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Research Article

Vertex Degrees and Isomorphic Properties in Complement of an m-Polar Fuzzy Graph

Ch. Ramprasad,[1,2] P. L. N. Varma,[2] S. Satyanarayana,[3] and N. Srinivasarao[4]

[1] *Department of Mathematics, Vasireddy Venkatadri Institute of Technology, Nambur 522 508, India*
[2] *Department of Mathematics, VFSTR University, Vadlamudi 522 237, India*
[3] *Department of CSE, KL University, Vaddeswaram 522502, India*
[4] *Department of Mathematics, Tirumala Engineering College, Narasaraopet 522034, India*

Correspondence should be addressed to S. Satyanarayana; s.satyans1@gmail.com

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Computational intelligence and computer science rely on graph theory to solve combinatorial problems. Normal product and tensor product of an m-polar fuzzy graph have been introduced in this article. Degrees of vertices in various product graphs, like Cartesian product, composition, tensor product, and normal product, have been computed. Complement and μ-complement of an m-polar fuzzy graph are defined and some properties are studied. An application of an m-polar fuzzy graph is also presented in this article.

1. Introduction

Akram [1] introduced the notion of bipolar fuzzy graphs describing various methods of their construction as well as investigating some of their important properties. Bhutani [2] discussed automorphism of fuzzy graphs. Chen et al. [3] generalized the concept of bipolar fuzzy set to obtain the notion of an m-polar fuzzy set. The notion of an m-polar fuzzy set is more advanced than fuzzy set and eliminates ambiguity more absolutely. Ghorai and Pal [4] studied some operations and properties of an m-polar fuzzy graph. Mordeson and Peng [5] defined join, union, Cartesian product, and composition of two fuzzy graphs. Rashmanlou et al. [6] discussed some properties of bipolar fuzzy graphs and their results. Sunitha and Vijaya Kumar [7] defined the complement of a fuzzy graph in another way which gives a better understanding about that concept. We have studied product m-polar fuzzy graph, product m-polar fuzzy intersection graph, and product m-polar fuzzy line graph [8].

In this article, we study the Cartesian product and composition of two m-polar fuzzy graphs and compute the degrees of the vertices in these graphs. The notions of normal product and tensor product of m-polar fuzzy graphs are introduced and some properties are studied. Also in the present work, we introduce the concept of complement, μ-complement of an m-polar fuzzy graph, and some properties are discussed.

These concepts strengthen the decision-making in critical situations. Some applications to decision-making are also studied.

In this article, unless and otherwise specified, all graphs considered are m-polar fuzzy graphs.

2. Preliminaries

Definition 1. The m-polar fuzzy graph of a graph $G^* = (V, E)$ is a pair $G = (W, F)$, where $W : V \to [0,1]^m$ is an m-polar fuzzy set in V and $F : V \tilde{\times} V \to [0,1]^m$ is an m-polar fuzzy set in $V \tilde{\times} V$ such that $F(qr) \leq \min\{W(q), W(r)\}$ for all $qr \in V \tilde{\times} V$ and $F(qr) = 0$ for all edges $qr \in (\widetilde{V}^2 - E)$ $(0 = 0, 0, \ldots, 0)$ is the smallest element in $[0,1]^m$. W is called the m-polar fuzzy vertex set of G and F is called m-polar fuzzy edge set of G.

Sometimes we denote the graph $G = (W, F)$ by $G = (V, W, F)$ also.

Definition 2. Given two graphs G_1, G_2 their Cartesian product, $G_1 \times G_2 = (V_1 \times V_2, W_1 \times W_2, F_1 \times F_2)$, is defined as follows:

for $i = 1, 2, \ldots, m$, we have

(i) $p_i \circ (W_1 \times W_2)(q_1, q_2) = \min\{p_i \circ W_1(q_1), p_i \circ W_2(q_2)\}$ for all $(q_1, q_2) \in V_1 \times V_2$,

(ii) $p_i \circ (F_1 \times F_2)((q, q_2)(q, r_2)) = \min\{p_i \circ W_1(q), p_i \circ F_2(q_2 r_2)\}$ for all $q \in V_1$, $q_2 r_2 \in E_2$,

(iii) $p_i \circ (F_1 \times F_2)((q_1, m)(r_1, m)) = \min\{p_i \circ F_1(q_1 r_1), p_i \circ W_2(m)\}$ for all $m \in V_2$, $q_1 r_1 \in E_1$,

(iv) $p_i \circ (F_1 \times F_2)((q_1, q_2)(r_1, r_2)) = 0$ for all $(q_1, q_2)(r_1, r_2) \in (V_1 \tilde{\times} V_2)^2 - E$.

Definition 3. Given two graphs G_1, G_2 their composition, $G_1[G_2] = (V_1 \times V_2, W_1[W_2], F_1[F_2])$, is defined as follows: for $i = 1, 2, \ldots, m$, we have

(i) $p_i \circ (W_1[W_2])(q_1, q_2) = \min\{p_i \circ W_1(q_1), p_i \circ W_2(q_2)\}$ for all $(q_1, q_2) \in V_1 \times V_2$,

(ii) $p_i \circ (F_1[F_2])((q, q_2)(q, r_2)) = \min\{p_i \circ W_1(q), p_i \circ F_2(q_2 r_2)\}$ for all $q \in V_1$, $q_2 r_2 \in E_2$,

(iii) $p_i \circ (F_1[F_2])((q_1, m)(r_1, m)) = \min\{p_i \circ F_1(q_1 r_1), p_i \circ W_2(m)\}$ for all $m \in V_2$, $q_1 r_1 \in E_1$,

(iv) $p_i \circ (F_1[F_2])((q_1, q_2)(r_1, r_2)) = \min\{p_i \circ W_2(q_2), p_i \circ W_2(r_2), p_i \circ F_1(q_1 r_1)\}$ for all $(q_1, q_2)(r_1, r_2) \in E^0 - E$, where $E^0 = E \cup \{(q_1, q_2)(r_1, r_2) \mid q_1 r_1 \in E_1,\ q_2 \neq r_2\}$.

Definition 4. The normal product of G_1 and G_2 is defined as the m-polar fuzzy graph $G_1 \bullet G_2 = (V_1 \times V_2, W_1 \bullet W_2, F_1 \bullet F_2)$ on $G^* = (V, E)$, where $E = \{((q, q_2)(q, r_2)) \mid q \in V_1,\ q_2 r_2 \in E_2\} \cup \{(q_1, m)(r_1, m) \mid q_1 r_1 \in E_1,\ m \in V_2\} \cup \{(q_1, q_2)(r_1, r_2) \mid q_1 r_1 \in E_1,\ q_2 r_2 \in E_2\}$ such that for $i = 1, 2, \ldots, m$, we have

(i) $p_i \circ (W_1 \bullet W_2)(q_1, q_2) = \min\{p_i \circ W_1(q_1), p_i \circ W_2(q_2)\}$ for all $(q_1, q_2) \in V_1 \times V_2$,

(ii) $p_i \circ (F_1 \bullet F_2)((q, q_2)(q, r_2)) = \min\{p_i \circ W_1(q), p_i \circ F_2(q_2 r_2)\}$ for all $q \in V_1$, $q_2 r_2 \in E_2$,

(iii) $p_i \circ (F_1 \bullet F_2)((q_1, m)(r_1, m)) = \min\{p_i \circ F_1(q_1 r_1), p_i \circ W_2(m)\}$ for all $m \in V_2$, $q_1 r_1 \in E_1$,

(iv) $p_i \circ (F_1 \bullet F_2)((q_1, q_2)(r_1, r_2)) = \min\{p_i \circ F_1(q_1 r_1), p_i \circ F_2(q_2 r_2)\}$ for all $q_1 q_2 \in E_1$ and $r_1 r_2 \in E_2$.

