set.

3. L-Contact Relations

In this section, our aim is to define the notion of a contact relation in fuzzy setting, present some examples, and show that all L-contact relations form an L-ordered set $\langle \langle \text{CR}(L^X), \approx \rangle, \leq \rangle$.

Suppose X is a universe set and L^X is the set of all L-sets in X . We know that $(L^X, \bigcup, \bigcap, *, 0_X, 1_X)$ might not be a Boolean algebra, where $0_X : X \rightarrow L$ and $1_X : X \rightarrow L$ are defined: $0_X(x) = 0$ and $1_X(x) = 1$, for every $x \in X$, respectively.

A mapping $\theta : L^X \times L^X \rightarrow L$ is called an L-binary relation on L^X . The truth degree to which elements A and B are related by an L-relation θ is denoted by $\theta(A, B)$. The collection of all L-relations is denoted by $\text{Rel}(L^X)$.

First, from the point of view of graded truth approach, we generalize the notion of a contact relation in fuzzy setting as follows.

Definition 6. Suppose X is a universe set, L^X is the set of all L-sets in X , $\theta \in \text{Rel}(L^X)$ is an L-relation on L^X , for $A, B, C \in L^X$, and consider the following properties:

- (C₀) $\theta(0_X, A) = 0$.
- (C₁) $\theta(A, A) = 1$ for $A \neq 0_X$.
- (C₂) $\theta(A, B) = \theta(B, A)$.
- (C₃) $B \subseteq C \Rightarrow \theta(A, B) \leq \theta(A, C)$.
- (C₄) $\theta(A, B \cup C) \leq \theta(A, B) \vee \theta(A, C)$.

θ is called an L-contact relation, and (L^X, θ) is called an L-contact algebra, if θ satisfies (C₀)–(C₄).

In the paper, we will denote the set of all L-contact relations on L^X by $\text{CR}(L^X)$. In [6–8], C denotes a contact relation. From now on, we write θ instead of C . Obviously, Definition 6 is a generalization of Definition 1 in fuzzy setting.

Second, we give some examples of L-contact relations.

Example 7. For $A, B \in L^X$, we define an L-contact relation:

$$1_{\text{CR}(L^X)}(A, B) = \begin{cases} 1, & \text{if } A \neq 0_X, B \neq 0_X, \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

Example 8. For $A, B \in L^X$, let

$$O_{\text{CR}(L^X)}(A, B) = \begin{cases} 1, & \text{if } A \wedge B \neq 0_X, \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

Then $O_{\text{CR}(L^X)}$ is a contact relation on L^X .

When $L = 2$, L^X is the power set of X ; that is, $L^X = 2^X$; then $(L^X, \bigcup, \bigcap, *, 0_X, 1_X) = (2^X, \bigcup, \bigcap, *, \emptyset, X)$ is a Boolean algebra. Clearly, we have

$$\begin{aligned} O_{\text{CR}(L^X)}(A, B) &= 1 \iff A \cap B \neq \emptyset, \\ O_{\text{CR}(L^X)}(A, B) &= 0 \iff A \cap B = \emptyset. \end{aligned} \quad (6)$$

Note 2. In [18], we introduced the relation ρ on L^X . We show that $O_{\text{CR}(L^X)}$ and ρ are the same in the classical case.

Definition 9. Suppose $A, B \in L^X$, and let $\rho(A, B) = \bigvee_{x \in X} A(x) \otimes B(x)$, which expresses the related degree of A and B with respect to $\otimes$ [18].

Obviously, ρ is not a contact relation on L^X , since, for every $A \in L^X$, $\rho(A, A) = 1$ does not hold. But when $L = 2$, $\rho(A, B) = 1 \iff A \cap B \neq \emptyset$; $\rho(A, B) = 0 \iff A \cap B = \emptyset$. So the relation ρ is the same with the relation $O_{\text{CR}(L^X)}$ in the classical case.

In the rest of the section, we define $\leq$ and $\approx$ on $\text{CR}(L^X)$. For $\theta_1, \theta_2 \in \text{CR}(L^X)$,

$$\begin{aligned} (\theta_1 \leq \theta_2) &= S(\theta_1, \theta_2) = \bigwedge_{A, B \in L^X} \theta_1(A, B) \longrightarrow \theta_2(A, B), \\ (\theta_1 \approx \theta_2) &= S(\theta_1, \theta_2) \wedge S(\theta_2, \theta_1) \\ &= \bigwedge_{A, B \in L^X} \theta_1(A, B) \longleftrightarrow \theta_2(A, B). \end{aligned} \quad (7)$$

Thus $\langle \langle \text{CR}(L^X), \approx \rangle, \leq \rangle$ is an L-ordered set; see [11].

In special, $\theta_1 \subseteq \theta_2 \iff S(\theta_1, \theta_2) = 1 \iff \theta_1(A, B) \leq \theta_2(A, B)$ for any $A, B \in L^X$.

