Research Article

An Application of Interval Arithmetic for Solving Fully Fuzzy Linear Systems with Trapezoidal Fuzzy Numbers

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We present a method for solving fully fuzzy linear systems using interval aspects of fuzzy numbers. This new method uses a decomposition technique to convert a fully fuzzy linear system into two types of decomposition in the form of interval matrices. It finds the solution of a fully fuzzy linear system by using interval operations. This new method uses interval arithmetic and two new interval operations $\ominus$ and $\oslash$. These new operations, which are inverses of basic interval operations $+$ and $\times$, will be presented in the middle of this paper. Some numerical examples are given to illustrate the ability of proposed methods.

1. Introduction

In various fields of sciences, in solving real-life problems, where the system of linear equations is noticed, there are situations where the values of the parameters cannot be stated exactly, but their estimation or some bounds on them can be measured. Modeling a lot of these problems leads to a fuzzy linear system because the inexact kind of real numbers can be modeled in fuzzy numbers.

Fuzzy linear systems have been studied by several authors. Friedman et al. [1] proposed a general model for solving an $n \times n$ fuzzy linear system whose coefficient matrix is crisp and the right-hand side column is an arbitrary fuzzy vector. Allahviranloo [2] proposed solution of a fuzzy linear system by using an iterative method and later suggested various numerical methods to solve fuzzy linear systems [3]. Some methods for solving these systems can be found in [4–7]. Later, a system, called fully fuzzy linear system (FFLS), is introduced wherein elements in the coefficient matrix, the right-hand side vector, and the vector of unknowns are fuzzy numbers.

This system is solved in [8, 9] using decomposition of the coefficient matrix. Some other methods used iterative methods for solving FFLS [3, 10, 11]. Many researchers

studied FFLS and numerical techniques to solve them [12–16]. Kochen et al. proposed a method for solving square and nonsquare FFLS with trapezoidal fuzzy numbers in 2016 [17]. In 2017, Edalatpanah reviewed some iterative methods for solving FFLS [18]. Recently, Allahviranloo and Babakordi introduced a new method for solving an extension of FFLS that has two coefficient matrices [19]. A new method for solving FFLS with hexagonal fuzzy numbers is introduced by Malkawi et al. in 2018 [20].

In Section 2, we present some basics about fuzzy numbers and α -cut arithmetic. Some basics of interval arithmetic, new interval operations and some properties of interval numbers, and also our method for solving FFLS are investigated in Section 3. In Section 4, some examples are solved to show the effectiveness of the proposed method and concluding remarks are contained in Section 5.

2. Preliminaries

In this section we review fuzzy numbers and α -cut arithmetic on them. α -cut method is a method for performing interval operations like addition, multiplication, and subtraction on fuzzy numbers. α -cut fuzzy arithmetic is described in detail in [21].

Let us consider an arbitrary trapezoidal fuzzy number $\mathbf{u} = (u_1, u_2, u_3, u_4)$, where $u_1, u_2, u_3, u_4 \in \mathbb{R}$ and $u_1 \leq u_2 \leq u_3 \leq u_4$. The membership function $\mu_{\mathbf{u}}$ of $\mathbf{u}$ is defined as follows:

$$\mu_{\mathbf{u}}(x) = \begin{cases} 0 & u_1 > x \\ \frac{x - u_1}{u_2 - u_1} & u_1 \leq x \leq u_2 \\ 1 & u_2 \leq x \leq u_3 \\ \frac{u_4 - x}{u_4 - u_3} & u_3 \leq x \leq u_4 \\ 0 & u_4 < x. \end{cases} \quad (1)$$

Two trapezoidal fuzzy numbers $\mathbf{u}$ and $\mathbf{v}$ are said to be equal, if and only if $u_i = v_i$ for $1 \leq i \leq 4$.

Given a trapezoidal fuzzy number $\mathbf{u}$ and membership function $\mu_{\mathbf{u}}$ and a real number $\alpha \in [0, 1]$, then the α -cut of a fuzzy number $\mathbf{u}$ is an interval, which is defined as

$$\mathbf{u}^\alpha = [\underline{\mathbf{u}}^\alpha, \overline{\mathbf{u}}^\alpha], \quad (2)$$

where $\underline{\mathbf{u}}^\alpha, \overline{\mathbf{u}}^\alpha$ are

$$\underline{\mathbf{u}}^\alpha = \inf \{x \in \mathbb{R} \mid \mu_{\mathbf{u}}(x) \geq \alpha\}, \quad (3)$$

$$\overline{\mathbf{u}}^\alpha = \sup \{x \in \mathbb{R} \mid \mu_{\mathbf{u}}(x) \geq \alpha\}. \quad (4)$$

Note that

$$\mathbf{u} = \bigcup_{\alpha \in [0,1]} \mathbf{u}^\alpha. \quad (5)$$

The basic arithmetic of two fuzzy numbers $\mathbf{u}$ and $\mathbf{v}$ is discussed in [21–23] based on interval arithmetic

$$(\mathbf{u} + \mathbf{v})^\alpha = \mathbf{u}^\alpha + \mathbf{v}^\alpha = [\underline{\mathbf{u}}^\alpha, \overline{\mathbf{u}}^\alpha] + [\underline{\mathbf{v}}^\alpha, \overline{\mathbf{v}}^\alpha], \quad (6)$$

$$(\mathbf{u} - \mathbf{v})^\alpha = \mathbf{u}^\alpha - \mathbf{v}^\alpha = [\underline{\mathbf{u}}^\alpha, \overline{\mathbf{u}}^\alpha] - [\underline{\mathbf{v}}^\alpha, \overline{\mathbf{v}}^\alpha], \quad (7)$$

$$(\mathbf{u}\mathbf{v})^\alpha = \mathbf{u}^\alpha \times \mathbf{v}^\alpha = [\underline{\mathbf{u}}^\alpha, \overline{\mathbf{u}}^\alpha] \times [\underline{\mathbf{v}}^\alpha, \overline{\mathbf{v}}^\alpha]. \quad (8)$$

Clearly, for a trapezoidal fuzzy number $\mathbf{u} = (u_1, u_2, u_3, u_4)$, we have $\mathbf{u}^0 = [u_1, u_4]$, $\mathbf{u}^1 = [u_2, u_3]$.

A matrix (vector) $\mathbf{A}$ is called a fuzzy matrix (fuzzy vector), if at least one element of matrix (vector) $\mathbf{A}$ is a fuzzy number. A matrix $\mathbf{A}$ is called a fully fuzzy matrix, whose elements of $\mathbf{A}$ are fuzzy numbers. An α -cut of an $n \times m$ matrix $\mathbf{A}$ is an interval matrix with the same dimension whose elements are α -cut of elements of matrix $\mathbf{A}$.

3. Materials and Methods

3.1. Interval Arithmetic. In this section, we briefly explain interval arithmetic and notion. Also, we present a new operation on interval arithmetic.

