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Research Article

On (L, M)-Double Fuzzy Filter Spaces

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We give in this paper the definitions of (L, M)-double fuzzy filter base and (L, M)-double fuzzy filter structures where L and M are strictly two-sided commutative quantales, and we also investigate the relations between them. Moreover, we propose second-order image and preimage operators of (L, M)-double fuzzy filter base and study some of its fundamental properties. Finally, we handle the given structures in the categorical aspect. For instance, we show that the category (L, M)-**DFIL** of (L, M)-double fuzzy filter spaces and filter maps between these spaces is a topological category over the category **SET**.

1. Introduction

Kubiak [1] and Šostak [2] introduced the notion of L-fuzzy topological space as a generalization of L-topological spaces introduced by Chang [3]. At the bottom of it lies the degree of openness of an L-fuzzy set. A general approach to the study of topological-type structures on fuzzy powersets was developed in [4–14].

On the other hand, Atanassov [15] introduced the idea of intuitionistic (double graded) fuzzy set. Çoker and his coworker(s) [16, 17] introduced the idea of topology of intuitionistic fuzzy sets. Recently, Mondal and Samanta [18] introduced the notion of intuitionistic gradation of openness which is a generalization of both fuzzy topological spaces [2] and the topology of intuitionistic fuzzy sets [16].

Working under the name "intuitionistic" did not continue because doubts were thrown about the suitability of this term, especially when working in the case of complete lattice L. These doubts were quickly ended in 2005 by Gutiérrez García and Rodabaugh [19]. They argued that this term is unsuitable in mathematics and applications. They concluded that they work under the name "double."

The notion of L-filter was introduced by Höhle and Šostak [7] as an expansion of fuzzy filter [20–25]. In recent years, L-filters were used to introduce many kinds of lattice-valued convergence spaces [26–28]. L-filter is an important tool to study L-fuzzy topology [29, 30] and L-fuzzy uniform space [26]. The structure of this paper is as follows. In Section 2, we recall some fundamental definitions related to quantale lattice by giving illustrative examples and also recall some definitions necessary for the main sections. In Section 3, we define (L, M)-double fuzzy filter and (L, M)-double fuzzy filter base and then study relations between them. In the next two sections, we consider two types of second-order Zadeh image and preimage operators of (L, M)-double fuzzy filter base and examine their characteristics by giving examples.

2. Preliminaries

Throughout this paper, let X be a nonempty set. Let $L = (L, \leq, \vee, \wedge)$ be a complete lattice with the least element 0_L and the greatest element 1_L. For $\alpha \in L$, $\underline{\alpha}(x) = \alpha$ for all $x \in X$. The second lattice belonging to the context of our work is denoted by M and $M_0 = M - \{0_M\}$ and $M_1 = M - \{1_M\}$.

A complete lattice $L = (L, \leq, \wedge, \vee)$ is called completely distributive, if for any family $\{\{a_{i,j} : j \in J_i\} : i \in I\}$ in L the following identity holds:

$$\text{(CD)} \quad \bigwedge_{i\in I}\left(\bigvee_{j\in J_i} a_{i,j}\right) = \bigvee_{\varphi\in\prod_{i\in I} J_i}\left(\bigwedge_{i\in I} a_{i,\varphi(i)}\right). \tag{1}$$

Definition 1 (see [24, 31–33]). A triple $L = (L, \leq, \odot)$ is called a strictly two-sided commutative quantale (stsc-quantale, for short) iff it satisfies the following properties:

(L1) $(L, \odot)$ is a commutative semigroup.

(L2) $x \odot 1_L = x$, for all $x \in L$.

(L3) $\odot$ is distributive over arbitrary joins:

$$x \odot \left(\bigvee_{i\in I} y_i\right) = \bigvee_{i\in I}(x \odot y_i), \quad \forall x \in L, \ \forall \{y_i\}_{i\in I} \subseteq L. \tag{2}$$

An stsc-quantale $L = (L, \leq, \odot)$ is an $\bigwedge$-distributive quantale (or stsc-biquantale [34]) if $\odot$ is distributive over nonempty meets:

$$x \odot \left(\bigwedge_{i\in I} y_i\right) = \bigwedge_{i\in I}(x \odot y_i), \quad \forall x \in L, \ \forall \{y_i\}_{i\in I} \subseteq L. \tag{3}$$

Remark 2 (see [24, 25, 31–33, 35]). (1) A complete lattice satisfying the infinite distributive law is an stsc-quantale. In particular, the unit interval $([0,1], \leq, \wedge, 0, 1)$ is an $\bigwedge$-distributive quantale.

(2) Every left-continuous t-norm T on $[0,1]$, $([0,1], \leq, T)$ is an stsc-quantale.

(3) Every continuous t-norm T on $[0,1]$, $([0,1], \leq, T)$ is an $\bigwedge$–distributive quantale.

(4) Every GL-monoid is an stsc-quantale.

(5) Let $(L, \leq, \odot)$ be an stsc-quantale. For each $x, y \in L$, we define

$$x \longmapsto y = \bigvee \{z \in L \mid x \odot z \leq y\}. \tag{4}$$

Then, it satisfies Galois correspondence; that is,

$$\begin{aligned} &x \odot z \leq y \Longleftrightarrow \\ &\quad z \leq x \longmapsto y, \\ &\qquad \forall x, y, z \in L. \end{aligned} \tag{5}$$

Definition 3 (see [1, 7, 24, 29, 31, 33, 35–38]). Let $(L, \leq, \odot)$ be an stsc-quantale. A mapping $\star : L \to L$ is called an order-reversing involution if it satisfies the following conditions:

(1) $x^{\star\star} = x$, for each $x \in L$.

(2) If $x \leq y$, then $y^\star \leq x^\star$, for each $x, y \in L$.

An stsc-quantale is called a Girard monoid [37] if $(x \mapsto 0_L) \mapsto 0_L = x$, $\forall x \in L$.

Hence, in case L is a Girard monoid, residuation $\mapsto$ induces an order-reversing involution $\star : L \to L$. In this paper, we always assume that $(L, \leq, \odot, \oplus, \star)$ (resp., $(M, \leq, \tilde{\odot}, \tilde{\oplus}, \star)$) is a Girard monoid with an order-reversing involution $\star$, and the operation $\oplus$ is defined by

$$\begin{aligned} x \oplus y &= (x^\star \odot y^\star)^\star, \\ x^\star &= x \longmapsto 0_L \end{aligned} \tag{6}$$

unless otherwise specified, where $\tilde{\odot}, \tilde{\oplus}$ denote the quantale operations on M.

Remark 4 (see [39]). When the underlying lattice L is the unit interval $[0,1]$ of the real numbers, the notion of a Girard monoid coincides with the notion of a left-continuous t-norm with strong induced negation $\star$, $(x^\star = x \mapsto 0)$.