Notice that $O_{\text{CR}(L^X)}$ is the smallest element in $\text{CR}(L^X)$. For any $A, B \in L^X$ and each $\theta \in \text{CR}(L^X)$,

$$\begin{aligned} A \wedge B &= 0_X \Rightarrow O_{\text{CR}(L^X)}(A, B) = 0 \Rightarrow O_{\text{CR}(L^X)}(A, B) \leq \theta(A, B); \\ A \wedge B &\neq 0_X, \text{ and let } C = A \wedge B \neq 0_X; \\ \Rightarrow 1 &= O_{\text{CR}(L^X)}(C, C) \leq O_{\text{CR}(L^X)}(A, B), \text{ and } 1 = \theta(C, C) \leq \theta(A, B); \\ \Rightarrow O_{\text{CR}(L^X)}(A, B) &= \theta(A, B) = 1. \end{aligned}$$

Hence $O_{\text{CR}(L^X)} \subseteq \theta$.

4. L-Filters and L-Topologies

This section is devoted to recall three notions: an L-filter on X , an L-relation, and an L-topology on the set of all L-ultrafilters, showing that $\langle \langle \text{Rel}^{\text{rsc}}(\text{Ult}(L^X)), \approx \rangle, \leq \rangle$ is an L-ordered set.

4.1. L-Filters on X . We recall the notion of an L-filter on X ; see [19, Chapter 3].

Definition 10. A map $\nu : L^X \rightarrow L$ is called an L-filter on X , if ν fulfills the following axioms:

- (F0) $\nu(1_X) = 1$.
- (F1) $A \subseteq B \Rightarrow \nu(A) \leq \nu(B)$.
- (F2) $\nu(A) \wedge \nu(B) \leq \nu(A \wedge B)$.
- (F3) $\nu(0_X) = 0$.

Now, we present some examples of L-filters.

Example 11. The function ν defined by $\nu(A) = \bigwedge_{x \in X} A(x)$ for every $A \in L^X$ is an L-filter.

The proof in detail is in [19, Chapter 3].

Example 12. For every $A \in L^X$ and $A \neq 0_X$, an **L**-filter is defined:

$$\nu_A(D) = \begin{cases} 1, & \text{if } A \subseteq D, \\ 0, & \text{otherwise.} \end{cases} \quad (8)$$

Let $F(L^X)$ be the set of all **L**-filters on X , on which we introduce the partial orderings $\leq$ and $\approx$, for $\nu_1, \nu_2 \in F(L^X)$:

$$(\nu_1 \leq \nu_2) = S(\nu_1, \nu_2) = \bigwedge_{A \in L^X} \nu_1(A) \longrightarrow \nu_2(A), \quad (9)$$

$$(\nu_1 \approx \nu_2) = S(\nu_1, \nu_2) \wedge S(\nu_2, \nu_1),$$

and thus $\langle \langle F(L^X), \approx \rangle, \leq \rangle$ is an **L**-ordered set; see [11].

In special, $\nu_1 \subseteq \nu_2 \Leftrightarrow S(\nu_1, \nu_2) = 1$; that is, $\nu_1(A) \leq \nu_2(A)$ for every $A \in L^X$.

Clearly, $0_{L^X} : L^X \rightarrow L$ is the smallest filter on $F(L^X)$, where $0_{L^X}(A) = 0$ for every $A \in L^X$, $A \neq 1_X$; and $0_{L^X}(1_X) = 1$.

For instance, when $L = 2$, $L^X = 2^X$ is the power set of X , $0_{L^X} = \{X\}$.

In [19, Chapter 3], the notion of an **L**-filter was investigated. By Theorem 6.2.9 [19, Chapter 3], we know that the partially ordered set $(F(L^X), \subseteq)$ has maximal elements. A maximal element in $(F(L^X), \subseteq)$ is also called an **L**-ultrafilter. An **L**-filter ν is an **L**-ultrafilter if and only if $\nu(A) = (\nu(A^*))^*$ for every $A \in L^X$ [19, Theorem 6.2.11].

For every $A \in L^X$, there exists an **L**-ultrafilter η_A , such that $\eta_A(A) = 1$, that is, Lemma 13.

Lemma 13. Suppose $A \in L^X$, $A \neq 0_X$, and then there exists an **L**-ultrafilter η_A , such that $\eta_A(A) = 1$.

Proof. For $A \in L^X$, $A \neq 0_X$,

- (1) by Example 12, we obtain an **L**-filter ν_A satisfying $\nu_A(A) = 1$,
- (2) by Corollary 6.2.10 in [19, Chapter 3], there exists an **L**-ultrafilter η_A , such that $\nu_A \subseteq \eta_A$. Certainly $\eta_A(A) = 1$.

□

Lemma 14. Suppose ν is an **L**-ultrafilter; then we have $\nu(B \cup C) \leq \nu(B) \vee \nu(C)$.

Proof. In [11], the notion of a filter in a residuated lattice L was introduced. In fact, ν is an **L**-ultrafilter on $X \Leftrightarrow \{\nu(A) \mid A \in L^X\}$ which is a maximal filter in L . By Lemma 2.64 [11, P55], Lemma 14 holds.

