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Research Article

Solidarity Value and Solidarity Share Functions for TU Fuzzy Games

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TU games under both crisp and fuzzy environments describe situations where players make full (crisp) or partial (fuzzy) binding agreements and generate worth in return. The challenge is then to decide how to distribute the profit among them in a rational manner: we call this a solution. In this paper, we introduce the notion of solidarity value and the solidarity share function as a suitable solution to TU fuzzy games. Two special classes of TU fuzzy games, namely, TU fuzzy games in Choquet integral form and in multilinear extension form, are studied and the corresponding solidarity value and the solidarity share functions are characterized.

1. Introduction

A cooperative game with transferable utility, or simply a TU game, is a pair (N, v), where N is a set of n players, called the grand coalition, and v is the characteristic function defined on 2^N that assigns to every subset (coalition) a real number called its worth which gives zero worth to the empty coalition. Let G_0 denote the class of TU games. A solution for any TU game is a function on G_0 which assigns to the TU game a distribution of payoffs for its players. If there is no ambiguity on the player set N, we denote by $G_0(N)$ the class of all TU games with the fixed N.

Among the various one-point solutions for TU games, the Shapley value [1] and the solidarity value [2] are perhaps the most popular ones. The Shapley value builds on the axioms of efficiency, linearity, anonymity, and the null player. The solidarity value on the other hand is characterized by efficiency, linearity, anonymity, and the axiom of v-null player. The null player axiom of the Shapley value rewards nothing to the nonperforming players. However in recent years solidarity has been considered as an important human attribute influencing both rationality (limited rationality) and social preference for fairness [3–5]. Therefore the role of solidarity in TU games is essentially discussed in the literature and the notion of the solidarity value was proposed as an alternative to the Shapley value. It follows that, unlike the Shapley value, the solidarity value expresses solidarity to both the nonperforming and performing players; see [2].

As an alternative to the values, the share functions are proposed in [6] as useful solution concepts for TU games that assign to every game a vector whose components add up to one. A share function determines how much share a player can get from the worth of the grand coalition and therefore is devoid of the efficiency requirement as opposed to the other standard value functions. Therefore a share function simplifies the model formulation to a great extent. In [7], it is shown that, on a ratio scale, meaningful statements can be made for a certain class of share functions, whereas all statements with respect to the value functions are meaningless. The share function corresponding to the solidarity value is called the solidarity share function. It is obtained by dividing the solidarity value of each player by the sum of the solidarity values of all the players. In [8] the solidarity share function for TU games is studied in detail.

Cooperative games with fuzzy coalitions or simply TU fuzzy games are a generalization of the ordinary TU games

in the sense that participation of the players in a fuzzy coalition belongs to the interval $[0,1]$; see [9]. A fuzzy coalition is a fuzzy subset of the player set N which assigns a membership grade to its members. This membership of a player in a coalition represents her rate of participation in it. When distinction between the two classes of games is needed, we call the standard TU game the crisp TU game or simply the TU game. TU fuzzy games derived from their crisp counterparts are found in the literature; see, for example, [9–15]. The Shapley share function for TU fuzzy games is studied in [16]. The relevance of the solidarity value and the corresponding share function for TU fuzzy games can be realized in situations where players with partial participations are marginally unproductive but being the part of the cooperative endeavor may be rewarded with some nonzero payoffs. In this paper, we introduce the notion of solidarity value and solidarity share functions for TU fuzzy games. A set of axioms to characterize these functions is proposed. We define two classes of TU fuzzy games, namely, the TU fuzzy games in Choquet integral form due to [15] and TU fuzzy games in multilinear extension form due to [17]. These two classes are continuous with respect to the standard metric and also monotonic when the associated crisp game is monotonic. Moreover they build on the idea of nonadditive interactions among the players; for more details we refer to [15–17].

The rest of the paper proceeds as follows. In Section 2, we compile the related definitions and results from the existing literature. Section 3 discusses the solidarity share functions for TU fuzzy games. In Section 4 we discuss the solidarity share functions for TU fuzzy games in Choquet integral form followed by some illustrative example. Section 5 concludes the paper.

2. Preliminaries

In this section we compile the definitions and results necessary for the development of the present study from [2, 6, 8, 9, 11, 12, 15, 16, 18]. We start with the notion of solidarity values and share functions in crisp games.

2.1. The Solidarity Value and the Share Function for TU Games. Let the player set N be fixed so that the class of TU games can be taken as $G_0(N)$. We define the following.

Definition 1. Let $\emptyset \neq T \in 2^N$ and $v \in G_0(N)$; the quantity $v^v(T) = (1/|T|)\sum_{k\in T}[v(T) - v(T\backslash k)]$ is called the average marginal contribution of a player of the coalition T.

Definition 2. Given a game $v \in G_0(N)$, player $i \in N$ is called a v-null player if $v^v(T) = 0$, for every coalition $T \subseteq N$ containing i.

Consider a function $\Phi : G_0(N) \to (\mathbb{R}^n_+)^{2^N}$ that assigns to any game $v \in G_0(N)$ a $2^N \to R^n_+$ mapping. For any fixed $v \in G_0(N)$ and set $W \subseteq 2^N$, we denote the corresponding n-ary vector in R^n_+ as $(\Phi_1(W,v),\ldots,\Phi_n(W,v))$. We define the solidarity value as follows.

Definition 3. A function $\Phi : G_0(N) \to (\mathbb{R}^n_+)^{2^N}$ is said to be the solidarity value on $G_0(N)$ if it satisfies the following four axioms.