Lemma 5 (see [34]). *Let L be a Girard monoid. For each $x, y, z, x_i, y_i \in L$, one has the following properties:*

(1) If $y \leq z$, then $x \odot y \leq x \odot z$, $x \oplus y \leq x \oplus z$, $x \mapsto y \leq x \mapsto z$, and $y \mapsto x \geq z \mapsto x$.

(2) $x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y$.

Let L be a complete lattice and $\varphi : X \to Y$ be a function. The Zadeh image and preimage operators $\varphi^{\rightarrow} : L^X \to L^Y$ and $\varphi^{\leftarrow} : L^Y \to L^X$ are defined by

$$\begin{aligned} &\varphi^{\rightarrow}(\lambda)(y) = \bigvee \{\lambda(x) \mid y = \varphi(x)\}, \\ &\varphi^{\leftarrow}(\mu)(x) = \mu(\varphi(x)), \\ &\qquad \forall x \in X, \ y \in Y. \end{aligned} \tag{7}$$

Lemma 6 (see [40]). *Let $(L, \leq, \odot)$ be an stsc-quantale and $\varphi : X \to Y$ be a function. For each $\lambda, \mu \in L^X$ and $\lambda_i \in L^Y$, one has the following properties:*

(1) *$\varphi^{\rightarrow}(\lambda \odot \mu) \leq \varphi^{\rightarrow}(\lambda) \odot \varphi^{\rightarrow}(\mu)$ with equality if φ is injective.*

(2) *$\varphi^{\leftarrow}(\odot_{i=1}^n \lambda_i) = \odot_{i=1}^n \varphi^{\leftarrow}(\lambda_i)$.*

Definition 7 (see [40]). Basic scheme for second-order image operators: let $\varphi : X \to Y$ be a function.

Case 1. Consider

$$[\varphi_L^{\rightarrow}]_L^{\rightarrow} : L^{L^X} \longrightarrow L^{L^Y}. \tag{8}$$

This is the Zadeh image operator of the Zadeh image operator. We denote it by $\varphi_1^{\rightrightarrows}$; that is, for all $\mathcal{U} \in L^{L^X}$ and $\mu \in L^Y$,

$$\begin{aligned} \varphi_1^{\rightrightarrows}(\mathcal{U})(\mu) &= [\varphi_L^{\rightarrow}]_L^{\rightarrow}(\mathcal{U})(\mu) \\ &= \bigvee \{\mathcal{U}(\lambda) : \mu = \varphi_L^{\rightarrow}(\lambda)\}. \end{aligned} \tag{9}$$

Case 2. Consider

$$[\varphi_L^{\leftarrow}]_L^{\leftarrow} : L^{L^X} \longrightarrow L^{L^Y}. \tag{10}$$

This is the Zadeh preimage operator of the Zadeh preimage operator. We denote it by $\varphi_2^{\rightrightarrows}$; that is, for all $\mathcal{U} \in L^{L^X}$ and $\mu \in L^Y$,

$$\varphi_2^{\rightrightarrows}(\mathcal{U})(\mu) = [\varphi_L^{\leftarrow}]_L^{\leftarrow}(\mathcal{U})(\mu) = \mathcal{U} \circ \varphi_L^{\leftarrow}(\mu). \tag{11}$$

Basic scheme for second-order preimage operators: let $\varphi : X \to Y$ be a function.

Case 1. Consider

$$[\varphi_L^{\leftarrow}]_L^{\rightarrow} : L^{L^X} \longleftarrow L^{L^Y}. \tag{12}$$

This is the Zadeh image operator of the Zadeh preimage operator. We denote it by $\varphi_1^{\Leftarrow}$; that is, for all $\mathscr{V} \in L^{L^Y}$ and $\lambda \in L^X$,

$$\begin{aligned}\varphi_1^{\Leftarrow}(\mathscr{V})(\lambda) &= [\varphi_L^{\leftarrow}]_L^{\rightarrow}(\mathscr{V})(\lambda)\\ &= \bigvee\{\mathscr{V}(\mu) : \lambda = \varphi_L^{\leftarrow}(\mu)\}.\end{aligned} \tag{13}$$

Case 2. Consider

$$[\varphi_L^{\rightarrow}]_L^{\leftarrow} : L^{L^X} \longleftarrow L^{L^Y}. \tag{14}$$

This is the Zadeh preimage operator of the Zadeh image operator. We denote it by $\varphi_2^{\Leftarrow}$; that is, for all $\mathscr{V} \in L^{L^Y}$ and $\lambda \in L^X$,

$$\varphi_2^{\Leftarrow}(\mathscr{V})(\lambda) = [\varphi_L^{\rightarrow}]_L^{\leftarrow}(\mathscr{V})(\lambda) = \mathscr{V} \circ \varphi_L^{\rightarrow}(\lambda). \tag{15}$$

In this paper, we consider additional operators as follows.

Define the operator $\varphi_1^{*\Leftarrow} : M^{L^X} \leftarrow M^{L^Y}$ as $\varphi_1^{*\Leftarrow}(\mathscr{V})(\lambda) = \bigwedge\{\mathscr{V}(\mu) : \lambda = \varphi_L^{\leftarrow}(\mu)\}$, for all $\mathscr{V} \in M^{L^Y}$ and $\lambda \in L^X$.

Define the operator $\varphi_1^{*\Rightarrow} : M^{L^X} \to M^{L^Y}$ as $\varphi_1^{*\Rightarrow}(\mathscr{V})(\mu) = \bigwedge\{\mathscr{V}(\lambda) : \mu = \varphi_L^{\rightarrow}(\lambda)\}$, for all $\mathscr{V} \in M^{L^X}$ and $\mu \in L^Y$.

All algebraic operations on L can be extended pointwise to the sets L^X and M^{L^X} as follows: for all $x \in X, \lambda, \mu \in L^X$, and $\mathscr{U}, \mathscr{V} \in M^{L^X}$;

(1) $\lambda \le \mu$ iff $\lambda(x) \le \mu(x)$.

(2) $(\lambda \odot \mu)(x) = \lambda(x) \odot \mu(x)$.

(3) $\mathscr{U} \le \mathscr{V}$ iff $\mathscr{U}(\lambda) \le \mathscr{V}(\lambda)$.