Suppose $\nu_1, \nu_2 \in \text{Ult}(L^X)$, and we define $\nu_1 \wedge \nu_2$ and $\nu_1 \vee \nu_2$ as follows:

$$\begin{aligned} (\nu_1 \wedge \nu_2)(A) &= \nu_1(A) \wedge \nu_2(A), \\ (\nu_1 \vee \nu_2)(A) &= \nu_1(A) \vee \nu_2(A), \end{aligned} \quad (10)$$

for any $A \in L^X$.

If $\{\nu_i \mid i \in I\} \subseteq \text{Ult}(L^X)$, $\bigvee_{i \in I} \nu_i$ is also defined: $(\bigvee_{i \in I} \nu_i)(A) = \bigvee_{i \in I} \nu_i(A)$ for any $A \in L^X$. □

4.2. L-Relations on $\text{Ult}(L^X)$. In the section, we turn to **L**-relations on $\text{Ult}(L^X)$.

A mapping $R : \text{Ult}(L^X) \times \text{Ult}(L^X) \rightarrow L$ is also an **L**-relation on $\text{Ult}(L^X)$. All of them will be denoted by $\text{Rel}(\text{Ult}(L^X))$. Suppose $R_1, R_2 \in \text{Rel}(\text{Ult}(L^X))$, and we define $R_1 \wedge R_2$ and $R_1 \vee R_2$ as follows:

$$\begin{aligned} (R_1 \wedge R_2)(\nu_1, \nu_2) &= R_1(\nu_1, \nu_2) \wedge R_2(\nu_1, \nu_2), \\ (R_1 \vee R_2)(\nu_1, \nu_2) &= R_1(\nu_1, \nu_2) \vee R_2(\nu_1, \nu_2), \end{aligned} \quad (11)$$

for any $\nu_1, \nu_2 \in \text{Ult}(L^X)$.

For example, $1_{\text{Ult}(L^X)}$ and $0_{\text{Ult}(L^X)}$ are defined as follows: $1_{\text{Ult}(L^X)}(\nu_1, \nu_2) = 1$ and $0_{\text{Ult}(L^X)}(\nu_1, \nu_2) = 0$ for $\nu_1, \nu_2 \in \text{Ult}(L^X)$. Clearly, $1_{\text{Ult}(L^X)}$ is the largest element in $\text{Rel}(\text{Ult}(L^X))$.

For $R \in \text{Rel}(\text{Ult}(L^X))$, it is symmetric if $R(\nu_1, \nu_2) = R(\nu_2, \nu_1)$ for any $\nu_1, \nu_2 \in \text{Ult}(L^X)$. It is reflexive if, for every $\nu \in \text{Ult}(L^X)$, $R(\nu, \nu) = 1$.

4.3. L-Topologies on $\text{Ult}(L^X)$. We introduce the notion of an **L**-topology on $\text{Ult}(L^X)$.

Definition 15. Suppose $T \subseteq \text{Ult}(L^X)$ is called an **L**-topology on $\text{Ult}(L^X)$, if

- (1) $\nu_1, \nu_2 \in T \Rightarrow \nu_1 \wedge \nu_2 \in T$,
- (2) for every subset $\{\nu_i \mid i \in I\}$ of T , $\bigvee_{i \in I} \nu_i \in T$ holds.

$\text{CL}(R)$ denotes the closure of R in the product topology $T \times T$ on $\text{Ult}(L^X) \times \text{Ult}(L^X)$. If R is a closed set in the product topology $T \times T$, then R is called a close relation.

The collection of all close, reflexive, and symmetric relations on $\text{Ult}(L^X)$ is denoted by $\text{Rel}^{\text{rsc}}(\text{Ult}(L^X))$. Clearly, $1_{\text{Ult}(L^X)} \in \text{Rel}^{\text{rsc}}(\text{Ult}(L^X))$.

For $R_1, R_2 \in \text{Rel}^{\text{rsc}}(\text{Ult}(L^X))$, we define **L**-order $\leq$ and $\approx$ on $\text{Rel}^{\text{rsc}}(\text{Ult}(L^X))$:

$$\begin{aligned} S(R_1, R_2) &= (R_1 \leq R_2) \\ &= \bigwedge_{\nu_1, \nu_2 \in \text{Ult}(L^X)} R_1(\nu_1, \nu_2) \longrightarrow R_2(\nu_1, \nu_2), \end{aligned} \quad (12)$$

$$(R_1 \approx R_2) = S(R_1, R_2) \wedge S(R_2, R_1).$$

Thus $\langle \langle \text{Rel}^{\text{rsc}}(\text{Ult}(L^X)), \approx \rangle, \leq \rangle$ is an **L**-ordered set; see [11].

For $A \in L^X$, let $h(A) = \{\nu \mid \nu(A) \neq 0\}$, and we assume that $\{h(A) \times h(B) \mid A, B \in L^X\}$ is a base for the product topology $T \times T$.

5. The Correspondence between $\text{Rel}^{\text{rsc}}(\text{Ult}(L^X))$ and $\text{CR}(L^X)$

In [6], Düntsch and Winter proved that there exists a bijective order preserving correspondence between $\text{Rel}^{\text{rsc}}(\text{Ult}(B))$ and $\text{CR}(B)$, that is, Theorem 2.