Definition 5. The tensor product of G_1 and G_2 is defined as $G_1 \otimes G_2 = (V_1 \times V_2, W_1 \otimes W_2, F_1 \otimes F_2)$ on $G^* = (V, E)$, where $E = \{((q_1, q_2)(r_1, r_2)) \mid q_1 r_1 \in E_1,\ q_2 r_2 \in E_2\}$ such that for $i = 1, 2, \ldots, m$, we have

(i) $p_i \circ (W_1 \otimes W_2)(q_1, q_2) = \min\{p_i \circ W_1(q_1), p_i \circ W_2(q_2)\}$ for all $(q_1, q_2) \in V_1 \times V_2$,

(ii) $p_i \circ (F_1 \otimes F_2)((q_1, q_2)(r_1, r_2)) = \min\{p_i \circ F_1(q_1 r_1), p_i \circ F_2(q_2 r_2)\}$ for all $q_1 q_2 \in E_1$ and $r_1 r_2 \in E_2$.

Definition 6. The union $G_1 \cup G_2 = (V_1 \cup V_2, W_1 \cup W_2, F_1 \cup F_2)$ of the graphs $G_1 = (V_1, W_1, F_1)$ and $G_2 = (V_2, W_2, F_2)$ of ${G_1}^*$ and ${G_2}^*$, respectively, is defined as follows: for $i = 1, 2, \ldots, m$, we have

(i)

$$p_i \circ (W_1 \cup W_2)(q) = \begin{cases} p_i \circ W_1(q) & \text{if } q \in V_1 - V_2 \\ p_i \circ W_2(q) & \text{if } q \in V_2 - V_1 \\ \max\{p_i \circ W_1(q), p_i \circ W_2(q)\} & \text{if } q \in V_1 \cap V_2. \end{cases} \quad (1)$$

(ii)

$$p_i \circ (F_1 \cup F_2)(qr) = \begin{cases} p_i \circ F_1(qr) & \text{if } qr \in E_1 - E_2 \\ p_i \circ F_2(qr) & \text{if } qr \in E_2 - E_1 \\ \max\{p_i \circ F_1(qr), p_i \circ F_2(qr)\} & \text{if } qr \in E_1 \cap E_2. \end{cases} \quad (2)$$

(iii)

$$p_i \circ (F_1 \cup F_2)(qr) = 0 \quad \text{if } qr \in (V_1 \tilde{\times} V_2)^2 - E_1 \cup E_2. \quad (3)$$

Definition 7. The join of the graphs $G_1 = (V_1, W_1, F_1)$ and $G_2 = (V_2, W_2, F_2)$ of ${G_1}^*$ and ${G_2}^*$, respectively, is defined as $G_1 + G_2 = (V_1 \cup V_2, W_1 + W_2, F_1 + F_2)$ such that for $i = 1, 2, \ldots, m$, we have the following:

(i) $p_i \circ (W_1 + W_2)(q) = p_i \circ (W_1 \cup W_2)(q)$ if $q \in V_1 \cup V_2$.

(ii) $p_i \circ (F_1 + F_2)(qr) = p_i \circ (F_1 \cup F_2)(qr)$ if $qr \in E_1 \cup E_2$.

(iii) $p_i \circ (F_1 + F_2)(qr) = \min\{p_i \circ W_1(q), p_i \circ W_2(r)\}$ if $qr \in E'$, where E' denotes the set of all edges joining the vertices of V_1 and V_2.

(iv) $p_i \circ (F_1 + F_2)(qr) = 0$ if $qr \in (V_1 \tilde{\times} V_2)^2 - E_1 \cup E_2 \cup E'$.

3. Degree of Vertices in m-Polar Fuzzy Graph

Definition 8. Let $G = (V, W, F)$ be an m-polar fuzzy graph. Then the degree of a vertex q in G is defined as $d_G(q) = \langle \sum_{qr \in E\ q \neq r} p_1 \circ F(qr), \sum_{qr \in E\ q \neq r} p_2 \circ F(qr), \sum_{qr \in E\ q \neq r} p_3 \circ F(qr), \ldots, \sum_{qr \in E\ q \neq r} p_m \circ F(qr) \rangle$.

Further, a highly irregular m-polar fuzzy graph is defined as an m-polar fuzzy graph $G = (V, W, F)$ in which every vertex of $G = (V, W, F)$ is adjacent to vertices with distinct degrees.

Example 9. Consider the graph $G = (V, W, F)$ of $G^* = (V, E)$, where $V = \{K, L, M, N\}$, $E = \{KL, LM, MN, NK\}$, $W = \{\langle 0.2, 0.5, 0.6 \rangle / K, \langle 0.3, 0.5, 0.7 \rangle / L, \langle 0.3, 0.6, 0.7 \rangle / M, \langle 0.4, 0.7, 0.8 \rangle / N\}$, and $F = \{\langle 0.2, 0.4, 0.5 \rangle / KL, \langle 0.3, 0.4, 0.6 \rangle / LM, \langle 0.3, 0.5, 0.6 \rangle / MN, \langle 0.2, 0.3, 0.6 \rangle / NK\}$ as in Figure 1.

In this graph, we have $d_G(K) = \langle 0.2, 0.4, 0.5 \rangle + \langle 0.2, 0.3, 0.6 \rangle = \langle 0.4, 0.7, 1.1 \rangle$, $d_G(L) = \langle 0.2, 0.4, 0.5 \rangle + \langle 0.3, 0.4, 0.6 \rangle = \langle 0.5, 0.8, 1.1 \rangle$, $d_G(M) = \langle 0.3, 0.4, 0.6 \rangle + \langle 0.3, 0.5, 0.6 \rangle = \langle 0.6, 0.9, 1.2 \rangle$, $d_G(N) = \langle 0.3, 0.5, 0.6 \rangle + \langle 0.2, 0.3, 0.6 \rangle = \langle 0.5, 0.8, 1.2 \rangle$.