An *interval number* $\hat{a}$ is the bounded, closed subset of real numbers, which is indicated by a hat, defined by $\hat{a} = [\underline{a}, \overline{a}]$, where $\underline{a}, \overline{a} \in \mathbb{R}$ and $\underline{a} \leq \overline{a}$. Let $\mathbb{IR}$ denote the set of such intervals. An interval $\hat{a}$ is a *degenerate interval* when $\underline{a} = \overline{a}$.

Infimum and *supremum* of an interval number $\hat{a}$ are $\underline{a}$ and $\overline{a}$, respectively. Let $a \in \hat{a}$, say $a \in \hat{a}$ when $\underline{a} \leq a \leq \overline{a}$.

Let $\circ$ denote a binary operation on real space $\mathbb{R}$. For $\hat{a}, \hat{b} \in \mathbb{IR}$

$$\hat{a} \circ \hat{b} = \{a \circ b \mid a \in \hat{a}, b \in \hat{b}\}. \quad (9)$$

Then, for basic arithmetic operations $\{+, -, \times, /\}$ in $\mathbb{IR}$, we have

$$\hat{a} + \hat{b} = [\underline{a} + \underline{b}, \overline{a} + \overline{b}] \quad (10)$$

$$\hat{a} - \hat{b} = [\underline{a} - \overline{b}, \overline{a} - \underline{b}] \quad (11)$$

$$\hat{a} \times \hat{b}$$

$$= \begin{cases} [\underline{a}\underline{b}, \overline{a}\overline{b}] & \underline{a} \geq 0, \underline{b} \geq 0 \\ [\overline{a}\underline{b}, \overline{a}\overline{b}] & \underline{a} \geq 0, \underline{b} < 0 < \overline{b} \\ [\overline{a}\underline{b}, \underline{a}\overline{b}] & \underline{a} \geq 0, \overline{b} \leq 0 \\ [\underline{a}\underline{b}, \overline{a}\overline{b}] & \underline{a} < 0 < \overline{a}, \underline{b} \geq 0 \\ [\overline{a}\underline{b}, \underline{a}\underline{b}] & \underline{a} < 0 < \overline{a}, \overline{b} \leq 0 \\ [\underline{a}\underline{b}, \overline{a}\overline{b}] & \overline{a} \leq 0, \underline{b} \geq 0 \\ [\underline{a}\underline{b}, \underline{a}\underline{b}] & \overline{a} \geq 0, \underline{b} < 0 < \overline{b} \\ [\overline{a}\underline{b}, \underline{a}\underline{b}] & \overline{a} \leq 0, \overline{b} \leq 0 \\ [\overline{a}\underline{b}, \underline{a}\underline{b}] & \overline{a} \leq 0, \overline{b} \leq 0 \\ [\min(\overline{a}\underline{b}, \underline{a}\overline{b}), \max(\underline{a}\underline{b}, \overline{a}\overline{b})] & \underline{a} < 0 < \overline{a}, \underline{b} < 0 < \overline{b} \end{cases} \quad (12)$$

$$= [\min(\underline{a}\underline{b}, \overline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\overline{b}), \max(\underline{a}\underline{b}, \overline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\overline{b})].$$

$$\frac{\hat{a}}{\hat{b}} = \hat{a} \times \left(\frac{1}{\hat{b}}\right) = \hat{a} \times \left[\frac{1}{\overline{b}}, \frac{1}{\underline{b}}\right] \quad \text{where } 0 \notin \hat{b} \quad (13)$$

$\mathbb{IR}^+$ is the set of all *improper intervals* which is defined in [24] as follows:

$$\mathbb{IR}^+ = \{\hat{a} = [\underline{a}, \overline{a}] \mid \underline{a}, \overline{a} \in \mathbb{R}, \underline{a} \geq \overline{a}\}. \quad (14)$$

The set

$$\mathbb{IR}^* = \{\hat{a} = [\underline{a}, \overline{a}] \mid \underline{a}, \overline{a} \in \mathbb{R}\} \quad (15)$$

contains all interval numbers and improper interval numbers called *general interval*. The basic arithmetic of general intervals is defined similar to intervals, but the $\times$ operation has some differences that are explained in [25].

By an *interval matrix*, we mean a matrix whose elements are interval numbers. Let $\hat{A}$ be an interval matrix with elements $\hat{A}_{ij}$ and let B be a matrix with real elements B_{ij} ; we consider $B \in \hat{A}$ if $B_{ij} \in \hat{A}_{ij}$ for all i and j . Basic arithmetic operations on interval matrices are defined similar to real matrices [26].

3.2. New Interval Operations. Let $\hat{a}, \hat{b} \in \mathbb{IR}^*$; we define two new operations $\ominus$ and $\oslash$ as follows:

$$\hat{a} \ominus \hat{b} = [\underline{a} - \underline{b}, \bar{a} - \bar{b}], \quad (16)$$

$$\hat{a} \oslash \hat{b}$$

$$= \begin{cases} [-\infty, +\infty] & \underline{a} = \bar{a} = 0, \underline{b} = \bar{b} = 0 \\ [0, 0] & \underline{a} = \bar{a} = 0, (\underline{b} \neq 0 \text{ or } \bar{b} \neq 0) \\ \text{undefined} & (\exists x \neq 0, x \in \hat{a}), \underline{b} = \bar{b} = 0 \\ \text{undefined} & (0 \notin \hat{a}), (0 \in \hat{b}) \\ \left[\frac{\underline{a}}{\underline{b}}, \frac{\bar{a}}{\bar{b}} \right] & \underline{a} \geq 0, \underline{b} > 0 \\ \left[0, \frac{\bar{a}}{\bar{b}} \right] & \underline{a} = 0, \underline{b} = 0 \\ \left[\frac{\underline{a}}{\underline{b}}, \frac{\bar{a}}{\bar{b}} \right] & \underline{a} \leq 0 \leq \bar{a}, \underline{b} \geq 0 \\ \left[\frac{\underline{a}}{\underline{b}}, \frac{\bar{a}}{\bar{b}} \right] & \bar{a} \leq 0, \underline{b} > 0 \\ \left[\frac{\underline{a}}{\underline{b}}, 0 \right] & \bar{a} = 0, \underline{b} = 0 \\ \hat{a} \oslash (-\hat{b}) & \bar{b} \leq 0 \\ \left[\frac{\underline{a}}{\underline{b}}, \frac{\bar{a}}{\bar{b}} \right] & \underline{a} \geq 0, \underline{b} < 0 < \bar{b} \\ \left[\max \left\{ \frac{\underline{a}}{\underline{b}}, \frac{\bar{a}}{\bar{b}} \right\}, \min \left\{ \frac{\bar{a}}{\bar{b}}, \frac{\underline{a}}{\underline{b}} \right\} \right] & \underline{a} \leq 0 \leq \bar{a}, \underline{b} < 0 < \bar{b} \\ \left[\frac{\bar{a}}{\bar{b}}, \frac{\underline{a}}{\underline{b}} \right] & \bar{a} \leq 0, \underline{b} < 0 < \bar{b} \end{cases} \quad (17)$$

The operation $\ominus$ is defined and used in some articles as an operation in extended interval arithmetic [27]. But, we use $\ominus$ in a different way. We illustrate that the operation $\ominus$ can act as an inverse of $+$ in interval arithmetic.