Axiom C_1 (Efficiency). If $v \in G_0(N)$ and $W \in 2^N$, then

$$\sum_{i\in W}\Phi_i(W,v) = v(W),$$
$$\Phi_i(W,v) = 0 \quad \forall i \notin W. \tag{1}$$

Axiom C_2 (v-Null Player). If $v \in G_0(N)$ and $i \in W \in 2^N$ are a v-null player, then

$$\Phi_i(W,v) = 0 \quad \forall i \in T \subset W. \tag{2}$$

Axiom C_3 (Symmetry). If $v \in G_0(N)$, $W \in 2^N$, and $i, j \in W$ are symmetric, that is, $v(S\cup i) = v(S\cup j)$ holds for any $S \in 2^{W\backslash\{i,j\}}$, then

$$\Phi_i(W,v) = \Phi_j(W,v). \tag{3}$$

Axiom C_4 (Additivity). For $v_1, v_2 \in G_0(N)$, define $v_1 + v_2 \in G_0(N)$ by $(v_1+v_2)(S) = v_1(S) + v_2(S)$ for each $S \in 2^N$. If $v_1, v_2 \in G_0(N)$ and $W \in 2^N$, then

$$\Phi_i(W, v_1+v_2) = \Phi_i(W,v_1) + \Phi_i(W,v_2). \tag{4}$$

Theorem 4. *Define a function $\Phi : G_0(N) \to (\mathbb{R}^n_+)^{2^N}$ by*

$$\Phi_i(W,v) = \begin{cases} \sum_{S\in P_i(W)} \delta(|S|,|W|)\, v^v(S), & \text{if } i \in W \\ 0, & \text{otherwise,} \end{cases} \tag{5}$$

where $P_i(W) = \{S \in 2^W \mid S \not\ni i\}$ and $\delta(|S|,|W|) = (|S|-1)!(|W|-|S|)!/|W|!$. Then the function Φ is the unique solidarity value on $G_0(N)$.

Proof. We refer to [8] for a detailed proof of Theorem 4. □

From now onward we denote the solidarity value by $\Phi^{\text{sol}}(W,v)$ where $W \subseteq N$ and $v \in G_0(N)$.

Definition 5. Let $C \subseteq G_0(N)$ be a set of TU games, and let $\mu : C \to \mathbb{R}$ be a given function. A μ-share function on a set of games $C \subseteq G_0(N)$ is a function $\Psi^\mu : C \to (\mathbb{R}^n_+)^{2^N}$ that satisfies the following Axioms CS_1, CS_2, and CS_3 and either Axiom CS_4 or CS_5:

Axiom CS_1 (μ-Efficiency). If $v \in C$ and $K \in 2^N$, then

$$\sum_{i\in K}\Psi^\mu_i(K,v) = 1,$$
$$\Psi^\mu_i(K,v) = 0, \tag{6}$$
$$i \notin K.$$

Axiom CS_2 (μ-Symmetry). If $v \in C$ and $K \in 2^N$, $i, j \in K$, and $v(S \cup \{i\}) = v(S \cup \{j\})$ hold for any $S \subseteq K \backslash \{i, j\}$, then $\Psi_i^\mu(K, v) = \Psi_j^\mu(K, v)$.

Axiom CS_3 (μv-Null Player). If $v \in C$ and $i \in K \in 2^N$ is an v-null player, that is, $v^v(T) = 0$, then

$$\Psi_i^\mu(K, v) = 0 \quad \forall i \in T \subset K. \tag{7}$$

Axiom CS_4 (μ-Additivity). For any pair $v_1, v_2 \in C$ such that $v_1 + v_2 \in C$, it holds that $\mu(K, v_1 + v_2)\Psi_i^\mu(K, v_1 + v_2) = \mu(K, v_1)\Psi_i^\mu(K, v_1) + \mu(K, v_2)\Psi_i^\mu(K, v_2)$.

Axiom CS_5 (μ-Linearity). For any pair v_1, v_2 of games in C and for any pair of real numbers a and b such that $av_1 + bv_2 \in C$, it holds that

$$\begin{aligned} &\mu(K, av_1 + bv_2)\Psi_i^\mu(K, av_1 + bv_2) \\ &\quad = a\mu(K, v_1)\Psi_i^\mu(K, v_1) + b\mu(K, v_2)\Psi_i^\mu(K, v_2). \end{aligned} \tag{8}$$

Theorem 6. *Let $\mu : C \to \mathbb{R}$ be a positive function on C. Then on the subclass C there exists a unique solidarity μ-share function $\Psi^\mu : C \to (\mathbb{R}_+{}^n)^{2^N}$ satisfying the axioms CS_1–CS_4 if and only if μ is additive on C.*

2.2. TU Games with Fuzzy Coalitions. Now we make a brief discourse of TU games with fuzzy coalitions or simply TU fuzzy games with the player set N. A fuzzy coalition is a fuzzy subset of N, which is identified with a characteristic function from N to $[0, 1]$. Let $L(N)$ be the set of all fuzzy coalitions in N. For a fuzzy coalition $S \in L(N)$ and player $i \in N$, $S(i)$ represents the membership grade of i in S. The empty fuzzy coalition denoted by $\emptyset$ is one where all the players provide zero membership. If no ambiguity arises we use the same notations to represent crisp and fuzzy coalitions as crisp coalitions are special fuzzy coalitions with memberships 0 or 1.

The support of a fuzzy coalition S is denoted by $\operatorname{supp} S = \{i \in N \mid S(i) > 0\}$. We use the notation $S \subseteq T$ if and only if $S(i) \leq T(i)$ for all $i \in N$. Let $\vee$ and $\wedge$, respectively, represent the maximum and the minimum operators. The union and intersections of fuzzy coalitions S and T given by $S \cup T$ and $S \cap T$ are defined as $(S \cup T)(i) = S(i) \vee T(i)$ and $(S \cap T)(i) = S(i) \wedge T(i)$, respectively, for each $i \in N$. Following are some special fuzzy coalitions.

For $i \in N$ and $S \in L(N)$, the fuzzy coalitions S_i and S_{-i} are given by the following:

$$\begin{aligned} S_i(j) &= \begin{cases} S(i) & \text{if } j = i \\ 0 & \text{otherwise,} \end{cases} \\ S_{-i}(j) &= \begin{cases} 0 & \text{if } j = i. \\ S(j) & \text{otherwise.} \end{cases} \end{aligned} \tag{9}$$

Definition 7. A TU game with fuzzy coalitions or simply a TU fuzzy game is a pair (U, v) where $U \in L(N)$ and $v : L(N) \to \mathbb{R}$ are a set function, satisfying $v(\emptyset) = 0$.

Let $\mathrm{FG}(N)$ denote the class of TU fuzzy games with player set N. Now we define the two classes of TU fuzzy games, namely, the TU fuzzy games in Choquet integral form due to [15] and TU fuzzy games in multilinear extension form due to [17]. As an a priori requirement, the following definition is given.