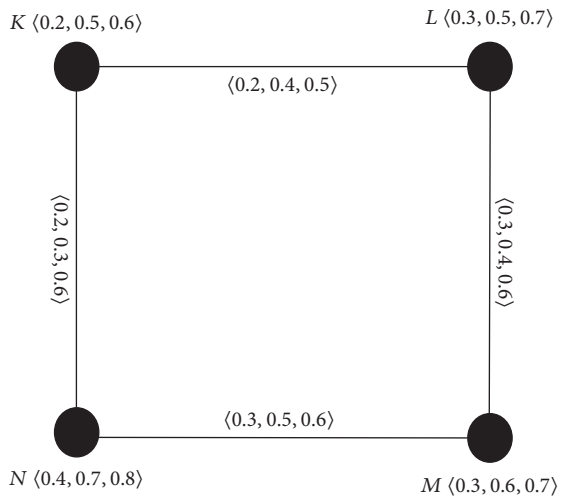

Figure 1: Highly irregular 3-polar fuzzy graph G.

4. Cartesian Product and Vertex Degree

From the definition of Cartesian product graphs, for every vertex $(q_1, q_2) \in V_1 \times V_2$, we have

$$\begin{aligned} d_{G_1\times G_2}(q_1,q_2) &= \sum_{(q_1,q_2)(r_1,r_2)\in E} p_i \circ (F_1 \times F_2)((q_1,q_2)(r_1,r_2)) \\ &= \sum_{q_1=r_1, q_2r_2\in E_2} \min\{p_i \circ W_1(q_1), p_i \circ F_2(q_2r_2)\} \\ &\quad + \sum_{q_2=r_2, q_1r_1\in E_1} \min\{p_i \circ W_2(q_2), p_i \circ F_1(q_1r_1)\} \\ &\qquad \text{for } i = 1, 2, \ldots, m. \end{aligned} \tag{4}$$

Theorem 10. *Let $G_1 = (V_1, W_1, F_1)$ and $G_2 = (V_2, W_2, F_2)$ be two graphs. If $p_i \circ W_1 \geq p_i \circ F_2$ and $p_i \circ W_2 \geq p_i \circ F_1$ then $d_{G_1\times G_2}(q_1, q_2) = d_{G_1}(q_1) + d_{G_2}(q_2)$ for $i = 1, 2, \ldots, m$.*

Proof. By definition, we get

$$\begin{aligned} d_{G_1\times G_2}(q_1,q_2) &= \sum_{(q_1,q_2)(r_1,r_2)} p_i \circ (F_1 \times F_2)((q_1,q_2)(r_1,r_2)) \\ &= \sum_{q_1=r_1, q_2r_2\in E_2} \min\{p_i \circ W_1(q_1), p_i \circ F_2(q_2r_2)\} \\ &\quad + \sum_{q_2=r_2, q_1r_1\in E_1} \min\{p_i \circ W_2(q_2), p_i \circ F_1(q_1r_1)\} \\ &= \sum_{q_2r_2\in E_2} p_i \circ F_2(q_2r_2) + \sum_{q_1r_1\in E_1} p_i \circ F_1(q_1r_1) \\ &\qquad (\text{Since } p_i \circ W_1 \geq p_i \circ F_2,\ p_i \circ W_2 \geq p_i \circ F_1) \\ &= d_{G_1}(q_1) + d_{G_2}(q_2). \end{aligned} \tag{5}$$

□

Example 11. Consider two graphs G_1, G_2 and their Cartesian product, $G_1 \times G_2$, as shown in Figure 2.

Since $p_i \circ W_1 \geq p_i \circ F_2$ and $p_i \circ W_2 \geq p_i \circ F_1$, by Theorem 10, we have

$$\begin{aligned} d_{G_1\times G_2}(s_1,s_2) &= d_{G_1}(s_1) + d_{G_2}(s_2) \\ &= \langle 0.2, 0.3, 0.1\rangle + \langle 0.2, 0.4, 0.1\rangle \\ &= \langle 0.4, 0.7, 0.2\rangle, \\ d_{G_1\times G_2}(s_1,t_2) &= d_{G_1}(s_1) + d_{G_2}(t_2) \\ &= \langle 0.2, 0.3, 0.1\rangle + \langle 0.2, 0.4, 0.1\rangle \\ &= \langle 0.4, 0.7, 0.2\rangle \\ d_{G_1\times G_2}(t_1,s_2) &= d_{G_1}(t_1) + d_{G_2}(s_2) \\ &= \langle 0.2, 0.3, 0.1\rangle + \langle 0.2, 0.4, 0.1\rangle \\ &= \langle 0.4, 0.7, 0.2\rangle, \\ d_{G_1\times G_2}(t_1,t_2) &= d_{G_1}(t_1) + d_{G_2}(t_2) \\ &= \langle 0.2, 0.3, 0.1\rangle + \langle 0.2, 0.4, 0.1\rangle \\ &= \langle 0.4, 0.7, 0.2\rangle. \end{aligned} \tag{6}$$

5. Composition Graph and Vertex Degree

From the definition of composition graphs, for every vertex $(q_1, q_2) \in V_1 \times V_2$, we have

$$\begin{aligned} d_{G_1[G_2]}(q_1,q_2) &= \sum_{(q_1,q_2)(r_1,r_2)\in E} p_i \circ (F_1[F_2])((q_1,q_2) \\ &\quad \cdot (r_1,r_2)) = \sum_{q_1=r_1, q_2r_2\in E_2} \min\{p_i \circ W_1(q_1), p_i \\ &\quad \circ F_2(q_2r_2)\} + \sum_{q_2=r_2, q_1r_1\in E_1} \min\{p_i \circ W_2(q_2), p_i \\ &\quad \circ F_1(q_1r_1)\} + \sum_{q_2\neq r_2, q_1r_1\in E_1} \min\{p_i \circ W_2(r_2), p_i \\ &\quad \circ W_2(q_2), p_i \circ F_1(q_1r_1)\} \quad \text{for } i = 1, 2, \ldots, m. \end{aligned} \tag{7}$$

Theorem 12. *Let $G_1 = (V_1, W_1, F_1)$ and $G_2 = (V_2, W_2, F_2)$ be two graphs. If $p_i \circ W_1 \geq p_i \circ F_2$ and $p_i \circ W_2 \geq p_i \circ F_1$ then $d_{G_1[G_2]}(q_1, q_2) = |V_2| d_{G_1}(q_1) + d_{G_2}(q_2)$ for all $(q_1, q_2) \in V_1 \times V_2$ for $i = 1, 2, \ldots, m$.*

Proof. By definition, we get

$$\begin{aligned} d_{G_1[G_2]}(q_1,q_2) &= \sum_{(q_1,q_2)(r_1,r_2)} p_i \circ (F_1[F_2])((q_1,q_2) \\ &\quad \cdot (r_1,r_2)) = \sum_{q_1=r_1, q_2r_2\in E_2} \min\{p_i \circ W_1(q_1), p_i \\ &\quad \circ F_2(q_2r_2)\} + \sum_{q_2=r_2, q_1r_1\in E_1} \min\{p_i \circ W_2(q_2), p_i \end{aligned}$$