Definition 8 (see [41]). The pair $(\mathscr{T}, \mathscr{T}^*)$ of maps $\mathscr{T}, \mathscr{T}^* : L^X \to M$ is called an (L, M)-double fuzzy topology on X if it satisfies the following conditions:

(LO1) $\mathscr{T}(\lambda) \le (\mathscr{T}^*(\lambda))^\star$, for each $\lambda \in L^X$,

(LO2) $\mathscr{T}(\underline{0}) = \mathscr{T}(\underline{1}) = 1_M$, $\mathscr{T}^*(\underline{0}) = \mathscr{T}^*(\underline{1}) = 0_M$,

(LO3) $\mathscr{T}(\lambda_1 \odot \lambda_2) \ge \mathscr{T}(\lambda_1) \tilde{\odot} \mathscr{T}(\lambda_2)$ and $\mathscr{T}^*(\lambda_1 \odot \lambda_2) \le \mathscr{T}^*(\lambda_1) \tilde{\oplus} \mathscr{T}^*(\lambda_2)$, for each $\lambda_1, \lambda_2 \in L^X$,

(LO4) $\mathscr{T}(\bigvee_{i\in\Delta}\lambda_i) \ge \bigwedge_{i\in\Delta}\mathscr{T}(\lambda_i)$ and $\mathscr{T}^*(\bigvee_{i\in\Delta}\lambda_i) \le \bigvee_{i\in\Delta}\mathscr{T}^*(\lambda_i)$, for each $\lambda_i \in L^X$, $i \in \Delta$.

The triplet $(X, \mathscr{T}, \mathscr{T}^*)$ is called an (L, M)-double fuzzy topological space ((L, M)-dfts, for short). $\mathscr{T}$ and $\mathscr{T}^*$ may be interpreted as gradation of openness and gradation of nonopenness, respectively.

Let $(\mathscr{U}, \mathscr{U}^*)$ and $(\mathscr{T}, \mathscr{T}^*)$ be (L, M)-double fuzzy topologies on X. We say that $(\mathscr{U}, \mathscr{U}^*)$ is finer than $(\mathscr{T}, \mathscr{T}^*)$ ($(\mathscr{T}, \mathscr{T}^*)$ is coarser than $(\mathscr{U}, \mathscr{U}^*)$) if $\mathscr{T}(\lambda) \le \mathscr{U}(\lambda)$ and $\mathscr{T}^*(\lambda) \ge \mathscr{U}^*(\lambda)$ for all $\lambda \in L^X$.

Let $(X, \mathscr{T}, \mathscr{T}^*)$ and $(Y, \mathscr{U}, \mathscr{U}^*)$ be (L, M)-dfts's. A function $\varphi : X \to Y$ is called LF-continuous iff $\mathscr{U}(\lambda) \le \mathscr{T}(\varphi^{\leftarrow}(\lambda))$ and $\mathscr{U}^*(\lambda) \ge \mathscr{T}^*(\varphi^{\leftarrow}(\lambda))$, for all $\lambda \in L^Y$.

Thus, we have the category $(\mathbf{L}, \mathbf{M})$-**DFTOP** where the objects are (L, M)-dfts's and the morphisms are LF-continuous maps between these spaces.

Example 9. Let $X = \{x, y\}$ be a set, $L = M = [0, 1]$ and $x \odot y = \max\{x + y - 1, 0\}$, $x \oplus y = \min\{x + y, 1\}$. Then, $([0, 1], \le, \odot)$ is a left-continuous t-norm (Lukasiewicz t-norm) with strong induced negation $x \mapsto 0 = \min\{1-x, 1\}$. Let $\mu, \rho \in [0, 1]^X$ be defined as follows: $\mu(x) = 0.6$, $\mu(y) = 0.3$, $\rho(x) = 0.5$, $\rho(y) = 0.7$. Define $\mathscr{T}, \mathscr{T}^\star : [0, 1]^X \to [0, 1]$ as follows:

$$\begin{aligned}\mathscr{T}(\lambda) &= \begin{cases} 1, & \text{if } \lambda = \underline{0}, \underline{1};\\ 0.8, & \text{if } \lambda = \mu;\\ 0.3, & \text{if } \lambda = \rho;\\ 0.7, & \text{if } \lambda = \mu \vee \rho;\\ 0.2, & \text{if } \lambda = \mu \wedge \rho;\\ 0, & \text{otherwise,}\end{cases}\\ \mathscr{T}^*(\lambda) &= \begin{cases} 0, & \text{if } \lambda = \underline{0}, \underline{1};\\ 0.2, & \text{if } \lambda = \mu;\\ 0.7, & \text{if } \lambda = \rho;\\ 0.3, & \text{if } \lambda = \mu \vee \rho;\\ 0.8, & \text{if } \lambda = \mu \wedge \rho;\\ 1, & \text{otherwise.}\end{cases}\end{aligned} \tag{16}$$

Then, the pair $(\mathscr{T}, \mathscr{T}^*)$ is a $([0, 1], [0, 1])$-dft on X.

Remark 10. (1) If $(L = M = [0, 1], \odot = \wedge, \oplus = \vee)$ with an order-reversing involution $\star$, $(a^\star = 1 - a)$ (L, M)-dfts is the concept of Mondal and Samanta [18].

(2) If L and M are frames with 0 and 1, (L, M)-dfts is the concept of Gutiérrez García and Rodabaugh [19].

(3) If $\odot = \wedge$, (L, M)-dfts is the concept of Abd El-latif [42].

Definition 11 (see [29, 30]). A map $\mathscr{F} : L^X \to L$ is called an L-filter if it fulfills the following conditions:

(LF1) $\mathscr{F}(\underline{0}) = 0_M$ and $\mathscr{F}(\underline{1}) = 1_M$.

(LF2) $\mathscr{F}(\lambda \odot \mu) \ge \mathscr{F}(\lambda) \odot \mathscr{F}(\mu)$, for each $\lambda, \mu \in L^X$.

(LF3) If $\lambda \le \mu$, then $\mathscr{F}(\lambda) \le \mathscr{F}(\mu)$.

The pair, $(X\mathscr{F})$, is called an L-filter space.

3. (L,M)-Double Fuzzy Filters and (L,M)-Double Fuzzy Filter Bases

Definition 12. The pair $(\mathscr{F}, \mathscr{F}^*)$ of maps $\mathscr{F}, \mathscr{F}^* : L^X \to M$ is called an (L, M)-double fuzzy filter (briefly, (L, M)-dff) on X if it fulfills the following axioms:

(DFF1) $\mathscr{F}(\lambda) \le (\mathscr{F}^*(\lambda))^\star$, for each $\lambda \in L^X$.

(DFF2) $\mathscr{F}(\underline{0}) = 0_M$, $\mathscr{F}(\underline{1}) = 1_M$ and $\mathscr{F}^*(\underline{0}) = 1_M$, $\mathscr{F}^*(\underline{1}) = 0_M$.

(DFF3) $\mathscr{F}(\lambda \odot \mu) \ge \mathscr{F}(\lambda) \tilde{\odot} \mathscr{F}(\mu)$ and $\mathscr{F}^*(\lambda \odot \mu) \le \mathscr{F}^*(\lambda) \tilde{\oplus} \mathscr{F}^*(\mu)$, for each $\lambda, \mu \in L^X$.

(DFF4) If $\lambda \le \mu$, then $\mathscr{F}(\lambda) \le \mathscr{F}(\mu)$ and $\mathscr{F}^*(\lambda) \ge \mathscr{F}^*(\mu)$.