Definition 8. Let $U \in L(N)$ and $i, j \in N$. For any $S \in L(U)$, define $\beta_{ij}[S]$ by

$$\beta_{ij}[S](k) = \begin{cases} S(j), & \text{if } k = i \\ S(i), & \text{if } k = j \\ S(k), & \text{otherwise.} \end{cases} \tag{10}$$

Definition 9. Given $S \in L(N)$, let $Q(S) = \{S(i) \mid S(i) > 0, i \in N\}$ and let $q(S)$ be the cardinality of $Q(S)$. Write the elements of $Q(S)$ in the increasing order as $h_1 < \cdots < h_{q(s)}$. Then a game $v \in \mathrm{FG}(N)$ is said to be a TU fuzzy game in Choquet integral form if and only if

$$v(S) = \sum_{l=1}^{q(S)} v\left([S]_{h_l}\right)(h_l - h_{l-1}) \quad \text{for any } S \in L(N), \text{ where } h_0 = 0. \tag{11}$$

The set of all TU fuzzy games in Choquet integral form is denoted by $\mathrm{FG}_C(N)$.

Definition 10 (see [12]). For any given $U \in L(N)$ and $v' \in G_0(N)$, a TU fuzzy game $v \in \mathrm{FG}(N)$ generated by v' and given by

$$v(U) = \min\left\{\sum_{T_0 \subseteq \operatorname{supp} U} \prod_{i \in T_0} U(i)\, v'(T_0), v'(\operatorname{supp} U)\right\} \tag{12}$$

is said to be a TU fuzzy game "in multilinear extension form." The set of all TU fuzzy games with form is denoted by multilinear extension form $\mathrm{FG}_M(N)$.

Following section includes the main contribution of the present study.

3. Solidarity Value for TU Fuzzy Games

We now discuss the notion of a solidarity value to the class $\mathrm{FG}(N)$ of TU fuzzy games with player set N along the line of [2]. Begin with the following definition.

Definition 11. Let $\emptyset \neq T \subseteq U \in L(N)$ and $v \in \mathrm{FG}(N)$, and the quantity

$$fv^v(T) = \frac{1}{|\operatorname{supp} T|} \sum_{k \in \operatorname{supp} T} [v(T) - v(T \backslash k)] \tag{13}$$

is called the average marginal contribution of a player of the fuzzy coalition T.

Definition 12. Given a game $v \in \mathrm{FG}(N)$, player $i \in N$ is called an fv-null player if $fv^v(T) = 0$, for every coalition $T \in L(U)$ with $T(i) \in (0, 1]$.

Note that Definitions 11 and 12 are the fuzzy extensions of their counterparts in crisp setting in the sense that if for all $T \in \mathrm{FG}(N)$, $T(i) = 1$ and all $j \in N$, $T(j) \in \{0, 1\}$, then $fv^v(T) = v^v(T)$.

Definition 13. A solidarity value function on $\mathrm{FG}(N)$ is a function $\Omega^{\mathrm{sol}} : \mathrm{FG}(N) \to (\mathbb{R}^n_+)^{L(N)}$ that satisfies the following four axioms, that is, Axioms F_1–F_4.

Axiom F_1 (Efficiency). If $v \in \mathrm{FG}(N)$ and $U \in L(N)$, then

$$\sum_{i\in N} \Omega_i^{\mathrm{sol}}(U, v) = v(U), \qquad \Omega_i^{\mathrm{sol}}(U, v) = 0 \quad \forall i \notin \operatorname{supp} U. \tag{14}$$

Axiom F_2 (Symmetry). If $v \in \mathrm{FG}(N)$, $U \in L(N)$ and $v(S) = v(\beta_{ij}[S])$. For any given $S \in L(U)$ and $i, j \in \operatorname{supp} U$, then $\Omega_i^{\mathrm{sol}}(U, v) = \Omega_j^{\mathrm{sol}}(U, v)$.

Axiom F_3 (Additivity). For any $u, v \in \mathrm{FG}(N)$, $\Omega^{\mathrm{sol}}(U, u + v) = \Omega^{\mathrm{sol}}(U, u) + \Omega^{\mathrm{sol}}(U, v)$.

Axiom F_4 (fv-Null Player). If $v \in \mathrm{FG}(N)$ and $i \in N$ are a fv-null player, that is, $fv^v(T) = 0$ for every fuzzy coalition $T \in L(U)$ with $T(i) \in (0, 1]$, then $\Omega_i^{\mathrm{sol}}(U, v) = 0$.

Note that Axioms F_1–F_4 are standard axioms derived from their crisp counterparts and therefore can be applied to any class of fuzzy games. Moreover, when we revert back to the class of crisp games these axioms become the standard Axioms CS_1–CS_4.

3.1. Solidarity Value for the Class $FG_C(N)$. We now find the solidarity value for the class $\mathrm{FG}_C(N)$ by use of the following theorem.

Theorem 14. *Let $v \in FG_C(N)$ and $U \in L(N)$. A function $\Omega^{sol} : FG_C(N) \to (\mathbb{R}^n_+)^{L(N)}$, defined by*

$$\Omega_i^{sol}(U, v) = \sum_{l=1}^{q(U)} \Phi^{sol}\left([U]_{h_l}, v\right) \cdot (h_l - h_{l-1}), \tag{15}$$

is a solidarity value function in U for $v \in FG_C(N)$, where

$$\Phi_i^{sol}\left([U]_{h_l}, v\right) = \begin{cases} \displaystyle\sum_{i\in T\subseteq [U]_{h_l}} \frac{(|T|-1)!\left(\left|[U]_{h_l}\right| - |T|\right)!}{\left|[U]_{h_l}\right|!} v^T(v), & \text{if } i \in [U]_{h_l} \\ 0, & \textit{elsewhere.} \end{cases} \tag{16}$$

Proof. Recall from Theorem 4 that there exists a unique function Φ^{sol} satisfying *Axioms* CS_1–CS_4. We use this to prove that the function Ω^{sol} satisfies *Axioms* F_1–F_4.