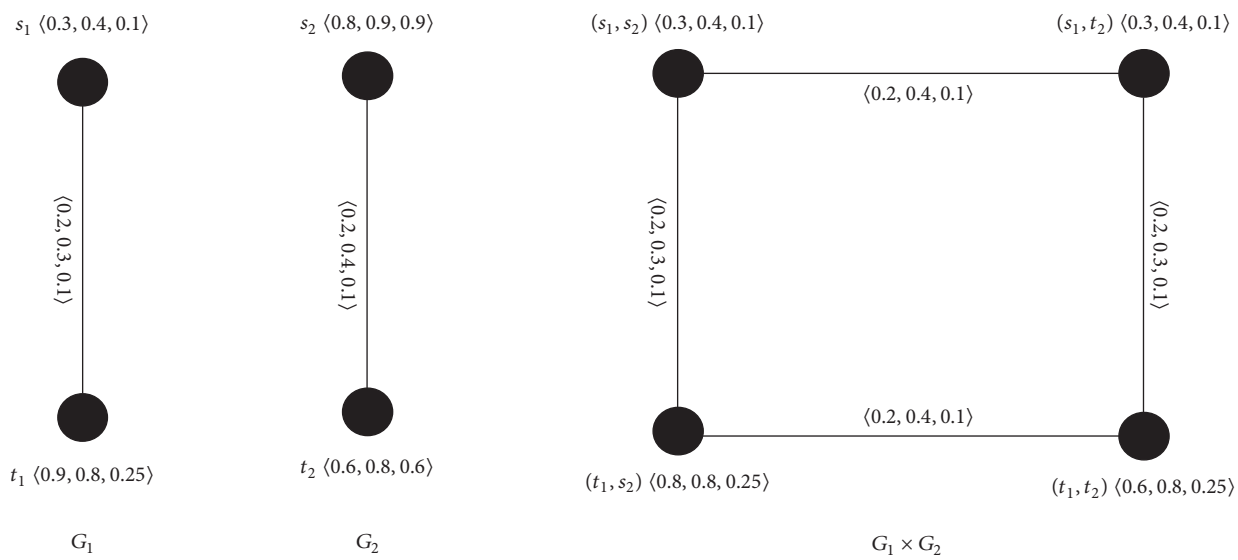

Figure 2: Cartesian product of two 3-polar fuzzy graphs G_1 and G_2.

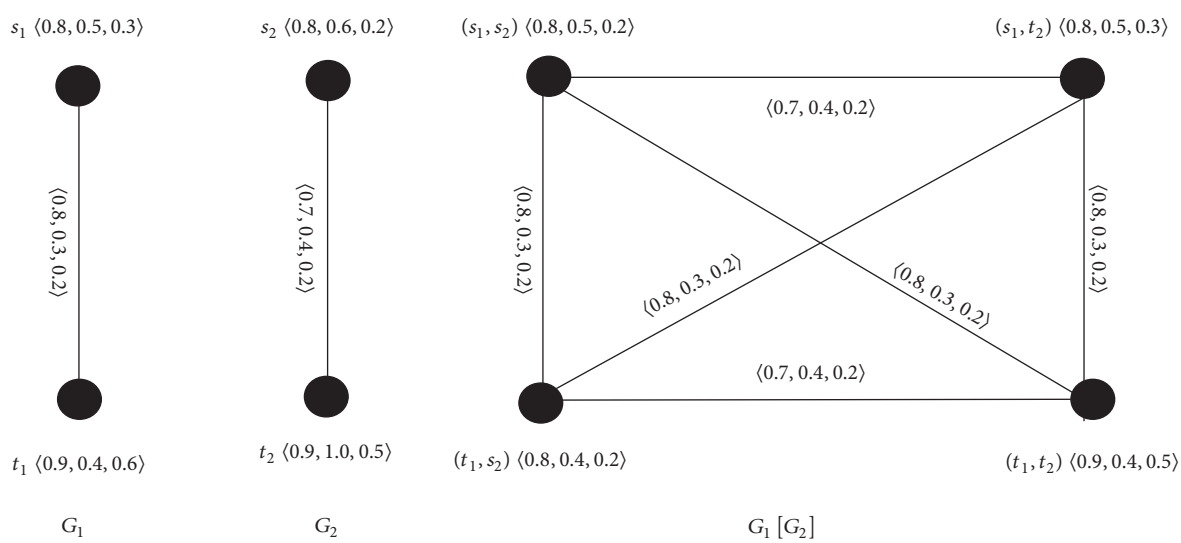

Figure 3: Composition of two 3-polar fuzzy graphs G_1 and G_2.

$$\begin{aligned}
&\circ F_1(q_1r_1)\} + \sum_{q_2\neq r_2, q_1r_1\in E_1} \min\{p_i \circ W_2(r_2), p_i \\
&\circ W_2(q_2), p_i \circ F_1(q_1r_1)\} = \sum_{q_2r_2\in E_2} p_i \circ F_2(q_2r_2) \\
&+ \sum_{q_2=r_2, q_1r_1\in E_1} p_i \circ F_1(q_1r_1) + \sum_{q_2\neq r_2, q_1r_1\in E_1} p_i \\
&\circ F_1(q_1r_1) = d_{G_2}(q_2) + |V_2| \sum_{q_1r_1\in E_1} p_i \circ F_1(q_1r_1) \\
&= |V_2| d_{G_1}(q_1) + d_{G_2}(q_2).
\end{aligned} \tag{8}$$

□

Example 13. Consider two graphs G_1, G_2 and their composition, $G_1[G_2]$, as shown in Figure 3.

Since $p_i \circ W_1 \geq p_i \circ F_2$ and $p_i \circ W_2 \geq p_i \circ F_1$, by Theorem 12, we have

$$\begin{aligned}
d_{G_1[G_2]}(s_1, s_2) &= |V_2| d_{G_1}(s_1) + d_{G_2}(s_2) \\
&= 2\langle 0.8, 0.3, 0.2\rangle + \langle 0.7, 0.4, 0.2\rangle \\
&= \langle 2.3, 1.0, 0.6\rangle, \\
d_{G_1[G_2]}(s_1, t_2) &= |V_2| d_{G_1}(s_1) + d_{G_2}(t_2) \\
&= 2\langle 0.8, 0.3, 0.2\rangle + \langle 0.7, 0.4, 0.2\rangle \\
&= \langle 2.3, 1.0, 0.6\rangle,
\end{aligned}$$