Theorem 1. Considering $\hat{a}, \hat{b} \in \mathbb{IR}$, the interval equation $\hat{a} + \hat{x} = \hat{b}$ has a unique solution $\hat{x} = \hat{b} \ominus \hat{a}$ and $\hat{x} \in \mathbb{IR}^*$.

Proof.

$$\begin{aligned} \hat{a} + \hat{x} &= \hat{b} \implies \\ [\underline{a}, \bar{a}] + [\underline{x}, \bar{x}] &= [\underline{b}, \bar{b}] \implies \end{aligned} \quad (18)$$

$$\begin{cases} \underline{a} + \underline{x} = \underline{b} \\ \bar{a} + \bar{x} = \bar{b} \end{cases} \quad (19)$$

$$\underline{a} + \underline{x} = \underline{b} \implies \underline{x} = \underline{b} - \underline{a} \quad (I)$$

$$\bar{a} + \bar{x} = \bar{b} \implies \bar{x} = \bar{b} - \bar{a} \quad (II)$$

$$(I), (II), (16) \implies \hat{x} = \hat{b} \ominus \hat{a} \quad (19)$$

□

Corollary 2. Supposing $\hat{x} = \hat{b} \ominus \hat{a}$ and $\hat{x} \in \mathbb{IR}^*$ and $\hat{x} \notin \mathbb{IR}$, the equation $\hat{a} + \hat{x} = \hat{b}$ has no solution in $\mathbb{IR}$ but has exactly one solution in $\mathbb{IR}^*$.

Corollary 3. If $\hat{a}, \hat{b} \in \mathbb{IR}$ and $\hat{c} = \hat{a} + \hat{b}$, then $\hat{b}$ can be computed by $\hat{c} \ominus \hat{a}$ uniquely. On the other hand, the operation $\ominus$ is the inverse operation of $+$ in interval arithmetic.

Theorem 4. Considering $\hat{a}, \hat{b} \in \mathbb{IR}$, the interval equation $\hat{a} \times \hat{x} = \hat{b}$ has a solution $\hat{x} = \hat{b} \oslash \hat{a}$, where $\hat{x} \in \mathbb{IR}^*$.

Remark 5. Solution $\hat{x}$ is not usually unique; e.g., the interval equation $[-1, 1] \times \hat{x} = [-2, 2]$ has solutions $[-1, 2], [-2, 0], [-2, 2]$. Actually this equation has unlimited solutions. All intervals $[u, 2]$ and $[-2, v]$ where $-2 \leq u \leq 0$ and $0 \leq v \leq 2$ can be solution of this equation.

In cases that $\hat{x}$ is not unique, we consider $\hat{x}$ as the longest interval as possible. In the above example, $\hat{x}$ will be $[-2, 2]$.

Proof. Let $\hat{a} \times \hat{x} = \hat{b}$; Table 1 shows sign of $\hat{b}$ related to signs of $\hat{a}$ and $\hat{x}$ and also shows formulas of multiplication result.

Now, given $\hat{a}, \hat{b} \in \mathbb{IR}$ such that $\hat{a} \times \hat{x} = \hat{b}$, $\bar{a} \leq 0$ and $\underline{b} \leq 0 \leq \bar{b}$. From Table 1, it is deduced that $0 \in \hat{x}$ and $\hat{b} = [\underline{a}\underline{x}, \bar{a}\bar{x}]$. Therefore

$$\begin{aligned} \hat{a}\hat{x} &= \hat{b} \iff \\ \hat{b} &= [\underline{a}\underline{x}, \bar{a}\bar{x}] \iff \\ \begin{cases} \underline{b} = \underline{a}\underline{x} \iff \underline{x} = \frac{\underline{b}}{\underline{a}} \\ \bar{b} = \bar{a}\bar{x} \iff \bar{x} = \frac{\bar{b}}{\bar{a}} \end{cases} &\iff \\ \hat{x} &= \hat{b} \oslash \hat{a}. \end{aligned} \quad (20)$$

In this case, $\hat{x}$ is calculated uniquely by operation $\oslash$. Other cases, except cases with condition $0 \in \hat{a}$, are provable as above.

In cases $0 \leq \underline{a}$ and $0 \geq \bar{a}$ among three possible choices, we could choose the appropriate interval according to the signs of $\underline{b}$ and $\bar{b}$, while in case $0 \in \hat{a}$ all three possible choices are similar in terms of signs of $\underline{b}$ and $\bar{b}$. To conquer this problem in case $0 \in \hat{a}$, we choose the longest interval as the solution from three possibilities $0 \leq \underline{x}$, $0 \in \hat{x}$, and $0 \geq \bar{x}$ which is $0 \in \hat{x}$.

Now, let $\hat{x}$ be an interval where $0 \in \hat{x}$ and let $\hat{x}$ be the solution of $\hat{a} \times \hat{x} = \hat{b}$; then $\hat{b} = [\min\{\underline{a}\underline{x}, \bar{a}\bar{x}\}, \max\{\underline{a}\underline{x}, \bar{a}\bar{x}\}]$. It follows that

$$\begin{aligned} \underline{b} &= \min\{\underline{a}\underline{x}, \bar{a}\bar{x}\} \implies \\ \underline{b} &\leq \bar{a}\bar{x} \implies \end{aligned} \quad (21)$$

$$\underline{x} \geq \frac{\underline{b}}{\bar{a}},$$

$$\bar{b} = \max\{\underline{a}\underline{x}, \bar{a}\bar{x}\} \implies$$

$$\bar{b} \geq \underline{a}\underline{x} \implies \quad (22)$$

$$\underline{x} \geq \frac{\bar{b}}{\underline{a}}.$$

TABLE 1: Interval multiplication.