The triplet $(X, \mathscr{F}, \mathscr{F}^*)$ is called an (L, M)-double fuzzy filter space (briefly, (L, M)-dffs).

If $(\mathscr{F}_1, \mathscr{F}_1^*)$ and $(\mathscr{F}_2, \mathscr{F}_2^*)$ are two (L, M)-dffs on X, we say that $(\mathscr{F}_1, \mathscr{F}_1^*)$ is finer than $(\mathscr{F}_2, \mathscr{F}_2^*)$ (or $(\mathscr{F}_2, \mathscr{F}_2^*)$ is coarser than $(\mathscr{F}_1, \mathscr{F}_1^*)$), denoted by $(\mathscr{F}_2, \mathscr{F}_2^*) \le (\mathscr{F}_1, \mathscr{F}_1^*)$ if and only if $\mathscr{F}_2(\lambda) \le \mathscr{F}_1(\lambda)$ and $\mathscr{F}_2^*(\lambda) \ge \mathscr{F}_1^*(\lambda)$, for each $\lambda \in L^X$.

Definition 13. Let $(X, \mathscr{F}_1, \mathscr{F}_1^*)$ and $(Y, \mathscr{F}_2, \mathscr{F}_2^*)$ be two (L, M)-dffs's. Then, a map $\varphi : X \to Y$ is said to be

(i) a filter map if and only if $\mathscr{F}_2 \le \mathscr{F}_1 \circ \varphi_L^{\leftarrow}$ and $\mathscr{F}_2^* \ge \mathscr{F}_1^* \circ \varphi_L^{\leftarrow}$;

(ii) a filter preserving map if and only if $\mathscr{F}_1 \le \mathscr{F}_2 \circ \varphi_L^{\rightarrow}$ and $\mathscr{F}_1^* \ge \mathscr{F}_2^* \circ \varphi_L^{\rightarrow}$.

Normally, the composition of filter maps (resp., filter preserving maps) is a filter map (resp., filter preserving map).

Hence, we get to the category $(\mathbf{L}, \mathbf{M})$-**DFIL** with objects of all the (L, M)-dffs's and the morphisms are filter maps between these spaces.

Remark 14. (i) Let $\mathscr{F}$ be an L-filter on X and $\mathscr{F}^* : L^X \to L$ defined by $\mathscr{F}^*(\lambda) = (\mathscr{F}(\lambda))^\star$. Then, the pair $(\mathscr{F}, \mathscr{F}^*)$ is an (L, L)-dff on X. Therefore, (L, M)-dff is a generalization of L-filter due to Höhle and Šostak [7, 29].

(ii) If $\odot = \wedge$, the definition of (L, M)-dff coincides with the definition of a proper (L, M)-intuitionistic fuzzy filter due to Abd El-latif [42].

Theorem 15. *Each (L, M)-dff $(X, \mathscr{F}, \mathscr{F}^*)$ produces an (L, M)-dfts $(X, \mathscr{T}_{\mathscr{F}}, \mathscr{T}^*_{\mathscr{F}^*})$.*

Proof. When contrasting the axioms of (L, M)-dff and (L, M)-dft, we find (DFF4) implying (DFT4).

Let $\{\lambda_i : i \in \Gamma\} \subseteq L^X$. Then, $\lambda_i \le \bigvee_{i\in\Gamma}\lambda_i$ for all $i \in \Gamma$; due to (DFF4), we have that $\mathscr{F}(\lambda_i) \le \mathscr{F}(\bigvee_{i\in\Gamma}\lambda_i)$ and $\mathscr{F}^*(\lambda_i) \ge \mathscr{F}^*(\bigvee_{i\in\Gamma}\lambda_i)$ for all $i \in \Gamma$. So,

$$\begin{aligned} \mathscr{F}\left(\bigvee_{i\in\Gamma}\lambda_i\right) &\ge \mathscr{F}(\lambda_i) \ge \bigwedge_{i\in\Gamma}\mathscr{F}(\lambda_i), \\ \mathscr{F}^*\left(\bigvee_{i\in\Gamma}\lambda_i\right) &\le \mathscr{F}^*(\lambda_i) \le \bigvee_{i\in\Gamma}\mathscr{F}^*(\lambda_i). \end{aligned} \tag{17}$$

Then, we can get an (L, M)-dft $(\mathscr{T}_{\mathscr{F}}, \mathscr{T}^*_{\mathscr{F}^*})$ defined by

$$\begin{aligned} \mathscr{T}_{\mathscr{F}}(\lambda) &= \begin{cases} \mathscr{F}(\lambda), & \text{if } \lambda \ne \underline{0} \\ 1_M, & \text{if } \lambda = \underline{0}, \end{cases} \\ \mathscr{T}^*_{\mathscr{F}^*}(\lambda) &= \begin{cases} \mathscr{F}^*(\lambda), & \text{if } \lambda \ne \underline{0} \\ 0_M, & \text{if } \lambda = \underline{0}. \end{cases} \end{aligned} \tag{18}$$

□

Theorem 16. *Let $(X, \mathscr{F}_1, \mathscr{F}_1^*)$ and $(Y, \mathscr{F}_2, \mathscr{F}_2^*)$ be (L, M)-dffs's. If $\varphi : (X, \mathscr{F}_1, \mathscr{F}_1^*) \to (Y, \mathscr{F}_2, \mathscr{F}_2^*)$ is a filter map, then $\varphi : (X, \mathscr{T}_{\mathscr{F}_1}, \mathscr{T}^*_{\mathscr{F}_1^*}) \to (Y, \mathscr{T}_{\mathscr{F}_2}, \mathscr{T}^*_{\mathscr{F}_2^*})$ is an LF-continuous map.*

Proof. Let $\mu \in L^Y$. If $\mu = 0_Y$ or $\varphi_L^{\leftarrow}(\mu) = 0_X$, then the proof is easy. Let $\mu \ne 0_Y$ and $\varphi_L^{\leftarrow}(\mu) \ne 0_X$. Then, from the definition of double filter map and Theorem 15, we have

$$\begin{aligned} \mathscr{T}_{\mathscr{F}_1}(\varphi_L^{\leftarrow}(\mu)) &= \mathscr{F}_1(\varphi_L^{\leftarrow}(\mu)) \ge \mathscr{F}_2(\mu) = \mathscr{T}_{\mathscr{F}_2}(\mu), \\ \mathscr{T}^*_{\mathscr{F}_1^*}(\varphi_L^{\leftarrow}(\mu)) &= \mathscr{F}_1^*(\varphi_L^{\leftarrow}(\mu)) \le \mathscr{F}_2^*(\mu) = \mathscr{T}^*_{\mathscr{F}_2^*}(\mu). \end{aligned} \tag{19}$$