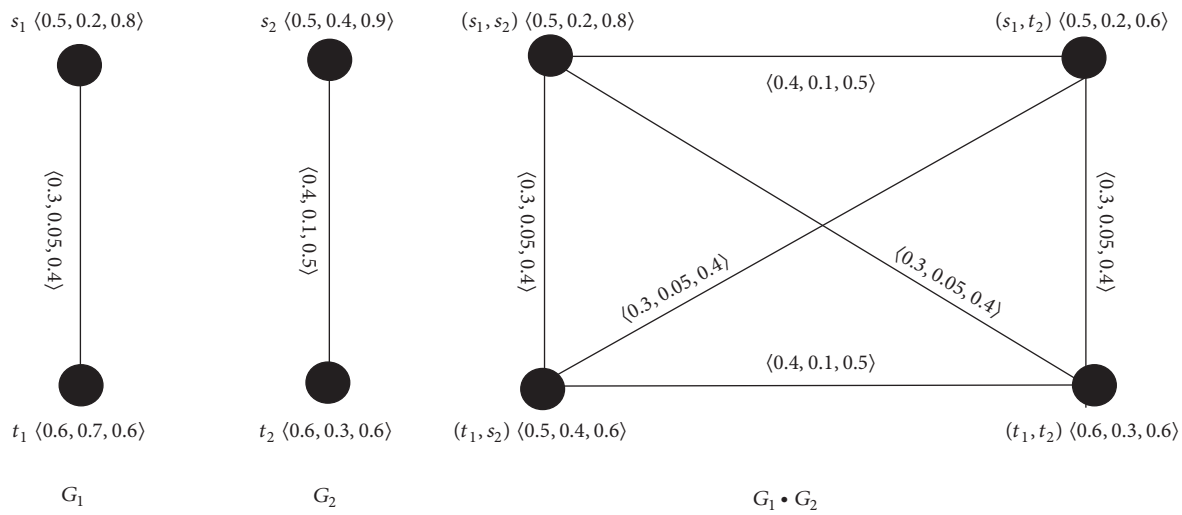

Figure 4: Normal product of two 3-polar fuzzy graphs G_1 and G_2.

$$\begin{aligned} d_{G_1[G_2]}(t_1, s_2) &= |V_2| d_{G_1}(t_1) + d_{G_2}(s_2) \\ &= 2\langle 0.8, 0.3, 0.2\rangle + \langle 0.7, 0.4, 0.2\rangle \\ &= \langle 2.3, 1.0, 0.6\rangle, \\ d_{G_1[G_2]}(t_1, t_2) &= |V_2| d_{G_1}(t_1) + d_{G_2}(t_2) \\ &= 2\langle 0.8, 0.3, 0.2\rangle + \langle 0.7, 0.4, 0.2\rangle \\ &= \langle 2.3, 1.0, 0.6\rangle. \end{aligned} \tag{9}$$

6. Normal Product and Vertex Degree

From the definition of normal product graphs, for every vertex $(q_1, q_2) \in V_1 \times V_2$, we have

$$\begin{aligned} & d_{G_1 \bullet G_2}(q_1, q_2) \\ &= \sum_{(q_1,q_2)(r_1,r_2)\in E} p_i \circ (F_1 \bullet F_2)((q_1, r_1)(q_2, r_2)) \\ &= \sum_{q_1=r_1, q_2r_2\in E_2} \min\{p_i \circ W_1(q_1), p_i \circ F_2(q_2r_2)\} \\ &\quad + \sum_{q_2=r_2, q_1r_1\in E_1} \min\{p_i \circ W_2(q_2), p_i \circ F_1(q_1r_1)\} \\ &\quad + \sum_{q_2r_2\in E_2, q_1r_1\in E_1} \min\{p_i \circ F_1(q_1r_1), p_i \circ F_2(q_2r_2)\} \\ &\qquad \text{for } i = 1, 2, \ldots, m. \end{aligned} \tag{10}$$

Theorem 14. *Let $G_1 = (V_1, W_1, F_1)$ and $G_2 = (V_2, W_2, F_2)$ be two graphs. If $p_i \circ W_1 \geq p_i \circ F_2$, $p_i \circ W_2 \geq p_i \circ F_1$ and $p_i \circ F_1 \leq p_i \circ F_2$, then $d_{G_1 \bullet G_2}(q_1, q_2) = |V_2| d_{G_1}(q_1) + d_{G_2}(q_2)$ for $i = 1, 2, \ldots, m$.*

Proof. By definition, we get

$$\begin{aligned} & d_{G_1 \bullet G_2}(q_1, q_2) = \sum_{(q_1,q_2)(r_1,r_2)\in E} p_i \circ (F_1 \times F_2)((q_1, r_1)(q_2, r_2)) \\ &= \sum_{q_1=r_1, q_2r_2\in E_2} \min\{p_i \circ W_1(q_1), p_i \circ F_2(q_2r_2)\} \\ &\quad + \sum_{q_2=r_2, q_1r_1\in E_1} \min\{p_i \circ W_2(q_2), p_i \circ F_1(q_1r_1)\} \\ &\quad + \sum_{q_2r_2\in E_2, q_1r_1\in E_1} \min\{p_i \circ F_1(q_1r_1), p_i \circ F_2(q_2r_2)\} \\ &= \sum_{q_2r_2\in E_2} p_i \circ F_2(q_2r_2) + \sum_{q_2=r_2, q_1r_1\in E_1} p_i \circ F_1(q_1r_1) \\ &\quad + \sum_{q_1r_1\in E_1} p_i \circ F_1(q_1r_1) \\ &\quad (\text{Since } p_i \circ W_1 \geq p_i \circ F_2, p_i \circ W_2 \geq p_i \circ F_1,\ p_i \circ F_1 \leq p_i \circ F_2) \\ &= |V_2| d_{G_1}(q_1) + d_{G_2}(q_2). \end{aligned} \tag{11}$$

□

Example 15. Consider two graphs G_1, G_2 and their normal product, $G_1 \bullet G_2$, is shown in Figure 4.

Since $p_i \circ W_1 \geq p_i \circ F_2$ $p_i \circ W_2 \geq p_i \circ F_1$ and $p_i \circ F_1 \leq p_i \circ F_2$, by Theorem 14, we have

$$\begin{aligned} d_{G_1 \bullet G_2}(s_1, s_2) &= |V_2| d_{G_1}(s_1) + d_{G_2}(s_2) \\ &= 2\langle 0.3, 0.05, 0.4\rangle + \langle 0.4, 0.1, 0.5\rangle \\ &= \langle 1.0, 0.2, 1.3\rangle, \\ d_{G_1 \bullet G_2}(s_1, t_2) &= |V_2| d_{G_1}(s_1) + d_{G_2}(t_2) \\ &= 2\langle 0.3, 0.05, 0.4\rangle + \langle 0.4, 0.1, 0.5\rangle \\ &= \langle 1.0, 0.2, 1.3\rangle \end{aligned}$$