$\hat{a} \times \hat{x} = \hat{b}$	$0 \leq \underline{a}$	$0 \in \hat{a}$	$0 \geq \bar{a}$
$0 \leq \underline{x}$	$[\underline{a}\underline{x}, \bar{a}\bar{x}]$ [+, +]	$[\underline{a}\bar{x}, \bar{a}\underline{x}]$ [-, +]	$[\underline{a}\bar{x}, \bar{a}\underline{x}]$ [-, -]
$0 \in \hat{x}$	$[\underline{a}\underline{x}, \bar{a}\bar{x}]$ [-, +]	$[\min\{\bar{a}\underline{x}, \underline{a}\bar{x}\}, \max\{\underline{a}\underline{x}, \bar{a}\bar{x}\}]$ [-, +]	$[\underline{a}\bar{x}, \bar{a}\underline{x}]$ [-, +]
$0 \geq \bar{x}$	$[\underline{a}\underline{x}, \bar{a}\bar{x}]$ [-, -]	$[\underline{a}\underline{x}, \bar{a}\underline{x}]$ [-, +]	$[\bar{a}\bar{x}, \underline{a}\underline{x}]$ [+, +]

Then $\underline{x} \geq \max\{\underline{b}/\bar{a}, \bar{b}/\underline{a}\}$. If $\underline{x} \neq \max\{\underline{b}/\bar{a}, \bar{b}/\underline{a}\}$, then $\underline{x} > \max\{\underline{b}/\bar{a}, \bar{b}/\underline{a}\}$, and $\bar{a}\underline{x} > \max\{\underline{b}, \bar{a}\bar{b}/\underline{a}\} \geq \underline{b}$. This contradicts with $\underline{b} = \min\{\bar{a}\underline{x}, \underline{a}\bar{x}\}$. Hence

$$\underline{x} = \max\left\{\frac{\underline{b}}{\bar{a}}, \frac{\bar{b}}{\underline{a}}\right\}. \quad (23)$$

Similarly, $\bar{x}$ is obtained as

$$\bar{x} = \min\left\{\frac{\bar{b}}{\underline{a}}, \frac{\underline{b}}{\bar{a}}\right\}. \quad (24)$$

It is observed simply that $\hat{x}$ is equal to $\hat{b} \oslash \hat{a}$.

In addition, two special cases should be considered separately:

- (i) If $\hat{a} = [0, 0]$ and $\hat{b} \neq [0, 0]$, there is no $\hat{x}$ such that $\hat{a} \times \hat{x} = \hat{b}$.
- (ii) If $\hat{a} = \hat{b} = [0, 0]$, then $\hat{x}$ is free and expression $\hat{a} \times \hat{x} = \hat{b}$ is consistent for all $\hat{x}$, and $\hat{x}$ is considered as unbounded interval $[-\infty, +\infty]$ that is the longest possible interval satisfying $\hat{a} \times \hat{x} = \hat{b}$.

□

Corollary 6. If $\hat{x} = \hat{b} \oslash \hat{a}$, $\hat{x} \in \mathbb{IR}^*$, and $\hat{x} \notin \mathbb{IR}$, then the equation $\hat{a} \times \hat{x} = \hat{b}$ has no solution in $\mathbb{IR}$ but has solution in $\mathbb{IR}^*$.

Corollary 7. If $\hat{a}, \hat{b} \in \mathbb{IR}$ and $\hat{c} = \hat{a} \times \hat{b}$, then $\hat{b}$ can be computed by $\hat{c} \oslash \hat{a}$.

The operation $\oslash$ is the inverse operation of $\times$ in equation $\hat{a} \times \hat{x} = \hat{b}$ where $0 \notin \hat{x}$. In case $0 \in \hat{x}$ the solution $\hat{a} \times \hat{x} = \hat{b}$ is not unique, but $\hat{z} = \hat{b} \oslash \hat{a}$ is the longest interval such that $\hat{a} \times \hat{z} = \hat{b}$. In other words, any $\hat{y} \in \mathbb{IR}$ that satisfies $\hat{a} \times \hat{y} = \hat{b}$ is a subset of $\hat{z}$.

3.3. Some Properties of Interval Numbers. Interval arithmetic and some of their properties over interval numbers and matrices are discussed in [26]. We note some of them here. For any $\hat{a}, \hat{b}, \hat{c} \in \mathbb{IR}$,

$$\begin{aligned} \hat{a} + \hat{b} &= \hat{b} + \hat{a}, \\ \hat{a} + (\hat{b} \pm \hat{c}) &= (\hat{a} + \hat{b}) \pm \hat{c}, \\ \hat{a}\hat{b} &= \hat{b}\hat{a}. \end{aligned} \quad (25)$$

Lemma 8. For all $\hat{a}, \hat{b} \in \mathbb{IR}$, $(\hat{a} + \hat{b}) \ominus \hat{c} = \hat{a} + (\hat{b} \ominus \hat{c})$.

Proof.

$$\begin{aligned} (\hat{a} + \hat{b}) \ominus \hat{c} &= [(a + b) - c, (\bar{a} + \bar{b}) - \bar{c}] \\ &= [\underline{a} + (\underline{b} - \underline{c}), \bar{a} + (\bar{b} - \bar{c})] \\ &= \hat{a} + (\hat{b} \ominus \hat{c}). \end{aligned} \quad (26)$$

All these properties are correct for interval matrices as well. □

3.4. A Method for Solving a Fully Fuzzy Linear System. Consider an $n \times n$ FFLS

$$\begin{aligned} \mathbf{a}_{11}\mathbf{x}_1 + \mathbf{a}_{12}\mathbf{x}_2 + \cdots + \mathbf{a}_{1n}\mathbf{x}_n &= \mathbf{b}_1 \\ \mathbf{a}_{21}\mathbf{x}_1 + \mathbf{a}_{22}\mathbf{x}_2 + \cdots + \mathbf{a}_{2n}\mathbf{x}_n &= \mathbf{b}_2 \\ &\vdots \\ \mathbf{a}_{n1}\mathbf{x}_1 + \mathbf{a}_{n2}\mathbf{x}_2 + \cdots + \mathbf{a}_{nn}\mathbf{x}_n &= \mathbf{b}_n, \end{aligned} \quad (27)$$

which is written as

$$\mathbf{A}\mathbf{x} = \mathbf{b} \quad (28)$$

where the coefficient matrix $\mathbf{A} = (\mathbf{a}_{ij})_{n \times n}$ is the fuzzy matrix of trapezoidal fuzzy numbers, and $\mathbf{x} = (\mathbf{x}_i)_{n \times 1}$, $\mathbf{b} = (\mathbf{b}_i)_{n \times 1}$ are column vectors of trapezoidal fuzzy numbers.

Allahviranloo et al. [14] defined the following solution sets for FFLS (27):

- (i) United trapezoidal fuzzy solution set:

$$\text{UTFSS} = \{\mathbf{x} \mid \mathbf{A}^1 \mathbf{x}^1 = \mathbf{b}^1, \mathbf{A}^0 \mathbf{x}^0 = \mathbf{b}^0\}, \quad (29)$$

- (ii) Tolerable trapezoidal fuzzy solution set:

$$\text{TTFSS} = \{\mathbf{x} \mid \mathbf{A}^1 \mathbf{x}^1 = \mathbf{b}^1, \mathbf{A}^0 \mathbf{x}^0 \leq \mathbf{b}^0\}, \quad (30)$$

- (iii) Controllable trapezoidal fuzzy solution set:

$$\text{CTFSS} = \{\mathbf{x} \mid \mathbf{A}^1 \mathbf{x}^1 = \mathbf{b}^1, \mathbf{A}^0 \mathbf{x}^0 \geq \mathbf{b}^0\}. \quad (31)$$

Note that $\mathbf{A}^1 = [\mathbf{a}_{ij}^1]_{n \times n}$ and $\mathbf{A}^0 = [\mathbf{a}_{ij}^0]_{n \times n}$ are interval matrices, and $\mathbf{b}^1 = [\mathbf{b}_i^1]_{n \times 1}$, $\mathbf{b}^0 = [\mathbf{b}_i^0]_{n \times 1}$ are known right-hand side interval vectors, and $\mathbf{x}^1 = [\mathbf{x}_i^1]_{n \times 1}$ and $\mathbf{x}^0 = [\mathbf{x}_i^0]_{n \times 1}$ are unknown interval vectors.