□

Corollary 17. *The function F : $(\mathbf{L}, \mathbf{M})$-**DFIL** $\to$ $(\mathbf{L}, \mathbf{M})$-**DFTOP** defined by $F(X, \mathscr{F}, \mathscr{F}^*) = (X, \mathscr{T}_{\mathscr{F}}, \mathscr{T}^*_{\mathscr{F}^*})$ and $F(\varphi) = \varphi$ is a functor.*

Notation 18. Let $\mathscr{B}, \mathscr{B}^* : L^X \to M$ be two maps and $\lambda \in L^X$. Then, $\langle\mathscr{B}\rangle$ and $\langle\mathscr{B}^*\rangle$ are defined as follows:

$$\begin{aligned} \langle\mathscr{B}\rangle(\lambda) &= \bigvee_{\mu\le\lambda}\mathscr{B}(\mu), \\ \langle\mathscr{B}^*\rangle(\lambda) &= \bigwedge_{\mu\le\lambda}\mathscr{B}^*(\mu). \end{aligned} \tag{20}$$

Definition 19. The pair $(\mathscr{B}, \mathscr{B}^*)$ of maps $\mathscr{B}, \mathscr{B}^* : L^X \to M$ is called an (L, M)-double fuzzy filter base (briefly, (L, M)-dffb) on X if it fulfills the following axioms:

(DFFB1) $\mathscr{B}(\lambda) \le (\mathscr{B}^*(\lambda))^\star$, for each $\lambda \in L^X$.

(DFFB2) $\mathscr{B}(\underline{0}) = 0_M$, $\mathscr{B}(\underline{1}) = 1_M$ and $\mathscr{B}^*(\underline{0}) = 1_M$, $\mathscr{B}^*(\underline{1}) = 0_M$.

(DFFB3) $\langle\mathscr{B}\rangle(\lambda \odot \mu) \ge \mathscr{B}(\lambda) \tilde{\odot} \mathscr{B}(\mu)$ and $\langle\mathscr{B}^*\rangle(\lambda \odot \mu) \le \mathscr{B}^*(\lambda) \tilde{\oplus} \mathscr{B}^*(\mu)$, for each $\lambda, \mu \in L^X$.

If $(\mathscr{B}_1, \mathscr{B}_1^*)$ and $(\mathscr{B}_2, \mathscr{B}_2^*)$ are two (L, M)-dffb's on X, we say $(\mathscr{B}_1, \mathscr{B}_1^*)$ is finer than $(\mathscr{B}_2, \mathscr{B}_2^*)$ (or $(\mathscr{B}_2, \mathscr{B}_2^*)$ is coarser than $(\mathscr{B}_1, \mathscr{B}_1^*)$) denoted by $(\mathscr{B}_2, \mathscr{B}_2^*) \le (\mathscr{B}_1, \mathscr{B}_1^*)$ if and only if $\langle\mathscr{B}_2\rangle(\lambda) \le \langle\mathscr{B}_1\rangle(\lambda)$ and $\langle\mathscr{B}_2^*\rangle(\lambda) \ge \langle\mathscr{B}_1^*\rangle(\lambda)$, for each $\lambda \in L^X$.

Remark 20. (i) An (L, M)-dffb is a generalization of L-filter base due to Kim and Ko [40].

(ii) If $(\mathscr{F}, \mathscr{F}^*)$ is an (L, M)-dff, then $(\mathscr{F}, \mathscr{F}^*)$ is an (L, M)-dffb with $\langle\mathscr{F}\rangle = \mathscr{F}$ and $\langle\mathscr{F}^*\rangle = \mathscr{F}^*$.

(iii) If $(\mathscr{B}, \mathscr{B}^*)$ is an (L, M)-dffb, then, by (DFFB3), $\lambda \odot \mu = \underline{0}$ implies $\mathscr{B}(\lambda) \tilde{\odot} \mathscr{B}(\mu) = 0_M$ and $\mathscr{B}^*(\lambda) \tilde{\oplus} \mathscr{B}^*(\mu) = 1_M$.

Theorem 21. *If $(\mathscr{B}, \mathscr{B}^*)$ is an (L, M)-dffb, then $(\langle\mathscr{B}\rangle, \langle\mathscr{B}^*\rangle)$ is the coarsest (L, M)-dff which satisfies $\mathscr{B} \le \langle\mathscr{B}\rangle$ and $\mathscr{B}^* \ge \langle\mathscr{B}^*\rangle$.*

Proof. (DFF1) For each $\lambda \in L^X$,

$$\begin{aligned} \langle\mathscr{B}\rangle(\lambda) &= \bigvee_{\mu\le\lambda}\mathscr{B}(\mu) \le \bigvee_{\mu\le\lambda}\mathscr{B}^*(\mu)^\star \\ &= \left(\bigwedge_{\mu\le\lambda}\mathscr{B}^*(\mu)\right)^\star = (\langle\mathscr{B}^*\rangle(\lambda))^\star. \end{aligned} \tag{21}$$

(DFF2) and (DFF4) are easily checked.

(DFF3) Suppose that there exist $\lambda, \mu \in L^X$ such that

$$\langle\mathscr{B}^*\rangle(\lambda \odot \mu) \nleq \langle\mathscr{B}^*\rangle(\lambda) \tilde{\oplus} \langle\mathscr{B}^*\rangle(\mu). \tag{22}$$

By the definition of $\langle\mathscr{B}^*\rangle$ and (L4′), there exist $\lambda_1, \mu_1 \in L^X$ with $\lambda_1 \le \lambda$ and $\mu_1 \le \mu$ such that

$$\langle\mathscr{B}^*\rangle(\lambda\odot\mu) \nleq \mathscr{B}^*(\lambda_1)\,\tilde{\oplus}\,\mathscr{B}^*(\mu_1). \tag{23}$$

Since $(\mathscr{B},\mathscr{B}^*)$ is an (L,M)-dffb,

$$\langle\mathscr{B}^*\rangle(\lambda_1\odot\mu_1) \le \mathscr{B}^*(\lambda_1)\,\tilde{\oplus}\,\mathscr{B}^*(\mu_1). \tag{24}$$

Since $\lambda_1\odot\mu_1 \le \lambda\odot\mu$, we have

$$\begin{aligned}\langle\mathscr{B}^*\rangle(\lambda\odot\mu) &\le \langle\mathscr{B}^*\rangle(\lambda_1\odot\mu_1)\\ &\le \mathscr{B}^*(\lambda_1)\,\tilde{\oplus}\,\mathscr{B}^*(\mu_1).\end{aligned} \tag{25}$$

It is a contradiction. Thus, $\langle\mathscr{B}^*\rangle(\lambda\odot\mu) \le \langle\mathscr{B}^*\rangle(\lambda)\,\tilde{\oplus}\,\langle\mathscr{B}^*\rangle(\mu)$, for each $\lambda,\mu \in L^X$. Similarly, $\langle\mathscr{B}\rangle(\lambda\odot\mu) \ge \langle\mathscr{B}\rangle(\lambda)\,\tilde{\odot}\,\langle\mathscr{B}\rangle(\mu)$, for each $\lambda,\mu\in L^X$.