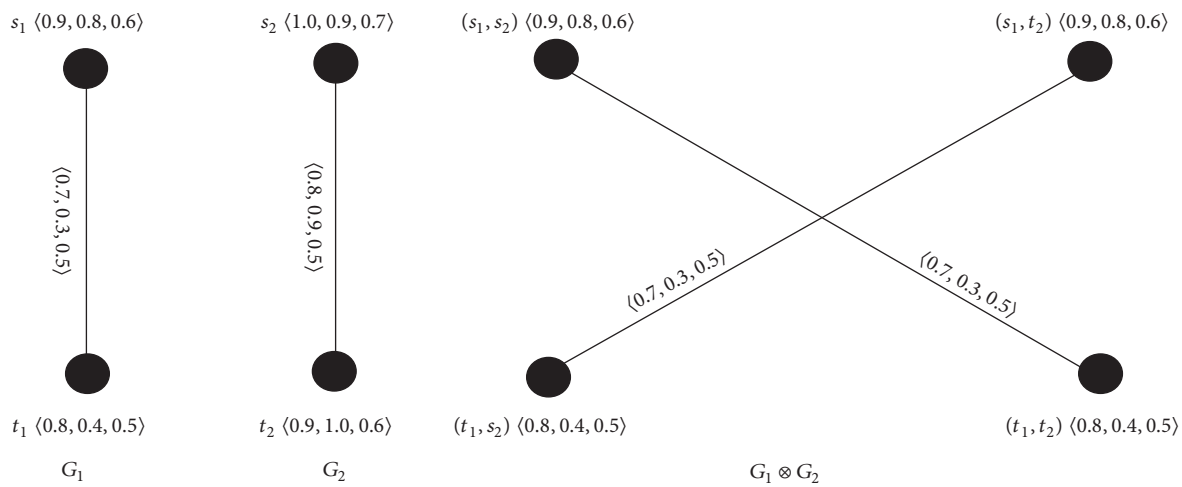

Figure 5: Tensor product of two 3-polar fuzzy graphs G_1 and G_2.

$$\begin{aligned} d_{G_1 \bullet G_2}(t_1, s_2) &= |V_2| d_{G_1}(t_1) + d_{G_2}(s_2) \\ &= 2\langle 0.3, 0.05, 0.4\rangle + \langle 0.4, 0.1, 0.5\rangle \\ &= \langle 1.0, 0.2, 1.3\rangle, \\ d_{G_1 \bullet G_2}(t_1, t_2) &= |V_2| d_{G_1}(t_1) + d_{G_2}(t_2) \\ &= 2\langle 0.3, 0.05, 0.4\rangle + \langle 0.4, 0.1, 0.5\rangle \\ &= \langle 1.0, 0.2, 1.3\rangle. \end{aligned} \tag{12}$$

7. Tensor Product and Vertex Degree

From the definition of tensor product, for every vertex $(q_1, q_2) \in V_1 \times V_2$ for $i = 1, 2, \ldots, m$

$$\begin{aligned} d_{G_1 \otimes G_2}(q_1, q_2) &= \sum p_i \circ (F_1 \otimes F_2)((q_1, q_2)(r_1, r_2)) \\ &= \sum_{q_1 r_1 \in E_1} \min\{p_i \circ F_1(q_1 r_1), p_i \circ F_2(q_2 r_2)\}. \end{aligned} \tag{13}$$

Theorem 16. *Let $G_1 = (V_1, W_1, F_1)$ and $G_2 = (V_2, W_2, F_2)$ be two m-polar fuzzy graphs. If $p_i \circ F_2 \geq p_i \circ F_1$, then $d_{G_1 \otimes G_2}(q_1, q_2) = d_{G_1}(q_1)$ and if $p_i \circ F_1 \geq p_i \circ F_2$, then $d_{G_1 \otimes G_2}(q_1, q_2) = d_{G_2}(q_2)$ for $i = 1, 2, \ldots, m$.*

Proof. Let $p_i \circ F_2 \geq p_i \circ F_1$. Then we get

$$\begin{aligned} & d_{G_1 \otimes G_2}(q_1, q_2) \\ &\quad = \sum_{q_1 r_1 \in E_1} \min\{p_i \circ F_1(q_1 r_1), p_i \circ F_2(q_2 r_2)\} \\ &\quad = \sum p_i \circ F_1(q_1 r_1) = d_{G_1}(q_1). \end{aligned} \tag{14}$$

So, $d_{G_1 \otimes G_2}(q_1, q_2) = d_{G_1}(q_1)$.

Let $p_i \circ F_1 \geq p_i \circ F_2$. Then we get

$$\begin{aligned} & d_{G_1 \otimes G_2}(q_1, q_2) \\ &\quad = \sum_{q_1 r_1 \in E_1} \min\{p_i \circ F_1(q_1 r_1), p_i \circ F_2(q_2 r_2)\} \\ &\quad = \sum p_i \circ F_2(q_2 r_2) = d_{G_2}(q_2). \end{aligned} \tag{15}$$

So, $d_{G_1 \otimes G_2}(q_1, q_2) = d_{G_2}(q_2)$. □

Example 17. Consider two graphs G_1, G_2 and their tensor product, $G_1 \otimes G_2$, is shown in Figure 5.

Since $p_i \circ F_2 \geq p_i \circ F_1$, by Theorem 16, we have

$$\begin{aligned} d_{G_1 \otimes G_2}(s_1, s_2) &= d_{G_1}(s_1) = \langle 0.7, 0.3, 0.5\rangle, \\ d_{G_1 \otimes G_2}(s_1, t_2) &= d_{G_1}(s_1) = \langle 0.7, 0.3, 0.5\rangle, \\ d_{G_1 \otimes G_2}(t_1, s_2) &= d_{G_1}(t_1) = \langle 0.7, 0.3, 0.5\rangle, \\ d_{G_1 \otimes G_2}(t_1, t_2) &= d_{G_1}(t_1) = \langle 0.7, 0.3, 0.5\rangle. \end{aligned} \tag{16}$$

8. μ-Complement of an m-Polar Fuzzy Graph

Now for an m-polar fuzzy graph G, we introduce the notion of its μ-complement.

Definition 18. The complement of an m-polar fuzzy graph $G = (V, W, F)$ is also an m-polar fuzzy graph $\overline{G} = (V, \overline{W}, \overline{F})$, where $\overline{W} = W$ and $\overline{F}$ is defined as follows:

$$p_i \circ \overline{F}(qr) = \wedge(p_i \circ W(q), p_i \circ W(r)) - p_i \circ F(qr) \quad \forall q, r \in V \text{ for } i = 1, 2, \ldots, m. \tag{17}$$

Definition 19. Let $G = (V, W, F)$ be an m-polar fuzzy graph. If G is isomorphic to $\overline{G}$ we say G is self-complementary. Similarly, if G is weak isomorphic to $\overline{G}$ then we say G is self-weak complementary.

Definition 20. The μ-complement of G is defined as $G^{\mu} = (V, W^{\mu}, F^{\mu})$, where $W^{\mu} = W$ and F^{μ} is given by

$$p_i \circ F^{\mu}(qr) = \begin{cases} \min\{p_i \circ W(q), p_i \circ W(r)\} - p_i \circ F(qr) & \text{if } p_i \circ F(qr) > 0 \\ 0 & \text{if } p_i \circ F(qr) = 0 \end{cases} \tag{18}$$

for $i = 1, 2, \ldots, m$.