Axiom F_1 (Efficiency). Let $v \in \mathrm{FG}_C(N)$ and $U \in L(N)$. Since $\sum_{i\in \operatorname{supp} U} \Phi_i^{\mathrm{sol}}([U]_{h_l}, v) = v([U]_{h_l})$ holds for any $l \in 1, \ldots, q(U)$, we obtain

$$\begin{aligned} \sum_{i\in N} \Omega_i^{\mathrm{sol}}(U, v) &= \sum_{l=1}^{q(U)} \sum_{i\in N} \Phi^{\mathrm{sol}}\left([U]_{h_l}, v\right) \cdot (h_l - h_{l-1}) \\ &= \sum_{l=1}^{q(U)} v\left([U]_{h_l}\right)(h_l - h_{l-1}) = v(U). \end{aligned} \tag{17}$$

Since $i \notin \operatorname{supp} U$ implies $i \notin [U]_{h_l}$, we must have $\Phi_i^{\mathrm{sol}}([U]_{h_l}, v) = 0$.

It follows that $\Omega_i^{\mathrm{sol}}(U, v) = \sum_{l=1}^{q(U)} \Phi^{\mathrm{sol}}([U]_{h_l}, v) \cdot (h_l - h_{l-1}) = 0$.

Axiom F_2 (Symmetry). Let $v \in \mathrm{FG}_C(N)$ and $U \in L(N)$. We have the following: $v(S) - v(\beta_{ij}[S]) = 0$, $\forall S \in L(U) \Leftrightarrow v(S) - v(\beta_{ij}[S]) = 0$, $\forall S \in L(U)$, such that $S(j) = 0$, $S(k) \in \{S(i), 0\}$ $\forall k \in \operatorname{supp} U$, $\Leftrightarrow v(S) - v(\beta_{ij}[S]) = 0$, $\forall S \in L(U)$, such that $S(i) = h$, $S(j) = 0$, and $S(k) \in \{h, 0\}$ $\forall k \in \operatorname{supp} U$, $\forall h \in (0, U(i)]$, $\Leftrightarrow \{v([S']_h \cup \{i\}) - v([S']_h \cup \{j\})\} \cdot h = 0$, $\forall S' \in L(U)$, such that $S'(i) = S'(j) = 0$, and $S'(k) \in \{h, 0\}$ $\forall k \in \operatorname{supp} U$, $\forall h \in (0, U(i)]$, $\Leftrightarrow \{v([S']_h \cup \{i\}) - v([S']_h \cup \{j\})\} = 0$, $\forall S' \in L(U)$, such that $S'(i) = S'(j) = 0$, and $S'(k) \in \{h, 0\}$ $\forall k \in \operatorname{supp} U$, $\forall h \in (0, U(i)]$, $\Leftrightarrow \{v(T \cup \{i\}) - v(T \cup \{j\})\} = 0$, $\forall T \in P([U]_h \backslash \{i, j\})$, $\forall h \in (0, U(i)]$. Consequently, if $v(S) = v(\beta_{ij}[S])$ for any $S \in L(U)$, then $v(T \cup \{i\}) = v(T \cup \{j\})$ for any $T \in P([U]_h \backslash \{i, j\})$ and $h \in (0, U(i)]$. Hence we have $\Phi_i^{\mathrm{sol}}([U]_h, v) = \Phi_j^{\mathrm{sol}}([U]_h, v)$ for any $h \in (0, U(i)]$ and $\Phi_i^{\mathrm{sol}}([U]_h, v) = \Phi_j^{\mathrm{sol}}([U]_h, v) = 0$ for any $h \in (U(i), 1]$. Therefore, $\Phi_i^{\mathrm{sol}}([U]_h, v) = \Phi_j^{\mathrm{sol}}([U]_h, v)$ for any $h \in (0, 1]$. It follows that $\Omega_i^{\mathrm{sol}}(U, v) = \Omega_j^{\mathrm{sol}}(U, v)$.

Axiom F_3 (Additivity). Since Φ^{sol} is additive so for any $u, v \in \mathrm{FG}_C(N)$ and by the definition of Ω_i^{sol} we can easily prove that $\Omega_i^{\mathrm{sol}}(U, u + v) = \Omega_i^{\mathrm{sol}}(U, u) + \Omega_i^{\mathrm{sol}}(U, v)$.

Axiom F_4 (fv-Null Player). Let $v \in \mathrm{FG}_C(N)$ and $i \in N$ is a fv-null player; that is,

$$fv^v(T) = 0,$$

for every fuzzy coalition $T \in L(U)$ with $T(i) \in (0, 1]$

$$\Downarrow$$

$$\frac{1}{|\operatorname{supp} T|} \sum_{k\in \operatorname{supp} T} [v(T) - v(T \backslash k)] = 0$$

$$\Downarrow$$

$$\frac{1}{\left|[T]_{h_l}\right|} \sum_{k\in [T]_{h_l}} \left[v\left([T]_{h_l}\right) - v\left([T]_{h_l} \backslash k\right)\right]$$

$$+\frac{1}{\left|[T]_{h_l}\right|}\sum_{k\notin[T]_{h_l}}\left[v\left([T]_{h_l}\right)-v\left([T]_{h_l}\backslash k\right)\right]=0$$

$$\Downarrow$$

$$\frac{1}{\left|[T]_{h_l}\right|}\sum_{k\in[T]_{h_l}}\left[v\left([T]_{h_l}\right)-v\left([T]_{h_l}\backslash k\right)\right.=0$$

$$\Downarrow$$

$$v^{v}\left([T]_{h_l}\right)=0$$

$$\Downarrow$$

$$\Phi^{\mathrm{sol}}\left([U]_{h_l},v\right)=0$$

$$\Downarrow$$

$$\sum_{l=1}^{q(U)}\Phi^{\mathrm{sol}}\left([U]_{h_l},v\right)\cdot\left(h_l-h_{l-1}\right)=0$$

$$\Downarrow$$

$$\Omega_i^{\mathrm{sol}}(U,v)=0. \tag{18}$$

This completes the proof. □

3.2. Solidarity Value for the Class $FG_M(N)$

Theorem 15. *For $\emptyset\neq T\subseteq U\in L(N)$, the game $\widehat{u}_T$, that is,*

$$\widehat{u}_T(S)=\begin{cases}\dbinom{|\mathrm{supp}\,S|}{|\mathrm{supp}\,T|}^{-1} & \text{if } S\supset T\\ 0 & \textit{otherwise},\end{cases} \tag{19}$$

has the following properties:

(i) $\widehat{u}_T(T)=1$;

(ii) *if* $\mathrm{supp}\,S=\mathrm{supp}\,T\cup E$ *with* $\emptyset\neq E\subset\mathrm{supp}\,U\backslash\mathrm{supp}\,T\in L(N)$, *then*

$$\widehat{u}_T(S)=\frac{1}{|\mathrm{supp}\,S|}\sum_{i\in\mathrm{supp}\,S}\widehat{u}_T(S_{-i}) \tag{20}$$

and every player $i\in\mathrm{supp}\,U\backslash\mathrm{supp}\,T$ is fv-null in the game $\widehat{u}_T$.

Theorem 16. *Let $v\in FG_M(N)$ and $U\in L(N)$. A function $\Omega^{sol}:FG_C(N)\to(\mathbb{R}_+^n)^{L(N)}$ defined by*

$$\begin{aligned}\Omega_i^{sol}(U,v)&=\sum_{T\subseteq U}\left(\frac{(|\mathrm{supp}\,T|-1)!\,(|\mathrm{supp}\,U|-|\mathrm{supp}\,T|)!}{|\mathrm{supp}\,U|!}\right)\\&\quad\cdot A^{v}(T),\quad\forall i\in\mathrm{supp}\,U,\end{aligned} \tag{21}$$

where

$$\begin{aligned}A^{v}(T)&=\frac{1}{|\mathrm{supp}\,T|}\\&\quad\cdot\sum_{k\in\mathrm{supp}\,T}\left\{\sum_{S_0\subseteq\mathrm{supp}\,T}\left\{\prod_{j\in S_0}U(j)\right\}v'(S_0)\right.\\&\quad\left.-\sum_{S_0\subseteq\mathrm{supp}\,T_{-k}}\left\{\prod_{j\in S_0}U(j)\right\}v'(S_0)\right\},\end{aligned} \tag{22}$$

is the unique solidarity value for $v\in FG_M(N)$ in U.

Proof. Let us construct any value function η on $\mathrm{FG}_M(N)$ satisfying efficiency, symmetry, additivity, and fv-null player axioms by

$$\eta_i(U,\kappa\widehat{u}_T)=\begin{cases}\dfrac{\kappa\left(\dbinom{|\mathrm{supp}\,U|}{|\mathrm{supp}\,T|}\right)^{-1}}{|\mathrm{supp}\,T|}, & \text{if } i\in\mathrm{supp}\,T\\ 0, & \text{otherwise.}\end{cases} \tag{23}$$

Now, we know that $v\in\mathrm{FG}_M(N)$ can be expressed by

$$v(S)=\left(\sum_{\phi\neq T\subseteq U}c_T(v)\,\widehat{u}_T(S)\right), \tag{24}$$

where

$$\begin{aligned}&c_T(v)\\&\quad=\sum_{S\subseteq T}(-1)^{|\mathrm{supp}\,T|-|\mathrm{supp}\,S|}\sum_{H_0\subseteq\mathrm{supp}\,S}\prod_{i\in H_0}U(i)\,v'(H_0);\end{aligned} \tag{25}$$

clearly Ω^{sol} given by (21) and (22) satisfies symmetry and fv-null player axioms. Moreover, Ω^{sol} is a linear mapping. Hence additivity is satisfied.

Using now linearity of Ω^{sol}, we get

$$\begin{aligned}\sum_{i\in\mathrm{supp}\,U}\Omega_i^{\mathrm{sol}}(U,v)&=\sum_{\emptyset\neq T\subseteq U}c_T\sum_{i\in\mathrm{supp}\,U}\Omega_i^{\mathrm{sol}}(U,\widehat{u}_T)\\&=\sum c_T\widehat{u}_T(U)=v(U)\end{aligned} \tag{26}$$

which proves that Ω^{sol} is efficient. It is obvious that $\eta(U,\widehat{u}_T)=\Omega_i^{\mathrm{sol}}(U,\widehat{u}_T)$ for each game $\widehat{u}_T$. Thus $\eta(U,v)=\Omega_i^{\mathrm{sol}}(U,v)$ for every $v\in\mathrm{FG}_M(N)$. □

4. Solidarity Share Functions for TU Fuzzy Games

We now extend the notion of a share function to the class $\mathrm{FG}(N)$ of TU fuzzy games with player set N. In the line of its crisp counterpart we assume here also that the share function assigns to each player her share in the payoff $v(U)$ of the fuzzy coalition $U\in L(N)$. Therefore we provide the following definitions as an extension to their crisp versions.

Definition 17. A real valued function $\mu : L(N) \times \mathrm{FG}(N) \to \mathbb{R}$ is called f-additive if, for $U \in L(N)$ and any pair $v_1, v_2 \in \mathrm{FG}(N)$ such that $v_1 + v_2 \in \mathrm{FG}(N)$, it holds that

$$\mu(U, v_1 + v_2) = \mu(U, v_1) + \mu(U, v_2). \tag{27}$$

Definition 18. A real valued function $\mu : L(N) \times \mathrm{FG}(N) \to \mathbb{R}$ is called f-linear on the class $\mathrm{FG}(N)$ of games if it is f-additive and if for any v on $\mathrm{FG}(N)$ and $U \in L(N)$ it holds that $\mu(U, \alpha v) = \alpha\mu(U, v)$ for any real number α such that $\alpha v \in \mathrm{FG}(N)$.

Definition 19. A real valued function $\mu : L(N) \times \mathrm{FG}(N) \to \mathbb{R}$ is called positive if $\mu(U, v) \geq 0$ $\forall v \in \mathrm{FG}(N)$, $U \in L(N)$.

Definition 20. Given a function $\mu : L(N) \times \mathrm{FG}(N) \to \mathbb{R}$, a solidarity μ-share function on $\mathrm{FG}(N)$ is a function $\Psi^\mu : \mathrm{FG}(N) \to (\mathbb{R}^n_+)^{L(N)}$ that satisfies the following axioms, that is, Axioms FS_1–FS_3 along with Axiom FS_4 or Axiom FS_5.