Let $(\mathscr{F},\mathscr{F}^*)$ be another (L,M)-dff which is finer than $(\mathscr{B},\mathscr{B}^*)$, that is, $\mathscr{B}\le\mathscr{F}$ and $\mathscr{B}^*\ge\mathscr{F}^*$. Then, we have

$$\begin{aligned}\langle\mathscr{B}\rangle(\lambda) &= \bigvee_{\mu\le\lambda}\mathscr{B}(\mu) \le \bigvee_{\mu\le\lambda}\mathscr{F}(\mu) = \mathscr{F}(\lambda),\\ \langle\mathscr{B}^*\rangle(\lambda) &= \bigwedge_{\mu\le\lambda}\mathscr{B}^*(\mu) \ge \bigwedge_{\mu\le\lambda}\mathscr{F}^*(\mu) = \mathscr{F}^*(\lambda).\end{aligned} \tag{26}$$

□

Theorem 22. *$\mathscr{H},\mathscr{H}^* : L^X \to M$ are maps fulfilling the following conditions:*

(C1) $\mathscr{H}(\lambda) \le (\mathscr{H}^*(\lambda))^\star$, for each $\lambda\in L^X$,

(C2) $\mathscr{H}(\underline{1}) = 1_M$ and $\mathscr{H}^*(\underline{1}) = 0_M$ and for each finite index set K, if $\bigodot_{i\in K}\lambda_i = \underline{0}$, then $\widetilde{\bigodot}_{i\in K}\mathscr{H}(\lambda_i) = 0_M$ and $\widetilde{\bigoplus}_{i\in K}\mathscr{H}^*(\lambda_i) = 1_M$.

We define the maps $\mathscr{B}_{\mathscr{H}},\mathscr{B}^*_{\mathscr{H}^*} : L^X \to M$ as

$$\begin{aligned}\mathscr{B}_{\mathscr{H}}(\lambda) &= \bigvee\left\{\widetilde{\bigodot_{i\in K}}\mathscr{H}(\lambda_i) : \lambda = \bigodot_{i\in K}\lambda_i\right\},\\ \mathscr{B}^*_{\mathscr{H}^*}(\lambda) &= \bigwedge\left\{\widetilde{\bigoplus_{i\in K}}\mathscr{H}^*(\lambda_i) : \lambda = \bigodot_{i\in K}\lambda_i\right\},\end{aligned} \tag{27}$$

where $\bigvee$ and $\bigwedge$ are taken for every finite index set K such that $\lambda = \bigodot_{i\in K}\lambda_i$, respectively. Then, the following properties are satisfied:

(i) $(\mathscr{B}_{\mathscr{H}},\mathscr{B}^*_{\mathscr{H}^*})$ is an (L,M)-dffb on X.

(ii) If $\mathscr{H}\le\mathscr{B}$, $\mathscr{H}^*\ge\mathscr{B}^*$ and $(\mathscr{B},\mathscr{B}^*)$ is an (L,M)-dffb on X, then $\langle\mathscr{B}_{\mathscr{H}}\rangle \le \langle\mathscr{B}\rangle$ and $\langle\mathscr{B}^*_{\mathscr{H}^*}\rangle \ge \langle\mathscr{B}^*\rangle$.

Proof. (i) (DFFB1) For each $\lambda\in L^X$, the following is valid:

$$\begin{aligned}\mathscr{B}_{\mathscr{H}}(\lambda) &= \bigvee\left\{\widetilde{\bigodot_{i\in K}}\mathscr{H}(\lambda_i) : \lambda = \bigodot_{i\in K}\lambda_i\right\}\\ &\le \bigvee\left\{\widetilde{\bigodot_{i\in K}}(\mathscr{H}^*(\lambda_i))^\star : \lambda = \bigodot_{i\in K}\lambda_i\right\}\\ &= \left(\bigwedge\left\{\widetilde{\bigoplus_{i\in K}}\mathscr{H}^*(\lambda_i) : \lambda = \bigodot_{i\in K}\lambda_i\right\}\right)^\star\\ &= (\mathscr{B}^*_{\mathscr{H}^*}(\lambda))^\star.\end{aligned} \tag{28}$$

(DFFB2) It is clear by condition (C2).

(DFFB3) For each $\lambda,\mu\in L^X$ and for any two finite index sets K,J with $\lambda = \bigodot_{k\in K}\lambda_k$ and $\mu = \bigodot_{j\in J}\mu_i$, since $\lambda\odot\mu = (\bigodot_{k\in K}\lambda_k)\odot(\bigodot_{j\in J}\mu_i)$, by the definition of $\mathscr{B}_{\mathscr{H}}$ and $\mathscr{B}^*_{\mathscr{H}^*}$, we get

$$\begin{aligned}\langle\mathscr{B}_{\mathscr{H}}\rangle(\lambda\odot\mu) &\ge \mathscr{B}_{\mathscr{H}}(\lambda\odot\mu)\\ &\ge \left(\widetilde{\bigodot_{k\in K}}\mathscr{H}(\lambda_k)\right)\\ &\quad\tilde{\odot}\left(\widetilde{\bigodot_{j\in J}}\mathscr{H}(\mu_j)\right),\\ \langle\mathscr{B}^*_{\mathscr{H}^*}\rangle(\lambda\odot\mu) &\le \mathscr{B}^*_{\mathscr{H}^*}(\lambda\odot\mu)\\ &\le \left(\widetilde{\bigoplus_{k\in K}}\mathscr{H}^*(\lambda_k)\right)\\ &\quad\tilde{\oplus}\left(\widetilde{\bigoplus_{j\in J}}\mathscr{H}^*(\mu_j)\right).\end{aligned} \tag{29}$$

If supremum and infimum are taken over finite index set K, respectively, then by (2) and (3),

$$\begin{aligned}\langle\mathscr{B}_{\mathscr{H}}\rangle(\lambda\odot\mu) &\ge \mathscr{B}_{\mathscr{H}}(\lambda)\,\tilde{\odot}\,\mathscr{B}_{\mathscr{H}}(\mu),\\ \langle\mathscr{B}^*_{\mathscr{H}^*}\rangle(\lambda\odot\mu) &\le \mathscr{B}^*_{\mathscr{H}^*}(\lambda)\,\tilde{\oplus}\,\mathscr{B}^*_{\mathscr{H}^*}(\mu).\end{aligned} \tag{30}$$

Thus, $(\mathscr{B}_{\mathscr{H}},\mathscr{B}^*_{\mathscr{H}^*})$ is an (L,M)-dffb on X.