Theorem 21. *Let $G = (V, W, F)$ be a self-weak complementary and highly irregular graph. Then, we have $\sum_{q\neq r} p_i \circ F(qr) \leq 0.5 \sum_{q\neq r} \wedge\{p_i \circ W(q), p_i \circ W(r)\}$.*

Proof. Let $G = (V, W, F)$ be a self-weak complementary, highly irregular graph of $G^* = (V, E)$. Then, there exists a weak isomorphism g from G to $\overline{G}$ such that for all $q, r \in V$, we get $p_i \circ W(q) = p_i \circ \overline{W}(g(q)) = p_i \circ W(g(q))$, $p_i \circ F(qr) \leq p_i \circ \overline{F}(g(q)g(r))$. From the definition of complement, from the above inequality, for all $q, r \in V$, we get that $p_i \circ F(qr) \leq p_i \circ \overline{F}(g(q)g(r)) = \wedge(p_i \circ W(q), p_i \circ W(r)) - p_i \circ F(g(q)g(r))$ and $p_i \circ F(qr) + p_i \circ F(g(q)g(r)) \leq \wedge(p_i \circ W(q), p_i \circ W(r))$. Hence, $\sum_{q\neq r} p_i \circ F(qr) + \sum_{q\neq r} p_i \circ F(g(q)g(r)) \leq \sum_{q\neq r} \wedge(p_i \circ W(q), p_i \circ W(r))$. Thus, we get

$$\begin{aligned} 2\sum_{q\neq r} p_i \circ F(qr) &\leq \sum_{q\neq r} \wedge(p_i \circ W(q), p_i \circ W(r)) \\ \text{or } \sum_{q\neq r} p_i \circ F(qr) &\leq 0.5 \sum_{q\neq r} \wedge\{p_i \circ W(q), p_i \circ W(r)\}. \end{aligned} \tag{19}$$

□

Theorem 22. *Let $G = (V, W, F)$ be a graph which is self-complementary and highly irregular. Then, we have $\sum_{q\neq r} p_i \circ F(qr) = 0.5 \sum_{q\neq r} \wedge\{p_i \circ W(q), p_i \circ W(r)\}$, for all $qr \in \widetilde{V}^2$, $i = 1, 2, \ldots, m$.*

Proof. Let $G = (V, W, F)$ be a self-complementary, highly irregular graph of $G^* = (V, E)$. Then, there exists an isomorphism g from G to $\overline{G}$ such that $p_i \circ W(q) = p_i \circ \overline{W}(q)$ for all $q \in V$ and $p_i \circ F(qr) = p_i \circ \overline{F}(g(q)g(r))$ for all $qr \in \widetilde{V}^2$.

Let $qr \in \widetilde{V}^2$. Then, for $i = 1, 2, \ldots, m$, we have

$$\begin{aligned} &p_i \circ \overline{F}(g(q)g(r)) \\ &\quad = \wedge(p_i \circ W(g(q)), p_i \circ W(g(r))) - p_i \\ &\qquad \circ F(g(q)g(r)); \end{aligned} \tag{20}$$

that is, $p_i \circ F(qr) = \wedge(p_i \circ W(g(q)), p_i \circ W(g(r))) - p_i \circ F(g(q)g(r))$. Therefore,

$$\begin{aligned} &\sum_{q\neq r} p_i \circ F(qr) + \sum_{q\neq r} p_i \circ F(g(q)g(r)) \\ &\quad = \sum_{q\neq r} \wedge\{p_i \circ W(g(q)), p_i \circ W(g(r))\} \\ &\quad = \sum_{q\neq r} \wedge\{p_i \circ W(q), p_i \circ W(r)\} \\ &2\sum_{q\neq r} p_i \circ F(qr) = \sum_{q\neq r} \wedge\{p_i \circ W(q), p_i \circ W(r)\} \end{aligned} \tag{21}$$

i.e. $\sum_{q\neq r} p_i \circ F(qr) = 0.5 \sum_{q\neq r} \wedge\{p_i \circ W(q), p_i \circ W(r)\}$. □

Theorem 23. *Let $G_1 = (V_1, W_1, F_1)$ and $G_2 = (V_2, W_2, F_2)$ be two graphs. If G_1 and G_2 are isomorphic, then their μ-complements, ${G_1}^{\mu}$ and ${G_2}^{\mu}$, are also isomorphic.*

Proof. Let $G_1 \cong G_2$ and let g be an isomorphism from G_1 to G_2. Then for $i = 1, 2, \ldots, m$

$$\begin{aligned} p_i \circ W_1(q) &= p_i \circ W_2(g(q)), \quad \forall q \in V_1, \\ p_i \circ F_1(qr) &= p_i \circ F_2(g(q)g(r)), \quad \forall qr \in E_1. \end{aligned} \tag{22}$$

If $p_i \circ F_1(qr) > 0$, then $p_i \circ F_2(g(q)g(r)) > 0$, and

$$\begin{aligned} p_i \circ {F_1}^{\mu}(qr) &= \min\{p_i \circ W_1(q), p_i \circ W_1(r)\} - p_i \\ &\quad \circ F_1(qr) \\ &= \min\{p_i \circ W_2(g(q)), p_i \circ W_2(g(r))\} \\ &\quad - p_i \circ F_2(g(q)g(r)) \\ &= p_i \circ {F_2}^{\mu}(g(q)g(r)). \end{aligned} \tag{23}$$

If $p_i \circ F_1(qr) = 0$, then $p_i \circ F_2(g(q)g(r)) = 0$, and $p_i \circ {F_1}^{\mu}(qr) = 0 = p_i \circ {F_2}^{\mu}(g(q)g(r))$.

Thus, $p_i \circ {F_1}^{\mu}(qr) = p_i \circ {F_2}^{\mu}(g(q)g(r))$ for all $qr \in E_1$. Therefore, g from ${G_1}^{\mu}$ to ${G_2}^{\mu}$ is an isomorphism; that is, ${G_1}^{\mu} \cong {G_2}^{\mu}$. □

Theorem 24. *Let $G_1 = (V_1, W_1, F_1)$ and $G_2 = (V_2, W_2, F_2)$ be two graphs such that $V_1 \cap V_2 = \phi$. Then $(G_1 + G_2)^{\mu} \cong {G_1}^{\mu} \cup {G_2}^{\mu}$.*

Proof. Let $I : V_1 \cup V_2 \to V_1 \cup V_2$ be the identity map. To show that $(G_1 + G_2)^{\mu} \cong {G_1}^{\mu} \cup {G_2}^{\mu}$, it is enough to prove that

$(p_i \circ W_1 + p_i \circ W_2)^\mu(q) = p_i \circ W_1{}^\mu(q) \cup p_i \circ W_2{}^\mu(q)$ and $(p_i \circ F_1 + p_i \circ F_2)^\mu(qr) = p_i \circ F_1{}^\mu(qr) \cup p_i \circ F_2{}^\mu(qr)$ for all $q, r \in V$ and for $i = 1, 2, \ldots, m$. Then for any $q, r \in V$, we have

$$(p_i \circ W_1 + p_i \circ W_2)^\mu(q) = \begin{cases} p_i \circ W_1(q) & \text{if } q \in V_1 \\ p_i \circ W_2(q) & \text{if } q \in V_2 \end{cases} = \begin{cases} p_i \circ W_1{}^\mu(q) & \text{if } q \in V_1 \\ p_i \circ W_2{}^\mu(q) & \text{if } q \in V_2 \end{cases} = (p_i \circ W_1{}^\mu \cup p_i \circ W_2{}^\mu)(q). \tag{24}$$