Now, we aim to propose a practical method to obtain the suitable solution of an FFLS. The suitable solution is defined in [14] as solution $\mathbf{x} \in \text{UTFSS}$. To find such a solution $\mathbf{x} \in \text{UTFSS}$, it is sufficient to solve the following two interval linear systems:

$$\mathbf{A}^0 \mathbf{x}^0 = \mathbf{b}^0 \quad (32)$$

$$\mathbf{A}^1 \mathbf{x}^1 = \mathbf{b}^1. \quad (33)$$

Now, consider the $n \times n$ linear system as

$$\begin{aligned} \hat{a}_{11}\hat{x}_1 + \hat{a}_{12}\hat{x}_2 + \cdots + \hat{a}_{1n}\hat{x}_n &= \hat{b}_1 \\ \hat{a}_{21}\hat{x}_1 + \hat{a}_{22}\hat{x}_2 + \cdots + \hat{a}_{2n}\hat{x}_n &= \hat{b}_2 \\ &\vdots \\ \hat{a}_{n1}\hat{x}_1 + \hat{a}_{n2}\hat{x}_2 + \cdots + \hat{a}_{nn}\hat{x}_n &= \hat{b}_n \end{aligned} \quad (34)$$

where $\hat{a}_{ij}, \hat{b}_i \in \mathbb{IR}$ for all $(1 \leq i, j \leq n)$ and $\hat{x}_i$ are unknown intervals. The interval linear system is represented as

$$\hat{A}\hat{x} = \hat{b} \quad (35)$$

Note that two systems (32) and (33) are in the form (34). Using operations $\ominus$ and $\oslash$, we can propose a new method to decompose an interval matrix $\hat{A}$ to two interval matrices, similar to LU decomposition method [28]. Then, one can solve systems (32) and (33) using this decomposition method.

Assume that $\hat{A} \in \mathbb{IR}^{n \times n}$, and there exist two interval matrices $\hat{L}, \hat{U} \in \mathbb{IR}^{n \times n}$ such that $\hat{A} = \hat{L} \times \hat{U}$, where $\hat{L}$ and $\hat{U}$ are lower triangular and upper triangular matrices, respectively. Then using this decomposition method, we can solve the interval linear system (34).

Consider the system

$$\hat{A} = \hat{L} \times \hat{U}, \quad (36)$$

where $\hat{A}_{n \times n}$ is a known interval matrix, $\hat{L}_{n \times n}$ is an unknown lower triangular interval matrix, and $\hat{U}_{n \times n}$ is an unknown upper triangular interval matrix. In this system, there exist n^2 linear interval equations, and there are $n^2 + n$ unknown variables. This system extends to $n^2 + n$ linear equations with $n^2 + n$ unknowns by adding n linear equations $\hat{e}_{ii} = 1$ for $1 \leq i \leq n$.

To solve this linear system, first, we set all diagonal elements of $\hat{L}$ with $[1, 1]$. Then, using reversing operations, all elements of $\hat{L}$ and $\hat{U}$ can be computed by solving linear interval equations $\hat{a}_{ij} = \sum_{k=1}^n (\hat{e}_{ik} \times \hat{u}_{kj})$. To solve such linear equations, we need to consider two cases:

Case 1 ($i \leq j$).

$$\hat{a}_{ij} = \sum_{k=1}^{i-1} (\hat{e}_{ik} \times \hat{u}_{kj}) + \hat{e}_{ii} \times \hat{u}_{ij} = \sum_{k=1}^{i-1} (\hat{e}_{ik} \times \hat{u}_{kj}) + \hat{u}_{ij} \quad (37)$$

which results in

$$\begin{aligned} \hat{a}_{ij} &= \sum_{k=1}^{i-1} (\hat{e}_{ik} \times \hat{u}_{kj}) + \hat{u}_{ij} \implies \\ \hat{u}_{ij} &= \hat{a}_{ij} \ominus \sum_{k=1}^{i-1} (\hat{e}_{ik} \times \hat{u}_{kj}). \end{aligned} \quad (38)$$

Case 2 ($i > j$).

$$\hat{a}_{ij} = \sum_{k=1}^{j-1} (\hat{e}_{ik} \times \hat{u}_{kj}) + \hat{e}_{ij} \times \hat{u}_{jj} \quad (39)$$

which results in

$$\begin{aligned} \hat{a}_{ij} &= \sum_{k=1}^{j-1} (\hat{e}_{ik} \times \hat{u}_{kj}) + \hat{e}_{ij} \times \hat{u}_{jj} \implies \\ \hat{e}_{ij} \times \hat{u}_{jj} &= \hat{a}_{ij} \ominus \sum_{k=1}^{j-1} (\hat{e}_{ik} \times \hat{u}_{kj}) \implies \\ \hat{e}_{ij} &= \left(\hat{a}_{ij} \ominus \sum_{k=1}^{j-1} (\hat{e}_{ik} \times \hat{u}_{kj}) \right) \oslash \hat{u}_{jj}. \end{aligned} \quad (40)$$

Using these results, we can compute interval LU decomposition for an interval matrix. The Algorithm ILU.Decomposition computes such interval LU decomposition and is given in Algorithm 1.

If at least one of $\hat{u}_{ii}$ or $\hat{u}_{ii}$ for $0 \leq i \leq n-1$ is $[0, 0]$, then a dividing-by-zero error occurs, and decomposition is not possible. Otherwise, there exist $\hat{L}, \hat{U} \in \mathbb{IR}^{n \times n}$ such that $\hat{A} = \hat{L}\hat{U}$.

Example 9. Consider the following 3×3 interval matrix:

$$\hat{A} = \begin{bmatrix} [1, 4] & [0.5, 3] & [-3, 2] \\ [0.5, 3] & [0.5, 4.5] & [-3, 3] \\ [-3, 2] & [-3, 3] & [-2, 4.25] \end{bmatrix}. \quad (41)$$

Initially, assign $[1, 1]$ to $\hat{e}_{11}, \hat{e}_{22}, \hat{e}_{33}$, and initialize all other elements of $\hat{L}$ and $\hat{U}$ to $[0, 0]$. Then, compute the rows of $\hat{U}$ and columns of $\hat{L}$ by annotating Algorithm 1. Compute the rows and columns index of $\hat{U}$ and $\hat{L}$, following the first loop (lines (4) to (14)).