(ii) For any finite family $\{\lambda_i : \bigodot_{i\in K}\lambda_i \le \lambda\}$, the following are true:

$$\begin{aligned}\langle\mathscr{B}\rangle(\lambda) &\ge \langle\mathscr{B}\rangle\left(\bigodot_{i\in K}\lambda_i\right) \ge \widetilde{\bigodot_{i\in K}}\langle\mathscr{B}\rangle(\lambda_i)\\ &\ge \widetilde{\bigodot_{i\in K}}\mathscr{B}(\lambda_i) \ge \widetilde{\bigodot_{i\in K}}\mathscr{H}(\lambda_i),\\ \langle\mathscr{B}^*\rangle(\lambda) &\le \langle\mathscr{B}^*\rangle\left(\bigodot_{i\in K}\lambda_i\right) \le \widetilde{\bigoplus_{i\in K}}\langle\mathscr{B}^*\rangle(\lambda_i)\\ &\le \widetilde{\bigoplus_{i\in K}}\mathscr{B}^*(\lambda_i) \le \widetilde{\bigoplus_{i\in K}}\mathscr{H}^*(\lambda_i).\end{aligned} \tag{31}$$

Then, $\langle\mathscr{B}_{\mathscr{H}}\rangle \le \langle\mathscr{B}\rangle$ and $\langle\mathscr{B}^*_{\mathscr{H}^*}\rangle \ge \langle\mathscr{B}^*\rangle$. □

Theorem 23. *Let $(\mathscr{B}_1,\mathscr{B}^*_1)$ and $(\mathscr{B}_2,\mathscr{B}^*_2)$ be two (L,M)-dffb's on X and Y, respectively, and $\varphi : X\to Y$ be a function. Then, one has the following properties:*

(i) $\varphi : (X,\langle\mathscr{B}_1\rangle,\langle\mathscr{B}^*_1\rangle) \to (Y,\langle\mathscr{B}_2\rangle,\langle\mathscr{B}^*_2\rangle)$ is a filter map if and only if $\mathscr{B}_2 \le \langle\mathscr{B}_1\rangle\circ\varphi_L^{\leftarrow}$ and $\mathscr{B}^*_2 \ge \langle\mathscr{B}^*_1\rangle\circ\varphi_L^{\leftarrow}$.

(ii) $\varphi : (X,\langle\mathscr{B}_1\rangle,\langle\mathscr{B}^*_1\rangle) \to (Y,\langle\mathscr{B}_2\rangle,\langle\mathscr{B}^*_2\rangle)$ is a filter preserving map if and only if $\mathscr{B}_1 \le \langle\mathscr{B}_2\rangle\circ\varphi_L^{\rightarrow}$ and $\mathscr{B}^*_1 \ge \langle\mathscr{B}^*_2\rangle\circ\varphi_L^{\rightarrow}$.

(iii) If $\mathscr{B}_2 \le \mathscr{B}_1\circ\varphi_L^{\leftarrow}$ and $\mathscr{B}^*_2 \ge \mathscr{B}^*_1\circ\varphi_L^{\leftarrow}$, then $\varphi : (X,\langle\mathscr{B}_1\rangle,\langle\mathscr{B}^*_1\rangle) \to (Y,\langle\mathscr{B}_2\rangle,\langle\mathscr{B}^*_2\rangle)$ is a filter map.

(iv) If $\mathscr{B}_1 \le \mathscr{B}_2\circ\varphi_L^{\rightarrow}$ and $\mathscr{B}^*_1 \ge \mathscr{B}^*_2\circ\varphi_L^{\rightarrow}$, then $\varphi : (X,\langle\mathscr{B}_1\rangle,\langle\mathscr{B}^*_1\rangle) \to (Y,\langle\mathscr{B}_2\rangle,\langle\mathscr{B}^*_2\rangle)$ is a filter preserving map.

Proof. Proving condition (i) is enough since the other conditions are similarly proved.

(i) ($\Rightarrow$:) Since $\mathscr{B}_2(\mu) \leq \langle\mathscr{B}_2\rangle(\mu)$ and $\mathscr{B}_2^*(\mu) \geq \langle\mathscr{B}_2^*\rangle(\mu)$, for each $\mu \in L^Y$, it is trivial.

($\Leftarrow$:) Let $\mathscr{B}_2(\mu) \leq \langle\mathscr{B}_1\rangle(\varphi_L^{\leftarrow}(\mu))$ and $\mathscr{B}_2^*(\mu) \geq \langle\mathscr{B}_1^*\rangle(\varphi_L^{\leftarrow}(\mu))$, for each $\mu \in L^Y$. We will show that φ is a filter map. For arbitrary $\mu \in L^Y$, we have

$$\begin{aligned}\langle\mathscr{B}_2\rangle(\mu) &= \bigvee_{\nu\leq\mu}\mathscr{B}_2(\nu) \leq \bigvee_{\varphi_L^{\leftarrow}(\nu)\leq\varphi_L^{\leftarrow}(\mu)}\langle\mathscr{B}_1\rangle(\varphi_L^{\leftarrow}(\nu)) \\ &\leq \langle\mathscr{B}_1\rangle(\varphi_L^{\leftarrow}(\mu)), \\ \langle\mathscr{B}_2^*\rangle(\mu) &= \bigwedge_{\nu\leq\mu}\mathscr{B}_2^*(\nu) \geq \bigwedge_{\varphi_L^{\leftarrow}(\nu)\leq\varphi_L^{\leftarrow}(\mu)}\langle\mathscr{B}_1^*\rangle(\varphi_L^{\leftarrow}(\nu)) \\ &\geq \langle\mathscr{B}_1^*\rangle(\varphi_L^{\leftarrow}(\mu)).\end{aligned} \tag{32}$$

Thus, φ is a filter map. □

Example 24. Let $X = \{x, y\}$ be a set, $L = M = [0,1]$ be the stsc-quantale with Lukasiewicz t-norm, and $\mu, \nu \in [0,1]^X$ be defined by $\mu(x) = 0.6$, $\mu(y) = 0.5$, $\nu(x) = 0.1$, $\nu(y) = 0$. Define the maps $\mathscr{B}_i, \mathscr{B}_i^* : L^X \to M$ $i = 1, 2, 3$ as follows:

$$\mathscr{B}_1(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{1}; \\ 0.6, & \text{if } \lambda = \mu; \\ 0.3, & \text{if } \lambda = \nu; \\ 0, & \text{otherwise,}\end{cases}$$

$$\mathscr{B}_1^*(\lambda) = \begin{cases} 0, & \text{if } \lambda = \underline{1}; \\ 0.4, & \text{if } \lambda = \mu; \\ 0.7, & \text{if } \lambda = \nu; \\ 1, & \text{otherwise,}\end{cases}$$

$$\mathscr{B}_2(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{1}; \\ 0.6, & \text{if } \lambda = \mu; \\ 0.5, & \text{if } \lambda = \mu\odot\mu; \\ 0, & \text{otherwise,}\end{cases}$$

$$\mathscr{B}_2^*(\lambda) = \begin{cases} 0, & \text{if } \lambda = \underline{1}; \\ 0.4, & \text{if } \lambda = \mu; \\ 0.5, & \text{if } \lambda = \mu\odot\mu; \\ 1, & \text{otherwise,}\end{cases}$$

$$\mathscr{B}_3(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{1}; \\ 0.4, & \text{if } \lambda = \mu; \\ 0.3, & \text{if } \lambda = \mu\odot\mu; \\ 0, & \text{otherwise,}\end{cases}$$

$$\mathscr{B}_3^*(\lambda) = \begin{cases} 0, & \text{if } \lambda = \underline{1}; \\ 0.6, & \text{if } \lambda = \mu; \\ 0.7, & \text{if } \lambda = \mu\odot\mu; \\ 1, & \text{otherwise.}\end{cases} \tag{33}$$