Moreover,

$$\begin{aligned}(p_i \circ F_1 + p_i \circ F_2)^\mu(qr) &= \min\{(p_i \circ W_1 + p_i \circ W_2)(q), (p_i \circ W_1 + p_i \circ W_2)(r)\} - (p_i \circ F_1 + p_i \circ F_2)(qr) \\ &= \begin{cases} p_i \circ W_1(q) \wedge p_i \circ W_1(r) - p_i \circ F_1(qr) & \text{if } qr \in E_1 \\ p_i \circ W_2(q) \wedge p_i \circ W_2(r) - p_i \circ F_2(qr) & \text{if } qr \in E_2 \\ p_i \circ W_1(q) \wedge p_i \circ W_2(r) - p_i \circ W_1(q) \wedge p_i \circ W_2(r) & \text{if } qr \in E' \end{cases} \\ &= \begin{cases} p_i \circ F_1{}^\mu(qr) & \text{if } qr \in E_1 \\ p_i \circ F_2{}^\mu(qr) & \text{if } qr \in E_2 \\ 0 & \text{if } qr \in E' \end{cases} \\ &= (p_i \circ F_1{}^\mu \cup p_i \circ F_2{}^\mu)(qr).\end{aligned} \tag{25}$$

Hence, the result is proved. □

Theorem 25. *Let $G_1 = (V_1, W_1, F_1)$ and $G_2 = (V_2, W_2, F_2)$ be two graphs such that $V_1 \cap V_2 = \phi$. Then $(G_1 \cup G_2)^\mu \cong G_1{}^\mu \cup G_2{}^\mu$.*

Proof. We shall prove that the identity map $I : V_1 \cup V_2 \to V_1 \cup V_2$ is the required isomorphism. Then for any i, $1 \le i \le m$ we have the following:

If $q \in V_1$, $(p_i \circ W_1 \cup p_i \circ W_2)^\mu(q) = (p_i \circ W_1 \cup p_i \circ W_2)(q) = p_i \circ W_1(q) = p_i \circ W_1{}^\mu(q) = (p_i \circ W_1{}^\mu \cup p_i \circ W_2{}^\mu)(q)$.

If $q \in V_2$, then $(p_i \circ W_1 \cup p_i \circ W_2)^\mu(q) = (p_i \circ W_1 \cup p_i \circ W_2)(q) = p_i \circ W_2(q) = p_i \circ W_2{}^\mu(q) = (p_i \circ W_2{}^\mu \cup p_i \circ W_2{}^\mu)(q)$.

If $qr \in E_1$, then $(p_i \circ F_1 \cup p_i \circ F_2)^\mu(qr) = p_i \circ W_1(q) \wedge p_i \circ W_1(r) - p_i \circ F_1(qr) = p_i \circ F_1{}^\mu(qr) = (p_i \circ F_1{}^\mu \cup p_i \circ F_2{}^\mu)(qr)$.

If $qr \in E_2$, then $(p_i \circ F_1 \cup p_i \circ F_2)^\mu(qr) = p_i \circ W_2(q) \wedge p_i \circ W_2(r) - p_i \circ F_2(qr) = p_i \circ F_2{}^\mu(qr) = (p_i \circ F_1{}^\mu \cup p_i \circ F_2{}^\mu)(qr)$.

Hence, the result is proved. □

9. m-Polar Fuzzy Graphs and Their Applications

One of the important applications of m-polar fuzzy graphs is guidance in decision-making. For example, let us consider a scenario where a college wants to select a principal. Let W = {Kasim, Leo, Modi, Nandhu} be the set of doctorates shortlisted and let F = {Krishna, Srinu, Mano, Ramesh} be the set of college management members. The management has to make a selection taking into account the set of qualities Q = {management skills, attitude, patents, and research}. Each member from set F can assign a value between 0 and 1 to each element listed in the set Q for each candidate mentioned in set W, such as $W(K) = \langle 0.3, 0.4, 0.6, 0.2\rangle$, $W(L) = \langle 0.9, 0.5, 0.7, 0.3\rangle$, $W(M) = \langle 0.4, 0.7, 0.3, 0.5\rangle$, and $W(N) = \langle 0.3, 0.6, 0.7, 0.6\rangle$. Then we have a 4-polar fuzzy graph, in which each member gives his opinion to w that belongs to W on the basis of his qualities. So $W(K)$, $W(L)$, $W(M)$, and $W(N)$ denote the degree of management skills, attitude, patents, and research of each person given by the members of college management and edges denote the identical qualities of two persons (see Figure 6).

The membership degree of the edge KL indicates that Kasim is 30% suitable, Leo is 40% suitable, Modi is 60% suitable, and Nandhu is 20% suitable. Thus, the edge KL specifies that Modi is the suitable person for the post of principal. In the same way, LM specifies Leo, MN specifies Leo, and NK also specifies Leo. Hence, according to all, Leo has a higher value of desirability among the shortlisted candidates and should be appointed as the principal.

10. Conclusions

Numerous uses can be harnessed from the theory of a fuzzy graph in the areas of number theory, algebra, topology, operation research, and so on. Not only researchers but also the common man can benefit from m-polar fuzzy data. It solves the day to day problems that are faced in the society related to probabilistic data. The novel approach discussed here enables decision makers to formulate better- preferred option with the help of unique case pattern than the conventional fuzzy

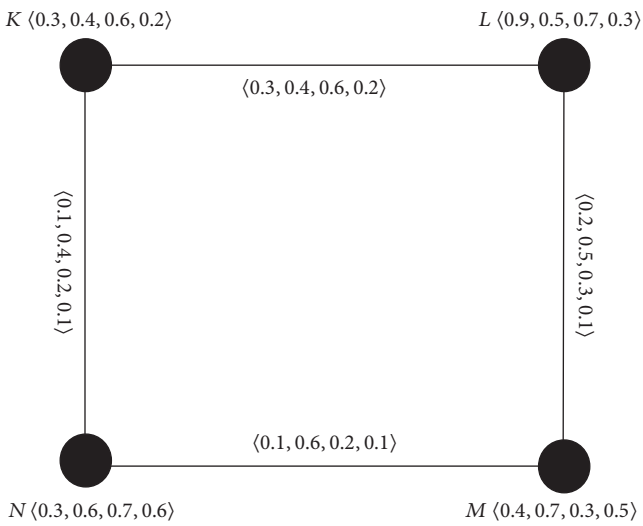

Figure 6: 4-polar fuzzy graph in decision-making.

graph approaches. The concepts complement, μ-complement of an m-polar fuzzy graph, Cartesian product, composition, tensor product, and normal product are applied to m-polar fuzzy graphs and many results have been obtained.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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