In case $r = 1$,

$$\begin{aligned} \hat{L} &= \begin{bmatrix} [1, 1] & [0, 0] & [0, 0] \\ [0.5, 0.75] & [1, 1] & [0, 0] \\ [-0.75, 0.5] & [0, 0] & [1, 1] \end{bmatrix}, \\ \hat{U} &= \begin{bmatrix} [1, 4] & [0.5, 3] & [-3, 2] \\ [0, 0] & [0, 0] & [0, 0] \\ [0, 0] & [0, 0] & [0, 0] \end{bmatrix}. \end{aligned} \quad (42)$$

```

(1) function ILU-Decomposition ( $\widehat{A}, \widehat{L}, \widehat{U}, n$ )
  (i) Input:  $\widehat{A} \in \mathbb{R}^{n \times n}$ 
  (ii) Output:  $\widehat{L}, \widehat{U} \in \mathbb{R}^{n \times n}$  such that  $\widehat{A} = \widehat{L}\widehat{U}$ 
(2)   for  $i = 1$  to  $n$  do
(3)      $\widehat{\ell}_{ii} \leftarrow [1, 1]$ 
(4)   for  $r = 1$  to  $n$  do
(5)      $i \leftarrow r$ 
(6)     for  $j = i$  to  $n$  do
(7)        $sum \leftarrow \sum_{k=1}^{j-1} (\widehat{\ell}_{ik} \times \widehat{u}_{kj})$ 
(8)        $h \leftarrow \widehat{a}_{ij} \ominus sum$ 
(9)        $\widehat{u}_{ij} \leftarrow h$ 
(10)     $j \leftarrow r$ 
(11)    for  $i = j + 1$  to  $n$  do
(12)       $sum \leftarrow \sum_{k=1}^{i-1} (\widehat{\ell}_{ik} \times \widehat{u}_{kj})$ 
(13)       $\widehat{u}_{ij} \leftarrow \widehat{a}_{ij} \ominus sum$ 
(14)       $\widehat{\ell}_{ij} \leftarrow sum \oslash \widehat{u}_{jj}$ 
  return  $\widehat{L}$  and  $\widehat{U}$ 
  If  $L \notin \mathbb{R}^{n \times n}$  or  $U \notin \mathbb{R}^{n \times n}$  there is no decomposition in
   $\mathbb{R}^{n \times n}$  but has decomposition in  $(\mathbb{R}^*)^{n \times n}$ .

```

ALGORITHM 1: Computing the interval LU decomposition.

In case $r = 2$,

$$\widehat{L} = \begin{bmatrix} [1, 1] & [0, 0] & [0, 0] \\ [0.5, 0.75] & [1, 1] & [0, 0] \\ [-0.75, 0.5] & [-0.333, 0.666] & [1, 1] \end{bmatrix},$$

$$\widehat{U} = \begin{bmatrix} [1, 4] & [0.5, 3] & [-3, 2] \\ [0, 0] & [0.25, 2.25] & [-0.75, 1.5] \\ [0, 0] & [0, 0] & [0, 0] \end{bmatrix}.$$

(43) To solve $\widehat{A}\widehat{x} = \widehat{b}$ we do the following steps:

- (1) Decompose $\widehat{A}$ into two matrices $\widehat{L}$ and $\widehat{U}$.
- (2) Solve $\widehat{L}\widehat{y} = \widehat{b}$ for $\widehat{y}$.
- (3) Solve $\widehat{U}\widehat{x} = \widehat{y}$ for $\widehat{x}$.

In case $r = 3$,

$$\widehat{L} = \begin{bmatrix} [1, 1] & [0, 0] & [0, 0] \\ [0.5, 0.75] & [1, 1] & [0, 0] \\ [-0.75, 0.5] & [-0.333, 0.666] & [1, 1] \end{bmatrix},$$

$$\widehat{U} = \begin{bmatrix} [1, 4] & [0.5, 3] & [-3, 2] \\ [0, 0] & [0.25, 2.25] & [-0.75, 1.5] \\ [0, 0] & [0, 0] & [0, 1] \end{bmatrix}.$$

(44)

Step (1) can be performed by Algorithm 1 and step (2) can be performed as follows:

$$\begin{bmatrix} [1, 1] & 0 & 0 & \dots & 0 \\ \widehat{\ell}_{21} & [1, 1] & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ \widehat{\ell}_{(n-1)1} & \widehat{\ell}_{(n-1)2} & \dots & [1, 1] & 0 \\ \widehat{\ell}_{n1} & \widehat{\ell}_{n2} & \dots & \dots & [1, 1] \end{bmatrix} \times \begin{bmatrix} \widehat{y}_1 \\ \widehat{y}_2 \\ \vdots \\ \widehat{y}_{n-1} \\ \widehat{y}_n \end{bmatrix}$$

It is observed simply that $\widehat{A} = \widehat{L} \times \widehat{U}$.

Now, to solve $\widehat{A}\widehat{x} = \widehat{b}$ for $\widehat{x}$ with $\widehat{A} = \widehat{L}\widehat{U}$, we have

$$\begin{aligned} \widehat{A}\widehat{x} &= \widehat{b} \implies \\ \widehat{L}\widehat{U}\widehat{x} &= \widehat{b} \implies \\ \widehat{L}\widehat{y} &= \widehat{b} \end{aligned} \quad (45)$$

$$= \begin{bmatrix} \widehat{b}_1 \\ \widehat{b}_2 \\ \vdots \\ \widehat{b}_{n-1} \\ \widehat{b}_n \end{bmatrix} \implies$$

```

(1) function ILU_Solve ( $\widehat{A}, \widehat{b}, n, \widehat{x}$ )
    (i) Input:  $\widehat{A} \in \mathbb{R}^{n \times n}, \widehat{b} \in \mathbb{R}^n$ 
    (ii) Output:  $\widehat{x} \in \mathbb{R}^n : \widehat{A}\widehat{x} = \widehat{b}$ 
(2)   call ILU_Decomposition( $\widehat{A}, \widehat{L}, \widehat{U}, n$ )
(3)   for  $i = 1$  to  $n$  do
(4)      $\widehat{y}_i = \widehat{b}_i \ominus \sum_{j=1}^{i-1} (\widehat{\ell}_{ij} \times \widehat{y}_j)$ 
(5)   for  $i = n$  to  $1$  do
(6)      $\widehat{x}_i = \left( \widehat{y}_i \ominus \sum_{j=i+1}^n (\widehat{u}_{ij} \times \widehat{y}_j) \right) \oslash \widehat{u}_{ii}$ 
return  $\widehat{x}$ 

```

ALGORITHM 2: Solving linear system of intervals using interval LU decomposition.