Axiom FS_1 (f-Efficiency). For $U \in L(N)$ we have $\sum_{i\in N} \Psi_i^\mu(U, v) = 1$ and $\Psi_i^\mu(U, v) = 0$, for each $i \notin \operatorname{supp} U$.

Axiom FS_2 (fv-Null Player). If $v \in \mathrm{FG}(N)$ and $i \in N$ are a fv-null player, that is, $fv^v(T) = 0$ for every fuzzy coalition $T \in L(U)$ with $T(i) \in (0, 1]$, then $\Psi_i^\mu(U, v) = 0$.

Axiom FS_3 (f-Symmetry). If $v \in \mathrm{FG}(N)$, $U \in L(N)$, and $v(S) = v(\beta_{ij}[S])$ for any given $S \in L(U)$ and $i, j \in \operatorname{supp} U$, then $\Psi_i^\mu(U, v) = \Psi_j^\mu(U, v)$.

Axiom FS_4 ($f\mu$-Additivity). For any pair $v_1, v_2 \in \mathrm{FG}(N)$ such that $v_1 + v_2 \in \mathrm{FG}(N)$, it holds that

$$\begin{aligned} &\mu(U, v_1 + v_2)\, \Psi_i^\mu(U, v_1 + v_2) \\ &\quad = \mu(U, v_1)\, \Psi_i^\mu(U, v_1) + \mu(U, v_2)\, \Psi_i^\mu(U, v_2), \\ &\qquad\qquad \forall i \in N. \end{aligned} \tag{28}$$

Axiom FS_5 ($f\mu$-Linearity). For any pair $v_1, v_2 \in \mathrm{FG}(N)$ such that $v_1 + v_2 \in \mathrm{FG}(N)$, it holds that $\mu(U, av_1 + bv_2)\Psi_i(U, av_1 + bv_2) = a\mu(U, v_1)\Psi_i^\mu(U, v_1) + b\mu(U, v_2)\Psi_i^\mu(U, v_2)$, for any pair of real numbers a and b such that $av_1 + bv_2 \in \mathrm{FG}(N)$ for all $i \in N$.

Note that *Axioms FS_1–FS_5* are intuitive of their crisp counterparts in the sense that reverting back to the crisp formulation we get the standard axioms of share functions. It follows that for any $v \in \mathrm{FG}(N)$ a solidarity μ-share function Ψ^μ gives a payoff $\Psi_i(U, v)\cdot v(U)$ to player i when she is involved in the fuzzy coalition U and satisfies the above-mentioned axioms.

4.1. Solidarity Share Functions for $FG_C(N)$. In this section we prove the existence and uniqueness of the solidarity μ-share function for the class $\mathrm{FG}_C(N)$ of fuzzy games in Choquet integral form. To discuss the existence and uniqueness of the solidarity μ-share function for TU fuzzy game in $\mathrm{FG}_C(N)$ we have to use some classical results from [1, 2]. Recall that, given a coalition $T \subset K \in P(N)$, the game w_T is defined as follows:

$$w_T(S) = \begin{cases} \dbinom{|S|}{|T|}^{-1}, & \text{if } S \supset T \\ 0, & \text{otherwise.} \end{cases} \tag{29}$$

Due to Theorem 6, for any $T \in 2^K$, each $v \in G_0(N)$ can be expressed as $v = (\sum_{T\in 2^K} c_T w_T)$ where $c_T(v) = \sum_{R\subseteq T}(-1)^{|T|-|R|} v(R)$. Denote $C^+ = \{T : c_T(v) \geq 0\}$ and $C^- = \{T : c_T(v) < 0\}$. Then

$$v = \left(\sum_{T\in C^+} c_T(v)\, w_T\right) - \left(\sum_{T\in C^-} c_T(v)\, w_T\right). \tag{30}$$

Following similar procedure as in Lemma 3.2. of [10], we can have, for $v \in \mathrm{FG}_C(N)$,

$$v = \left(\sum_{T\in 2^N} c_T(v)\, z_T\right), \tag{31}$$

where

$$\begin{aligned} c_T(v) &= \sum_{R\subseteq T} (-1)^{|T|-|R|} v(R), \\ z_T(U) &= \sum_{l=1}^{q(U)} w_T\left([U]_{h_l}\right) \cdot (h_l - h_{l-1}) \quad \forall U \in L(N). \end{aligned} \tag{32}$$

It follows from the above discussion that $v(U)$ can be rewritten as

$$\begin{aligned} v(U) = &\left(\sum_{T\in 2^N c_T(v)\geq 0} c_T(v)\, z_T(U)\right) \\ &- \left(\sum_{T\in 2^N c_T(v)<0} (-c_T(v))\, z_T(U)\right). \end{aligned} \tag{33}$$

Theorem 21. *Let $\mu : FG_C(N) \to \mathbb{R}$ be a real valued function. There exists a unique solidarity μ-share function $\Psi^\mu : FG_C(N) \to (\mathbb{R}^n_+)^{L(N)}$ that satisfies the axioms of f-efficiency (FS_1), fv- null player (FS_2), f-symmetry (FS_3), and $f\mu$- additivity (FS_4) if and only if μ is f-additive on $FG_C(N)$.*

Proof. The proof proceeds in the line of [6]. First we suppose that Ψ^μ satisfies f-efficiency and $f\mu$- additivity. It follows that Ψ^μ is μ-additive on $G_0(N)$. Thus we have

$$\begin{aligned}\mu(U, v_1 + v_2) \sum_{i\in N} \Psi_i^{\mu}(U, v_1 + v_2) &= \mu(U, v_1 + v_2) \\ &\cdot \left[\Psi_1^{\mu}(U, v_1 + v_2) + \cdots + \Psi_{|\operatorname{supp}(U)|}^{\mu}(U, v_1 + v_2)\right] \\ &= \mu(U, v_1) \sum_{i\in N} \Psi_i(U, v_1) + \mu(U, v_2) \sum_{i\in N} \Psi_i(U, v_2)\end{aligned} \tag{34}$$

for any $v_1, v_2 \in \mathrm{FG}_C(N)$ such that $v_1 + v_2 \in \mathrm{FG}_C(N)$. f-efficiency then implies that $\mu(U, v_1 + v_2) = \mu(U, v_1) + \mu(U, v_2)$. Hence μ is f-additive.