In Sections 3 and 4, we know that two $\mathbf{L}$ -ordered sets $\langle \langle \text{Rel}^{\text{rsc}}(\text{Ult}(L^X)), \approx, \leq \rangle$ and $\langle \langle \text{CR}(L^X), \approx, \leq \rangle$ are generalizations of $\text{Rel}^{\text{rsc}}(\text{Ult}(B))$ and $\text{CR}(B)$ in fuzzy setting, respectively.

In this section, we continue to investigate the relation between two $\mathbf{L}$ -ordered sets $\langle \langle \text{Rel}^{\text{rsc}}(\text{Ult}(L^X)), \approx, \leq \rangle$ and $\langle \langle \text{CR}(L^X), \approx, \leq \rangle$, obtaining a fuzzification of Theorem 2 [6]. For this, we define two mappings:

$$\begin{aligned} q : \text{Rel}^{\text{rsc}}(\text{Ult}(L^X)) &\longrightarrow \text{CR}(L^X), \\ p : \text{CR}(L^X) &\longrightarrow \text{Rel}^{\text{rsc}}(\text{Ult}(L^X)), \end{aligned} \quad (13)$$

and we prove that they are the order preserving correspondences. We divide the work into three steps.

Step 1. We define a mapping q from $\text{Rel}^{\text{rsc}}(\text{Ult}(L^X))$ to $\text{CR}(L^X)$ and prove that it is one-to-one.

Given $R \in \text{Rel}^{\text{rsc}}(\text{Ult}(L^X))$, for $A, B \in L^X$, put

$$\theta_R(A, B) = \bigvee_{\nu_1, \nu_2 \in \text{Ult}(L^X)} R(\nu_1, \nu_2) \wedge \nu_1(A) \wedge \nu_2(B). \quad (14)$$

Then we have the following.

Lemma 16. *Suppose R is a close, reflexive, and symmetric $\mathbf{L}$ -relation on $\text{Ult}(L^X)$; then θ_R is an $\mathbf{L}$ -contact relation on L^X .*

Proof. we have to show that θ_R is an $\mathbf{L}$ -contact relation, that is, to verify θ_R satisfies the conditions of Definition 6. Consider the following:

$$(C_0) \theta_R(0_X, A) = 0.$$

$$(C_1) \text{ For } A \neq 0_X, \text{ by Lemma 13, } \eta_A(A) = 1, \text{ and then}$$

$$\begin{aligned} \theta_R(A, A) &= \bigvee_{\nu_1, \nu_2 \in \text{Ult}(L^X)} R(\nu_1, \nu_2) \wedge \nu_1(A) \wedge \nu_2(A) \\ &\geq R(\eta_A, \eta_A) \wedge \eta_A(A) \wedge \eta_A(A) = 1. \end{aligned} \quad (15)$$

$$(C_2) \theta_R(A, B) = \theta_R(B, A) \text{ is obvious.}$$

$$(C_3) B \leq C \Rightarrow \nu(B) \leq \nu(C), \text{ and then}$$

$$\begin{aligned} \theta_R(A, B) &= \bigvee_{\nu_1, \nu_2 \in \text{Ult}(L^X)} R(\nu_1, \nu_2) \wedge \nu_1(A) \wedge \nu_2(B) \\ &\leq \bigvee_{\nu_1, \nu_2 \in \text{Ult}(L^X)} R(\nu_1, \nu_2) \wedge \nu_1(A) \wedge \nu_2(C) \\ &= \theta_R(A, C). \end{aligned} \quad (16)$$

(C₄) By Lemma 14, we have

$$\begin{aligned} \theta_R(A, B \cup C) &= \bigvee_{\nu_1, \nu_2 \in \text{Ult}(L^X)} R(\nu_1, \nu_2) \wedge \nu_1(A) \wedge \nu_2(B \cup C) \\ &\leq \bigvee_{\nu_1, \nu_2 \in \text{Ult}(L^X)} R(\nu_1, \nu_2) \wedge \nu_1(A) \wedge [\nu_2(B) \vee \nu_2(C)] \\ &= \bigvee_{\nu_1, \nu_2 \in \text{Ult}(L^X)} [R(\nu_1, \nu_2) \wedge \nu_1(A) \wedge \nu_2(B)] \\ &\quad \vee [R(\nu_1, \nu_2) \wedge \nu_1(A) \wedge \nu_2(C)] \end{aligned} \quad (17)$$

($\mathbf{L}$ is a complete Heyting algebra)

$$\begin{aligned} &= \left[\bigvee_{\nu_1, \nu_2 \in \text{Ult}(L^X)} R(\nu_1, \nu_2) \wedge \nu_1(A) \wedge \nu_2(B) \right] \\ &\quad \vee \left[\bigvee_{\nu_1, \nu_2 \in \text{Ult}(L^X)} R(\nu_1, \nu_2) \wedge \nu_1(A) \wedge \nu_2(C) \right] \\ &= \theta_R(A, B) \vee \theta_R(A, C). \end{aligned}$$

So, θ_R is an $\mathbf{L}$ -contact relation on L^X ; that is, $\theta_R \in \text{CR}(L^X)$.