$$\widehat{b}_i = \widehat{y}_i + \sum_{j=1}^{i-1} (\widehat{\ell}_{ij} \times \widehat{y}_j) \Rightarrow$$

$$\widehat{y}_i = \widehat{b}_i \ominus \sum_{j=1}^{i-1} (\widehat{\ell}_{ij} \times \widehat{y}_j),$$

and finally, step (3) can be performed as

$$\begin{bmatrix} \widehat{u}_{11} & \widehat{u}_{12} & \widehat{u}_{13} & \dots & \widehat{u}_{1n} \\ 0 & \widehat{u}_{22} & \widehat{u}_{23} & \dots & \widehat{u}_{2n} \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \dots & \widehat{u}_{(n-1)(n-1)} & \widehat{u}_{(n-1)n} \\ 0 & 0 & \dots & 0 & \widehat{u}_{nn} \end{bmatrix} \times \begin{bmatrix} \widehat{x}_1 \\ \widehat{x}_2 \\ \vdots \\ \widehat{x}_{n-1} \\ \widehat{x}_n \end{bmatrix} = \begin{bmatrix} \widehat{y}_1 \\ \widehat{y}_2 \\ \vdots \\ \widehat{y}_{n-1} \\ \widehat{y}_n \end{bmatrix} \Rightarrow$$

$$\widehat{x}_i = \widehat{u}_{ii} \times \widehat{x}_i + \sum_{j=i+1}^n (\widehat{u}_{ij} \times \widehat{y}_j) \Rightarrow$$

$$\widehat{x}_i = \left(\widehat{y}_i \ominus \sum_{j=i+1}^n (\widehat{u}_{ij} \times \widehat{y}_j) \right) \oslash \widehat{u}_{ii}.$$

Algorithm 2 shows our approach to solve a system of linear interval equation $\widehat{A}\widehat{x} = \widehat{b}$.

Example 10. Considering the interval matrix $\widehat{A}$ in Example 9 and the following $\widehat{b}$, we aim to solve $\widehat{A}\widehat{x} = \widehat{b}$ for $\widehat{x}$.

$$\widehat{b} = \begin{bmatrix} [-50.00, 47.50] \\ [-52.50, 48.75] \\ [-47.50, 51.25] \end{bmatrix}. \quad (49)$$

In Example 9, ILU decomposition of $\widehat{A}$ is obtained. So, we need to solve $\widehat{L}\widehat{y} = \widehat{b}$ and $\widehat{U}\widehat{x} = \widehat{y}$. By solving $\widehat{L}\widehat{y} = \widehat{b}$ and $\widehat{U}\widehat{x} = \widehat{y}$, $\widehat{y}$ and $\widehat{x}$ are obtained, respectively, as follows:

$$\widehat{y} = \begin{bmatrix} [-50, 47.5] \\ [-15, 13.1251] \\ [-1.8750, 5] \end{bmatrix} \quad (50)$$

$$\text{and } \widehat{x} = \begin{bmatrix} [-5.00, 7.50] \\ [-5.00, 2.50] \\ [-1.875, 5.00] \end{bmatrix},$$

4. Numerical Results

Example 11. Consider the following FFLS which is solved in [8, 9, 29]:

$$\begin{bmatrix} (18, 19, 20) & (10.5, 12, 13.5) & (5.5, 6, 6.2) \\ (1.9, 2, 2.1) & (3.9, 4, 4.4) & (1.3, 1.5, 1.7) \\ (1.9, 2, 2.2) & (1.9, 2, 2.3) & (4.4, 4.5, 4.6) \end{bmatrix} \begin{bmatrix} (\mathbf{x}_{11}, \mathbf{x}_{12}, \mathbf{x}_{13}) \\ (\mathbf{x}_{21}, \mathbf{x}_{22}, \mathbf{x}_{23}) \\ (\mathbf{x}_{31}, \mathbf{x}_{32}, \mathbf{x}_{33}) \end{bmatrix} = \begin{bmatrix} (1469.3, 1897, 2433.2) \\ (358, 434, 543) \\ (447.2, 535.5, 667.4) \end{bmatrix} \quad (51)$$

First, we solve 1-cut system which is obtained as follows:

$$\begin{bmatrix} 19 & 12 & 6 \\ 2 & 4 & 1.5 \\ 2 & 2 & 4.5 \end{bmatrix} \begin{bmatrix} \mathbf{x}_{12} \\ \mathbf{x}_{22} \\ \mathbf{x}_{32} \end{bmatrix} = \begin{bmatrix} 1897 \\ 434 \\ 535.5 \end{bmatrix}. \quad (52)$$

The 1-cut crisp solution will be $(37, 62, 75)^T$.

Now, we solve 0-cut system which is obtained as follows:

$$\begin{bmatrix} [18, 20] & [10.5, 13.5] & [5.5, 6.2] \\ [1.9, 2.1] & [3.9, 4.4] & [1.3, 1.7] \\ [1.9, 2.2] & [1.9, 2.3] & [4.4, 4.6] \end{bmatrix} \begin{bmatrix} [\mathbf{x}_{11}, \mathbf{x}_{13}] \\ [\mathbf{x}_{21}, \mathbf{x}_{23}] \\ [\mathbf{x}_{31}, \mathbf{x}_{33}] \end{bmatrix} = \begin{bmatrix} (1469.3, 2433.2) \\ (358, 543) \\ (447., 667.4) \end{bmatrix}. \quad (53)$$

The left-hand side matrix decomposed into the following two matrices:

$$\widehat{L} = \begin{bmatrix} [1, 1] & [0, 0] & [0, 0] \\ [0.1050, 0.1056] & [1, 1] & [0, 0] \\ [0.1055, 0.1101] & [0.2732, 0.2836] & [1, 1] \end{bmatrix}, \quad (54)$$

$$\widehat{U} = \begin{bmatrix} [18, 20] & [10.5, 13.5] & [5.5, 6.2] \\ [0, 0] & [2.7916, 2.9826] & [0.7194, 1.0490] \\ [0, 0] & [0, 0] & [3.6154, 3.6314] \end{bmatrix}$$

We solve equations $\widehat{L}\widehat{y} = \widehat{b}$ and $\widehat{U}\widehat{x} = \widehat{y}$ and the solutions are

$$\widehat{y} = \begin{bmatrix} [1469.2, 2433.3] \\ [203.2, 288.4] \\ [234.4, 321] \end{bmatrix}, \quad (55)$$

$$\widehat{x} = \begin{bmatrix} [29.0993, 49.9928] \\ [56.0765, 65.5814] \\ [64.8558, 88.3868] \end{bmatrix}.$$

Finally, solution of System (53) is obtained as

$$\mathbf{x} = \begin{bmatrix} (29.0993, 37, 49.9928) \\ (56.0765, 62, 65.5814) \\ (64.8558, 75, 88.3868) \end{bmatrix} \quad (56)$$