Secondly we will show that we can have at most one solidarity share function $\Psi^{\mu} : \mathrm{FG}_C(N) \to (\mathbb{R}_+^n)^{L(N)}$ satisfying the four axioms. Let $\Psi^{\mu} : \mathrm{FG}_C(N) \to (\mathbb{R}_+^n)^{L(N)}$ be a function satisfying the four axioms. For a positively scaled unanimity game $\alpha z_T \in \mathrm{FG}_C(N)$, $\alpha > 0$ and consequently for $\alpha z_T \in \mathrm{FG}_C(N)$, we obtain

(i) $\Psi_i^{\mu}(U, \alpha z_T) = 1/|\operatorname{supp} U|$, when $i \in \operatorname{supp} U$.

(ii) $\Psi_i^{\mu}(U, \alpha z_T) = 0$, when $i \notin \operatorname{supp} U$.

Again for αz_T, $\alpha > 0$ from (i) and (ii) clearly Ψ^{μ} satisfies all the four axioms. Thus it follows that for any $\alpha z_T, \alpha > 0$ the function Ψ^{μ} given by (i) and (ii) is the solidarity μ-share function satisfying the axioms of f-efficiency, $f\nu$-null player, and f-symmetry if and only if μ is f-additive.

The uniqueness of $\Psi^{\mu}(U, v)$ follows immediately. We next show that $\Psi^{\mu}(U, v)$ satisfies the four axioms for an arbitrary v. The assumption of f-additivity of μ ensures f-efficiency as in the case of crisp games. Consequently the $f\nu$-null player axiom also follows. Third, for any $U \in L(N)$ and $S \in L(U)$ with $i, j \in \operatorname{supp} U$, $v(S) = v(\beta_{ij}[S])$ implies $v(T \cup \{i\}) = v(T \cup \{j\})$, $\forall T \in P([U]_h \backslash \{i, j\})]$, $\forall h \in (0, U(i)]$, then $c_i(v) = c_j(v)$, whereas for any other $U \in L(N)$ with nonzero weight $c_T(v)$, i and j either both have nonzero memberships in U or both have zero memberships in U. Hence it follows that $\Psi_i^{\mu}(U, v) = \Psi_j^{\mu}(U, v) = 0$ when, $j \notin \operatorname{supp} U$.

Next for $i, j \in \operatorname{supp} U$,

$$\begin{aligned}\Psi_i^{\mu}(U, v) &= \frac{1}{\mu(U, v)} \left[\sum_{T\in C^+} \mu(U, c_T(v) z_T) \frac{1}{|\operatorname{supp}(U)|} - \sum_{T\in C^-} \mu(U, -c_T(v) z_T) \frac{1}{|\operatorname{supp}(U)|}\right], \\ \Psi_j^{\mu}(U, v) &= \frac{1}{\mu(U, v)} \left[\sum_{T\in C^+} \mu(U, c_T(v) z_T) \frac{1}{|\operatorname{supp}(U)|} - \sum_{T\in C^-} \mu(U, -c_T(v) z_T) \frac{1}{|\operatorname{supp}(U)|}\right].\end{aligned} \tag{35}$$

So Ψ^{μ} satisfies the symmetry (FS_3) axiom. Finally for any two games $v_1, v_2 \in \mathrm{FG}_C(N)$ we have that $(v_1 + v_2)(U) = (\sum_{T\in 2^N}(c_T(v_1) + c_T(v_2))z_T)(U)$. Following f-additivity of μ this implies $\mu(U, v_1 + v_2)\Psi^{\mu}(U, v_1 + v_2) = \mu(U, v_1)\Psi^{\mu}(U, v_1) + \mu(U, v_2)\Psi(U, v_2)$ and hence Ψ^{μ} is $f\mu$-additive. □

Theorem 22. *For given positive numbers ω_k with $k = 1, 2, \ldots, n$, let the function μ^{ω} be defined by*

$$\mu_{\omega}(U, v) = \sum_{i\in N} \sum_{l=1}^{q(U)} \left\{\sum_{i\in T\subseteq [U]_{h_l}} \omega_k v^{v}(T)\right\} (h_l - h_{l-1}). \tag{36}$$

Then the solidarity μ-share function $\Psi^{\mu_{\omega}}$ defined by

$$\Psi_i^{\mu_{\omega}}(U, v) = \frac{\sum_{l=1}^{q(U)} \left\{\sum_{i\in T\subseteq [U]_{h_l}} \omega_k v^{v}(T)\right\} (h_l - h_{l-1})}{\mu_{\omega}(U, v)} \tag{37}$$

is the unique solidarity μ-share function satisfying the axioms of f-efficiency, $f\nu$-null player, f-symmetry, and $f\mu_{\omega}$- additivity on $FG_C(N)$ wherever μ_{ω} is positive.

Proof. By definition, μ_{ω} is f-additive. Hence the existence and uniqueness of the solidarity μ-share function follows from Theorem 21. We show that $\Psi^{\mu_{\omega}}$ satisfies the four axioms with respect to μ_{ω} on the class $\mathrm{FG}_C(N)$ of μ_{ω}-positive games. Next we show that $\Psi^{\mu_{\omega}}$ satisfies the above four axioms. The f-efficiency and $f\nu$-null player axioms are direct consequences of their crisp counterparts. Now for any $U \in L(N)$ and $S \in L(U)$ with $i, j \in \operatorname{supp} U$, $v(S) = v(\beta_{ij}[S])$ implies $v(T \cup \{i\}) = v(T \cup \{j\})$, $\forall T \in P([U]_h \backslash \{i, j\})]$, $\forall h \in (0, U(i)]$, then we have that $f\nu^{v}(S) = f\nu^{v}(\beta_{ij}[S])$ implies $\nu^{v}(T \cup \{i\}) = \nu^{v}(T \cup \{j\})$ $\forall T \in P([U]_h \backslash \{i, j\})$. Following the fact that ω_k depends only on the size of T, the symmetry axiom holds. Finally we have $\mu_{\omega}(U, v)\Psi_i^{\mu_{\omega}}(U, v) = \sum_{l=1}^{q(U)} \{\sum_{i\in T\subseteq [U]_{h_l}} \omega_k \nu^{v}(T)\}(h_l - h_{l-1})$. For all T containing i, it holds that $\nu^{(au+bv)}(T) = a\nu^{v}(T) + b\nu^{v}(T)$; it follows that $\Psi^{\mu_{\omega}}$ is μ_{ω}-additive. □

In the following theorem, we take a particular form of the function μ and obtain the corresponding solidarity share function for the class $\mathrm{FG}_C(N)$. This exemplifies the existence of a wide range of such share functions generated by the various choices of the function μ.