Furthermore, for every $R \in \text{Rel}^{\text{rsc}}(\text{Ult}(L^X))$, we define $q(R) = \theta_R \in \text{CR}(L^X)$, and thus we obtain a mapping q from $\text{Rel}^{\text{rsc}}(\text{Ult}(L^X))$ to $\text{CR}(L^X)$. The key is to prove that q is one-to-one. $\square$

Lemma 17. *Suppose R is a close, reflexive, and symmetric $\mathbf{L}$ -relation on $\text{Ult}(L^X)$, and then $q : \text{Rel}^{\text{rsc}}(\text{Ult}(L^X)) \rightarrow \text{CR}(L^X)$ and $q(R) = \theta_R \in \text{CR}(L^X)$ is an injective mapping.*

Proof. By the above analysis, the remainder is to prove that q is one-to-one, which is equivalent to show for any two close relations $R_1, R_2 \in \text{Rel}^{\text{rsc}}(\text{Ult}(L^X))$; if $R_1 \neq R_2$, then their images $\theta_{R_1}, \theta_{R_2}$ are not equal.

Suppose R_1, R_2 are two distinct close, reflexive, and symmetric relations, and then there exist $\nu_1^0, \nu_2^0 \in \text{Ult}(L^X)$, such that $R_1(\nu_1^0, \nu_2^0) \neq R_2(\nu_1^0, \nu_2^0)$. Without loss of generality, we may assume $R_1(\nu_1^0, \nu_2^0) \not\leq R_2(\nu_1^0, \nu_2^0)$.

(1) Consider $R_1(\nu_1^0, \nu_2^0) \not\leq R_2(\nu_1^0, \nu_2^0)$, clearly $\nu_1^0 \neq \nu_2^0$, and there exist $A, B \in L^X$, $\nu_1^0(A) \neq 0$, $\nu_2^0(B) \neq 0$, such that

$$\begin{aligned} &R_1(\nu_1^0, \nu_2^0) \wedge \nu_1^0(A) \wedge \nu_2^0(B) \\ &\not\leq R_2(\nu_1^0, \nu_2^0) \wedge \nu_1^0(A) \wedge \nu_2^0(B). \end{aligned} \quad (18)$$

(2) Since R_2 is closed, we have that $R_1(\nu_1^0, \nu_2^0) \wedge \nu_1^0(A) \wedge \nu_2^0(B) \not\leq R_2(\nu_1, \nu_2) \wedge \nu_1(A) \wedge \nu_2(B)$ holds for any $\nu_1, \nu_2 \in \text{Ult}(L^X)$ with $\nu_1(A) \neq 0$, $\nu_2(B) \neq 0$.

If not, there exist $\omega_1, \omega_2 \in \text{Ult}(L^X)$, $\omega_1(A) \neq 0$, $\omega_2(B) \neq 0$, such that

$$\begin{aligned} &R_1(\nu_1^0, \nu_2^0) \wedge \nu_1^0(A) \wedge \nu_2^0(B) \\ &\leq R_2(\omega_1, \omega_2) \wedge \omega_1(A) \wedge \omega_2(B), \end{aligned} \quad (19)$$

and we have

$$\begin{aligned} R_1(\nu_1^0, \nu_2^0) &\wedge \nu_1^0(A) \wedge \nu_2^0(B) \\ &\leq \text{CL}(R_2)(\nu_1^0, \nu_2^0) \wedge \nu_1^0(A) \wedge \nu_2^0(B) \\ &= R_2(\nu_1^0, \nu_2^0) \wedge \nu_1^0(A) \wedge \nu_2^0(B), \end{aligned} \quad (20)$$

which leads to a contradiction.

(3) On the other hand, if $\theta_{R_1} = \theta_{R_2}$, then

$$\begin{aligned} R_1(\nu_1^0, \nu_2^0) &\wedge \nu_1^0(A) \wedge \nu_2^0(B) \\ &\leq \bigvee_{\nu_1, \nu_2 \in \text{Ult}(L^X)} R_1(\nu_1, \nu_2) \wedge \nu_1(A) \wedge \nu_2(B) \\ &= \theta_{R_1}(A, B) = \theta_{R_2}(A, B) \\ &= \bigvee_{\nu_1, \nu_2 \in \text{Ult}(L^X)} R_2(\nu_1, \nu_2) \wedge \nu_1(A) \wedge \nu_2(B). \end{aligned} \quad (21)$$

Since R_2 is closed, there exist $\eta_1, \eta_2 \in \text{Ult}(L^X)$ satisfying

$$\begin{aligned} R_1(\nu_1^0, \nu_2^0) &\wedge \nu_1^0(A) \wedge \nu_2^0(B) \\ &\leq R_2(\eta_1, \eta_2) \wedge \eta_1(A) \wedge \eta_2(B), \end{aligned} \quad (22)$$

a contradiction.

So $\theta_{R_1} \not\leq \theta_{R_2}$ holds, which implies that q is an injective mapping. $\square$

Note 3. When $\mathbf{L} = \mathbf{2}$, $q(R) = \{\nu_1 \times \nu_2 \mid R(\nu_1, \nu_2) = 1\}$, and $\theta_R(A, B) = 1 \Leftrightarrow$ there exist $\nu_1, \nu_2 \in \text{Ult}(L^X)$, such that $A \in \nu_1, B \in \nu_2$, and $R(\nu_1, \nu_2) = 1$; that is, $\nu_1 \times \nu_2 \leq q(R)$.