Example 12. Consider a 3×3 FFLS:

$$\mathbf{Ax} = \mathbf{b} \quad (57)$$

where

$$\mathbf{A} = \begin{bmatrix} (-7.2, -1.9, -1.9, 5.4) & (-4.4, -2.9, 1, 5) & (-0.7, 0, 0.2, 1.7) & (-0.2, 0, 0.5, 1) \\ (-26, 6.7, 8.4, 34.7) & (-23, -3, 15.9, 24.3) & (-9, -1.1, 0.4, 3.9) & (-6.7, -3.5, 0, 1.1) \\ (-17.4, 3.5, 4.1, 23.2) & (-23, -5.9, 5, 17.3) & (-9, -1.7, 0.7, 5.5) & (-7, -2, 2, 6.2) \\ (-24, -4.6, 0, 18) & (-15, -6.8, 5.6, 25) & (-8, -0.5, 1.4, 12.7) & (-11, -2, 1, 8.4) \end{bmatrix}, \quad (58)$$

$$\mathbf{b} = \begin{bmatrix} (-13.9, -2.6, 1.3, 12.3) \\ (-63.7, -7.4, 13.0, 70.6) \\ (-51, -4.9, 7.4, 54.8) \\ (-67.8, -7.9, 4.3, 58.6) \end{bmatrix}. \quad (59)$$

This system is converted to two interval systems $\mathbf{A}^0 \mathbf{x}^0 = \mathbf{b}^0$ and $\mathbf{A}^1 \mathbf{x}^1 = \mathbf{b}^1$ as follows:

$$\begin{bmatrix} [-7.2, 5.4] & [-4.4, 5] & [-0.7, 1.7] & [-0.2, 1] \\ [-26, 34.7] & [-23, 24.3] & [-9, 3.9] & [-6.7, 1.1] \\ [-17.4, 23.2] & [-23, 17.3] & [-9, 5.5] & [-7, 6.2] \\ [-24, 18] & [-15, 25] & [-8, 12.7] & [-11, 8.4] \end{bmatrix} \mathbf{x}^0 \quad (60)$$

$$= \begin{bmatrix} [-13.9, 12.3] \\ [-63.7, 70.6] \\ [-51.0, 54.8] \\ [-67.8, 58.6] \end{bmatrix}.$$

$$\begin{bmatrix} [-1.9, -1.9] & [-2.9, 1] & [0, 0.2] & [0, 0.5] \\ [6.7, 8.4] & [-3, 15.9] & [-1.1, 0.4] & [-3.5, 0] \\ [3.5, 4.1] & [-5.9, 5] & [-1.7, 0.7] & [-2, 2] \\ [-4.6, 0] & [-6.8, 5.6] & [-0.5, 1.4] & [-2, 1] \end{bmatrix} \mathbf{x}^0 \quad (61)$$

$$= \begin{bmatrix} [-2.6, 1.3] \\ [-7.4, 13.0] \\ [-4.9, 7.4] \\ [-7.9, 4.3] \end{bmatrix}.$$

Interval matrix $\mathbf{A}^0$ is decomposed to $\widehat{L}_0, \widehat{U}_0$:

$$\widehat{L}_0 = \begin{bmatrix} [1, 1] & [0, 0] & [0, 0] & [0, 0] \\ [-4.8184, -0.6484] & [1, 1] & [0, 0] & [0, 0] \\ [-3.2230, 0.7073] & [-2.2224, 1.0102] & [1, 1] & [0, 0] \\ [-1.2122, 3.3335] & [-0.1101, 2.7049] & [-2.4038, 0.8945] & [1, 1] \end{bmatrix}, \quad (62)$$

$$\widehat{U}_0 = \begin{bmatrix} [-7.2, 5.4] & [-4.4, 5] & [-0.7, 1.7] & [-0.2, 1] \\ [0, 0] & [1.0885, 3.0808] & [-0.8088, 0.5272] & [-1.8817, 0.1364] \\ [0, 0] & [0, 0] & [-2.3497, 1.4468] & [-1.8764, 1.3112] \\ [0, 0] & [0, 0] & [0, 0] & [-1.5464, 0.1876] \end{bmatrix}.$$

Algorithm 2 solves $\widehat{L}_0 \widehat{U}_0 \mathbf{x}^0 = \mathbf{b}^0$ and obtains

$$\mathbf{x}^0 = \begin{bmatrix} [-0.4024, 1.5463] \\ [-0.0229, 0.0851] \\ [-0.4682, 0.8773] \\ [-1.5900, 2.0357] \end{bmatrix}. \quad (63)$$

Similarly, interval matrix $\mathbf{A}^1$ is decomposed to $\widehat{L}_1, \widehat{U}_1$ as follows:

$$\widehat{L}_1 = \begin{bmatrix} [1, 1] & [0, 0] & [0, 0] & [0, 0] \\ [-4.4256, -3.5253] & [1, 1] & [0, 0] & [0, 0] \\ [-2.1571, -1.8455] & [-1.2222, -0.8880] & [1, 1] & [0, 0] \\ [0, 2.4261] & [0.1666, 1.0327] & [-0.6347, 0.1303] & [1, 1] \end{bmatrix}, \quad (64)$$

$$\widehat{U}_1 = \begin{bmatrix} [-1.9001, -1.9] & [-2.9, 1] & [0, 0.2001] & [0, 0.5] \\ [0, 0] & [1.4142, 3.0642] & [-0.1799, 0.4037] & [-1.2872, 0] \\ [0, 0] & [0, 0] & [-0.7857, 0.4715] & [-0.9224, 0.4344] \\ [0, 0] & [0, 0] & [0, 0] & [-0.3958, -0.7927] \end{bmatrix}.$$

Then, $\mathbf{x}^1$ is obtained from $\widehat{L}_1 \widehat{U}_1 \mathbf{x}^1 = \mathbf{b}^1$ by Algorithm 2 as

$$\mathbf{x}^1 = \begin{bmatrix} [-0.2608, 1.0021] \\ [-0.0145, 0.0538] \\ [-0.3374, 0.6323] \\ [-0.9569, 1.2251] \end{bmatrix}. \quad (65)$$

Finally, $\mathbf{x}$ is found as

$$\begin{bmatrix} [-0.4024, -0.2608, 1.0021, 1.5463] \\ [-0.0229, -0.0145, 0.0538, 0.0851] \\ [-0.4682, -0.3374, 0.6323, 0.8773] \\ [-1.5900, -0.9569, 1.2251, 2.0357] \end{bmatrix}. \quad (66)$$

5. Conclusion

In the present paper, we proposed a new method for solving the fully fuzzy linear system with trapezoidal fuzzy numbers. This problem is converted into two interval linear systems; then by introducing a new interval method based on decomposition, we solved these systems. Our method is using two inverse operations $\ominus$ and $\oslash$ which are presented in this paper for the first time. Numerical results demonstrated the efficiency of our new method.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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