Theorem 23. *Let the function μ_{sol} be defined by $\mu_{sol}(U, v) = v(U)$. Then the solidarity μ-share function $\Psi^{\mu_{sol}}$ is the unique solidarity μ-share function satisfying the axioms of f-efficiency, $f\nu$-null player, f-symmetry, and $f\mu_{sol}$-linearity on $FG_C(N)$.*

Proof. For $T \subseteq [U]_{h_l}$ with $|T| = k$, take $\omega_k = (k-1)!(|[U]_{h_l}| - k)!/|U_{h_l}|!$. Then, we have that $\mu_{\omega_{\mathrm{sol}}}$ as defined in Theorem 15 given by

$$\begin{aligned}\mu_{\omega}(U, v) &= \sum_{i\in N} \left\{\sum_{l=1}^{q(U)} \sum_{i\in T\subseteq [U]_{h_l}} \omega_k \nu^{v}(T)\right\} (h_l - h_{l-1}) \\ &= v(U) = \mu_{\mathrm{sol}}(U, v).\end{aligned} \tag{38}$$

Further, the share function $\Psi^{\mu_{\mathrm{sol}}}$ as defined in Theorem 15 is given by

$$\Psi_i^{\mu_{\text{sol}}}(U,v) = \frac{\left\{\sum_{l=1}^{q(U)} \sum_{i\in T\subseteq [U]_{h_l}} \omega_k v^v(T)\right\}(h_l - h_{l-1})}{\mu_{\text{sol}}(U,v)}$$
$$= \frac{\sum_{l=1}^{q(U)} \left\{\sum_{i\in T\subseteq [U]_{h_l}} \left((k-1)!\left(\left|[U]_{h_l}\right| - k\right)!/\left|[U]_{h_l}\right|!\right) v^v(T)\right\}(h_l - h_{l-1})}{\mu_{\text{sol}}(U,v)} = \frac{\Phi_i^{\text{sol}}(U,v)}{v(U)} = \Psi_i^{\mu_{\text{sol}}}(U,v), \quad i \in \operatorname{supp} U. \tag{39}$$

This completes the proof. □

4.2. Solidarity Share Functions for $FG_M(N)$. Here we discuss the existence and uniqueness of the solidarity μ-share function for TU fuzzy games in $\text{FG}_M(N)$ following the definition of the game $\widehat{u}_T$.

Theorem 24. *Let $\mu : G_M(N) \to \mathbb{R}$ be a real valued function on the class $G_M(N)$ of games. Then on $G_M(N)$ there exists a unique μ-share function $\Psi^\mu : G_M(N) \to (\mathbb{R})^{L(N)}$ that satisfies the axioms of f-efficiency, $f\nu$-null player property, f-symmetry, and $f\mu$-additivity if and only if μ is f-additive on $G_M(N)$.*

Proof. The proof goes exactly in the same line of Theorem 16 and hence is omitted. □

Theorem 25. *For given positive numbers ω_s with $s = 1, 2, \ldots, n$, let the function μ^ω be defined by*

$$\mu^\omega(U,v) = \sum_{i\in\operatorname{supp} U} \sum_{T\subseteq U} \omega_s A^v(T), \tag{40}$$

where

$$A^v(T) = \frac{1}{|\operatorname{supp} T|} \cdot \sum_{k\in\operatorname{supp} T} \left\{ \sum_{S_0\subseteq\operatorname{supp} T} \left\{\prod_{j\in S_0} U(j)\right\} v'(S_0) - \sum_{S_0\subseteq\operatorname{supp} T_{-k}} \left\{\prod_{j\in S_0} U(j)\right\} v'(S_0)\right\}. \tag{41}$$

Then the solidarity μ-share function Ψ^{μ_ω} defined by

$$\Psi_i^{\mu_\omega}(U,v) = \frac{\sum_{T\subseteq U} \omega_s A^v(T)}{\mu^\omega(U,v)} \tag{42}$$

is the unique solidarity μ-share function satisfying the axioms of f-efficiency (FS_1), $f\nu$-null player (FS_2), f-symmetry (FS_3), and $f\mu_\omega$-additivity (FS_4) on $FG_M(N)$ wherever μ_ω is positive.

Proof. Along the line of Theorem 21, we can easily get the result. □

Next we obtain a particular μ to exemplify the wide variety of the class of μ-share functions.

Theorem 26. *Let the function μ_{sol} be defined by $\mu_{sol}(U,v) = v(U) = W(v)$. Then the solidarity μ-share function $\Psi^{\mu_{sol}}$ is the unique solidarity μ-share function corresponding to the solidarity value function satisfying the axioms of f-efficiency, $f\nu$-null player, f-symmetry, and $f\mu_{sol}$-linearity on $FG_M(N)$.*

Proof. The proof proceeds exactly in the same line of Theorem 22 so it is omitted. □

5. Conclusion

We have discussed the notion of solidarity value and solidarity share function on a class of TU fuzzy games. The solidarity share function on the two classes $\text{FG}_C(N)$ and $\text{FG}_M(N)$ is illustrated. Few consequent properties and relationships have been investigated. Other solution concepts of TU fuzzy games can also be studied in a similar way which is kept for our future work.

Conflicts of Interest

The authors declare that funding listed in "Acknowledgments" did not lead to any conflicts of interest regarding the publication of this manuscript. There are also no other conflicts of interest in the manuscript.

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