Step 2. We define a mapping p from $\text{CR}(L^X)$ to $\text{Rel}^{\text{rsc}}(\text{Ult}(L^X))$; that is, for each $\mathbf{L}$ -contact relation $\theta \in \text{CR}(L^X)$, we consider defining the corresponding $R_\theta \in \text{Rel}^{\text{rsc}}(\text{Ult}(L^X))$.

For an $\mathbf{L}$ -ultrafilter ν , an $\mathbf{L}$ -set in X is defined; that is,

$$\begin{aligned} \bigcap \nu : X &\longrightarrow L, \\ \bigcap \nu(x) &= \bigwedge_{A \in L^X} \nu(A) \longrightarrow A(x), \quad \text{for every } x \in X. \end{aligned} \quad (23)$$

Obviously, for an $\mathbf{L}$ -ultrafilter ν , we have $\bigcap \nu \neq 0_X$.

Suppose $\theta \in \text{CR}(L^X)$; in other words, θ is an $\mathbf{L}$ -contact relation on L^X , for any $\mathbf{L}$ -ultrafilters $\nu_1, \nu_2 \in \text{Ult}(L^X)$, and let

$$R_\theta(\nu_1, \nu_2) = \theta\left(\bigcap \nu_1, \bigcap \nu_2\right). \quad (24)$$

Therefore we have the following lemma.

Lemma 18. Suppose $\theta \in \text{CR}(L^X)$, then $p : \text{CR}(L^X) \rightarrow \text{Rel}^{\text{rsc}}(\text{Ult}(L^X))$, for $\theta \in \text{CR}(L^X)$, and $p(\theta) = \text{CL}(R_\theta) \in \text{Rel}^{\text{rsc}}(\text{Ult}(L^X))$ is an injective mapping.

Proof. The closeness, symmetry, and reflexivity of R_θ follow directly from the above definition.

By the above proof and Lemma 17, we know that p is also an injective mapping. This completes the proof. $\square$

Note 4. When $\mathbf{L} = \mathbf{2}$, $p(\theta) = \{A \times B \mid \theta(A, B) = 1\}$, and $R(\nu_1, \nu_2) = 1 \Leftrightarrow \forall A \in \nu_1, B \in \nu_2, \theta(A, B) = 1$ holds.

Step 3. By Lemmas 16 and 17, we obtain two mappings p, q , in the step, and we wish to prove that q, p preserve the order $\leq$.

First, we prove that q preserves the order $\leq$.

Lemma 19. $q : \text{Rel}^{\text{rsc}}(\text{Ult}(L^X)) \rightarrow \text{CR}(L^X)$ preserves the order $\leq$.

Proof. For each $R \in \text{Rel}^{\text{rsc}}(\text{Ult}(L^X))$, we define

$$\begin{aligned} q : \text{Rel}^{\text{rsc}}(\text{Ult}(L^X)) &\longrightarrow \text{CR}(L^X), \\ q(R) &= \theta_R \in \text{CR}(L^X). \end{aligned} \quad (25)$$

We have to prove that q preserves the order $\leq$; that is, $(R_1 \leq R_2) \leq (q(R_1) \leq q(R_2))$. Consider

$$\begin{aligned} (q(R_1) \leq q(R_2)) &= (\theta_{R_1} \leq \theta_{R_2}) \\ &= \bigwedge_{A, B \in L^X} \theta_{R_1}(A, B) \longrightarrow \theta_{R_2}(A, B) \\ &= \bigwedge_{A, B \in L^X} \left[\bigvee_{\nu_1, \nu_2 \in \text{Ult}(L^X)} R_1(\nu_1, \nu_2) \wedge \nu_1(A) \wedge \nu_2(B) \right] \\ &\longrightarrow \left[\bigvee_{\nu_1, \nu_2 \in \text{Ult}(L^X)} R_2(\nu_1, \nu_2) \wedge \nu_1(A) \wedge \nu_2(B) \right] \\ &= \bigwedge_{A, B \in L^X} \bigwedge_{\nu_1, \nu_2 \in \text{Ult}(L^X)} \left\{ R_1(\nu_1, \nu_2) \wedge \nu_1(A) \wedge \nu_2(B) \right. \\ &\longrightarrow \left. \left[\bigvee_{\nu_1, \nu_2 \in \text{Ult}(L^X)} R_2(\nu_1, \nu_2) \wedge \nu_1(A) \wedge \nu_2(B) \right] \right\} \\ &\geq \bigwedge_{A, B \in L^X} \bigwedge_{\nu_1, \nu_2 \in \text{Ult}(L^X)} [R_1(\nu_1, \nu_2) \wedge \nu_1(A) \wedge \nu_2(B) \\ &\longrightarrow R_2(\nu_1, \nu_2) \wedge \nu_1(A) \wedge \nu_2(B)] \\ &\geq \bigwedge_{A, B \in L^X} \bigwedge_{\nu_1, \nu_2 \in \text{Ult}(L^X)} [R_1(\nu_1, \nu_2) \longrightarrow R_2(\nu_1, \nu_2)] \\ &\geq (R_1 \leq R_2). \end{aligned} \quad (26)$$

Second, we also prove that p preserves the order $\leq$. $\square$

Lemma 20. $p : \text{CR}(L^X) \rightarrow \text{Rel}^{\text{rsc}}(\text{Ult}(L^X))$ preserves the order $\leq$.

Proof. For $\theta \in \text{CR}(L^X)$, let

$$\begin{aligned} p : \text{CR}(L^X) &\longrightarrow \text{Rel}^{\text{rsc}}(\text{Ult}(L^X)), \\ p(\theta) &= \text{CL}(R_\theta) \in \text{Rel}^{\text{rsc}}(\text{Ult}(L^X)). \end{aligned} \quad (27)$$

We have to prove that p preserves the order $\leq$; that is to say, $(\theta_1 \leq \theta_2) \Rightarrow (p(\theta_1) \leq p(\theta_2))$. Consider

$$\begin{aligned} &(p(\theta_1) \leq p(\theta_2)) \\ &= \bigwedge_{v_1, v_2 \in \text{Ult}(L^X)} p(\theta_1)(v_1, v_2) \longrightarrow p(\theta_2)(v_1, v_2) \\ &= \bigwedge_{v_1, v_2 \in \text{Ult}(L^X)} \text{CL}(R_{\theta_1})(v_1, v_2) \longrightarrow \text{CL}(R_{\theta_2})(v_1, v_2) \\ &\geq \bigwedge_{v_1, v_2 \in \text{Ult}(L^X)} R_{\theta_1}(v_1, v_2) \longrightarrow R_{\theta_2}(v_1, v_2) \\ &= \bigwedge_{v_1, v_2 \in \text{Ult}(L^X)} \theta_1(\bigcap v_1, \bigcap v_2) \longrightarrow \theta_2(\bigcap v_1, \bigcap v_2) \\ &\geq \bigwedge_{A, B \in L^X} \theta_1(A, B) \longrightarrow \theta_2(A, B) = (\theta_1 \leq \theta_2). \end{aligned} \quad (28)$$

□

Note 5. Suppose B is a Boolean algebra, and by Lemmas 18, 19, and 20, we obtain the corresponding results: for $\theta_1, \theta_2 \in \text{CR}(B)$, if $\theta_1 \subseteq \theta_2$, then $p(\theta_1) \subseteq p(\theta_2)$ holds. Similarly, for $R_1, R_2 \in \text{Rel}^{\text{rsc}}(\text{Ult}(L^X))$, if $R_1 \subseteq R_2$, then $q(R_1) \subseteq q(R_2)$ also holds.

So, there exists a bijective order preserving correspondence between the set of L -contact relations on L^X and the set of all close, reflexive, symmetric relations on $\text{Ult}(L^X)$, that is, the following proposition, which is a fuzzification of Theorem 2.

Proposition 21. Suppose L^X is the set of all L -sets on X , and then there exists a bijective order preserving correspondence between the set of all L -contact relations on L^X and the set of all close, reflexive, symmetric relations on $\text{Ult}(L^X)$.

6. The Structure of Contact Relations

In this section, we consider two problems; the first is discussing the structure of $\text{Rel}^{\text{rsc}}(\text{Ult}(L^X))$, and the second is obtaining the corresponding results about the $\text{CR}(L^X)$.

First, we focus on the structure of $\text{Rel}^{\text{rsc}}(\text{Ult}(L^X))$.

Let $\text{Rel}^{\text{rsc}}(\text{Ult}(L^X))$ be the set of all close, reflexive, and symmetric relations on $\text{Ult}(L^X)$. For each $R \in \text{Rel}^{\text{rsc}}(\text{Ult}(L^X))$, R is a close set in the product topology $T \times T$ on $\text{Ult}(L^X) \times \text{Ult}(L^X)$. From the topological point of view, we know that $\text{Rel}^{\text{rsc}}(\text{Ult}(L^X))$ is closed with finite join and infinity meet. In other words, suppose $\{R_i \mid i \in I\} \subseteq \text{Rel}^{\text{rsc}}(\text{Ult}(L^X))$, and we have $\bigvee_{j=1}^m R_j \in \text{Rel}^{\text{rsc}}(\text{Ult}(L^X))$ and $\bigwedge_{i \in I} R_i \in \text{Rel}^{\text{rsc}}(\text{Ult}(L^X))$.

Furthermore, $\text{Rel}^{\text{rsc}}(\text{Ult}(L^X))$ is also closed with the operator defined as follows:

$$\sum_{i \in I} R_i = \text{CL}\left(\bigvee_{i \in I} R_i\right), \quad (29)$$

that is to say, $\sum_{i \in I} R_i \in \text{Rel}^{\text{rsc}}(\text{Ult}(L^X))$.

On $\text{Rel}^{\text{rsc}}(\text{Ult}(L^X))$, the largest element is $1_{\text{Ult}(L^X)}$; the smallest element is $1_{\text{Ult}(L^X)}^{\text{rsc}}$, where, for $v_1, v_2 \in \text{Ult}(L^X)$, $1_{\text{Ult}(L^X)}^{\text{rsc}}(v_1, v_2) = 1$, if $v_1 = v_2$; $1_{\text{Ult}(L^X)}^{\text{rsc}}(v_1, v_2) = 0$, otherwise.

Second, with the help of Proposition 21, we obtain the corresponding results on $\text{CR}(L^X)$.

Let $\text{CR}(L^X)$ be the set of all L -contact relations on L^X . It is closed with the two operators: $\sum$ and $\prod$. That is, suppose $\{\theta_i \mid i \in I\} \subseteq \text{CR}(L^X)$:

$$\begin{aligned} \sum_{i \in I} \theta_i &= q\left(\text{CL}\left(\bigvee_{i \in I} p(\theta_i)\right)\right), \\ \prod_{i \in I} \theta_i &= q\left(\bigwedge_{i \in I} p(\theta_i)\right). \end{aligned} \quad (30)$$

Thus we obtain the following.

Proposition 22. Suppose $\text{CR}(L^X)$ is the collection of all L -contact relations on L^X , and then it forms a complete lattice with respect to $\sum$ and $\prod$, on which the largest element is $1_{\text{CR}(L^X)}$; the smallest element is $O_{\text{CR}(L^X)}$.

Note 6. Consider

$$(1) \quad q(1_{\text{Ult}(L^X)}^{\text{rsc}}) = O_{\text{CR}(L^X)}.$$

For any $A, B \in L^X$,

$$(i) \quad O_{\text{CR}(L^X)}(A, B) = 1$$

$$\begin{aligned} &\Rightarrow A \wedge B \neq 0_X \\ &\Rightarrow 1_{\text{Ult}(L^X)}^{\text{rsc}}(\eta_{A \wedge B}, \eta_{A \wedge B}) = 1, \text{ and } \eta_{A \wedge B}(A) = 1, \\ &\quad \eta_{A \wedge B}(B) = 1 \\ &\Rightarrow q(1_{\text{Ult}(L^X)}^{\text{rsc}})(A, B) = \bigvee_{v_1, v_2 \in \text{Ult}(L^X)} 1_{\text{Ult}(L^X)}^{\text{rsc}}(v_1, v_2) \wedge v_1(A) \wedge v_2(B) \\ &\geq 1_{\text{Ult}(L^X)}^{\text{rsc}}(\eta_{A \wedge B}, \eta_{A \wedge B}) \wedge \eta_{A \wedge B}(A) \wedge \eta_{A \wedge B}(B) = 1; \end{aligned}$$

$$(ii) \quad q(1_{\text{Ult}(L^X)}^{\text{rsc}})(A, B) = 0$$

$$\begin{aligned} &\Rightarrow \bigvee_{v_1, v_2 \in \text{Ult}(L^X)} 1_{\text{Ult}(L^X)}^{\text{rsc}}(v_1, v_2) \wedge v_1(A) \wedge v_2(B) = 0 \\ &\Rightarrow A \wedge B = 0_X. \end{aligned}$$

If not, $A \wedge B \neq 0_X$; by (i), we obtain $1_{\text{Ult}(L^X)}^{\text{rsc}}(\eta_{A \wedge B}, \eta_{A \wedge B}) \wedge \eta_{A \wedge B}(A) \wedge \eta_{A \wedge B}(B) = 1$, a contradiction. Thus $\Rightarrow O_{\text{CR}(L^X)}(A, B) = 0$.

(2) Consider $p(O_{CR(L^X)}) = 1_{\text{Ult}(L^X)}^{\text{rsc}}$.

For any $\nu_1, \nu_2 \in \text{Ult}(L^X)$,

(i) $1_{\text{Ult}(L^X)}^{\text{rsc}}(\nu_1, \nu_2) = 1$

$$\Rightarrow \nu_1 = \nu_2$$

$$\Rightarrow \bigcap \nu_1 = \bigcap \nu_2,$$

$$\Rightarrow p(O_{CR(L^X)})(\nu_1, \nu_2) = \text{CL}(O_{CR(L^X)})(\nu_1, \nu_2) \geq O_{CR(L^X)}(\bigcap \nu_1, \bigcap \nu_2) = 1;$$

(ii) $p(O_{CR(L^X)})(\nu_1, \nu_2) = 0$

$$\Rightarrow \bigcap \nu_1 \neq \bigcap \nu_2$$

$$\Rightarrow \nu_1 \neq \nu_2$$

$$\Rightarrow 1_{\text{Ult}(L^X)}^{\text{rsc}}(\nu_1, \nu_2) = 0.$$

7. Conclusion

In the paper, from the point of view of graded truth approach, we introduced the notion of a contact relation in fuzzy setting, proved all contact relations on L^X form an $\mathbf{L}$ -ordered complete lattice with the two operators $\sum$ and $\prod$, and investigated the correspondence between the contact relations on L^X and the close, reflexive, symmetric relations on $\text{Ult}(L^X)$.

Competing Interests

The author declares that they have no competing interests.

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