It can be seen by easy computation that

(1) $(\mathscr{B}_1, \mathscr{B}_1^*)$ and $(\mathscr{B}_3, \mathscr{B}_3^*)$ are not (L,M)-double fuzzy filters but they are (L,M)-double fuzzy filter bases, so they generate (L,M)-double fuzzy filters $(\langle\mathscr{B}_1\rangle, \langle\mathscr{B}_1^*\rangle)$ and $(\langle\mathscr{B}_3\rangle, \langle\mathscr{B}_3^*\rangle)$.

(2) $(\mathscr{B}_2, \mathscr{B}_2^*)$ is not an (L,M)-double fuzzy filter base and it does not satisfy condition (C2) of Theorem 22.

(3) Since $(\langle\mathscr{B}_3\rangle, \langle\mathscr{B}_3^*\rangle) \leq (\langle\mathscr{B}_1\rangle, \langle\mathscr{B}_1^*\rangle)$, $id_X : (X, \langle\mathscr{B}_1\rangle, \langle\mathscr{B}_1^*\rangle) \to (X, \langle\mathscr{B}_3\rangle, \langle\mathscr{B}_3^*\rangle)$ is a filter map and $id_X : (X, \langle\mathscr{B}_3\rangle, \langle\mathscr{B}_3^*\rangle) \to (X, \langle\mathscr{B}_1\rangle, \langle\mathscr{B}_1^*\rangle)$ is a filter preserving map though $\mathscr{B}_3 \nleq \mathscr{B}_1, \mathscr{B}_3^* \ngeq \mathscr{B}_1^*$ and $\mathscr{B}_1 \nleq \mathscr{B}_3, \mathscr{B}_1^* \ngeq \mathscr{B}_3^*$.

We also note that if $L = M = [0,1]$ is considered as a frame, then $(\mathscr{B}_1, \mathscr{B}_1^*)$ is an (L,M)-double fuzzy filter base.

4. The Types $(\varphi_1^{\Leftarrow}, \varphi_1^{*\Leftarrow})$, $(\varphi_2^{\Rightarrow}, \varphi_2^{\Rightarrow})$ of Preimages and Images of (L,M)-Double Fuzzy Filter Bases

Theorem 25. *Let $\varphi : X \to Y$ be a function and $(\mathscr{B}, \mathscr{B}^*)$ be an (L,M)-dffb on Y. Then, the following properties are satisfied.*

(i) If $\varphi_L^{\leftarrow}(\mu) = \underline{0}$ implies $\mathscr{B}(\mu) = 0_M$ and $\mathscr{B}^*(\mu) = 1_M$, then $(\varphi_1^{\Leftarrow}(\mathscr{B}), \varphi_1^{*\Leftarrow}(\mathscr{B}^*))$ is an (L,M)-dffb on X and $(\langle\phi_1^{\Leftarrow}(\mathscr{B})\rangle, \langle\varphi_1^{*\Leftarrow}(\mathscr{B}^*)\rangle)$ is the coarsest (L,M)-dff on X for which $\varphi : (X, \langle\phi_1^{\Leftarrow}(\mathscr{B})\rangle, \langle\varphi_1^{*\Leftarrow}(\mathscr{B}^*)\rangle) \to (Y, \langle\mathscr{B}\rangle, \langle\mathscr{B}^*\rangle)$ is a filter map.

(ii) If φ is surjective, then $(\varphi_1^{\Leftarrow}(\mathscr{B}), \varphi_1^{*\Leftarrow}(\mathscr{B}^*))$ is an (L,M)-dffb.

(iii) If $\varphi_L^{\leftarrow}(\mu) = \underline{0}$ implies $\mathscr{B}(\mu) = 0_M$ and $\mathscr{B}^*(\mu) = 1_M$, φ is injective, and $(\mathscr{B}, \mathscr{B}^*)$ is an (L,M)-dff on Y, then $(\varphi_1^{\Leftarrow}(\mathscr{B}), \varphi_1^{*\Leftarrow}(\mathscr{B}^*))$ is an (L,M)-dff on X.

Proof. (i) (DFFB1) For each $\lambda \in L^X$, we have

$$\begin{aligned}\varphi_1^{\Leftarrow}(\mathscr{B})(\lambda) &= \bigvee\{\mathscr{B}(\mu) : \lambda = \varphi_L^{\leftarrow}(\mu)\} \\ &\leq \bigvee\left\{(\mathscr{B}^*(\mu))^\star : \lambda = \varphi_L^{\leftarrow}(\mu)\right\} \\ &= \left(\bigwedge\{\mathscr{B}^*(\mu) : \lambda = \varphi_L^{\leftarrow}(\mu)\}\right)^\star \\ &= (\varphi_1^{*\Leftarrow}(\mathscr{B}^*)(\lambda))^\star.\end{aligned} \tag{34}$$

(DFFB2) Since $\varphi_L^{\leftarrow}(\underline{1}) = \underline{1}$, then $\varphi_1^{\Leftarrow}(\mathscr{B})(\underline{1}) = 1_M$ and $\varphi_1^{*\Leftarrow}(\mathscr{B}^*)(\underline{1}) = 0_M$. By assumption, $\varphi_1^{\Leftarrow}(\mathscr{B})(\underline{0}) = 0_M$ and $\varphi_1^{*\Leftarrow}(\mathscr{B}^*)(\underline{0}) = 1_M$.

(DFFB3) Suppose that there exist $\lambda_1, \lambda_2 \in L^X$ such that

$$\begin{aligned}&\langle\varphi_1^{*\Leftarrow}(\mathscr{B}^*)\rangle(\lambda_1\odot\lambda_2) \\ &\quad\nleq \varphi_1^{*\Leftarrow}(\mathscr{B}^*)(\lambda_1) \,\tilde{\oplus}\, \varphi_1^{*\Leftarrow}(\mathscr{B}^*)(\lambda_2).\end{aligned} \tag{35}$$

By the definition of $\varphi_1^{*\Leftarrow}(\mathscr{B}^*)$ and (L4′), there exist $\mu_1, \mu_2 \in L^Y$ with $\lambda_1 = \varphi_L^{\leftarrow}(\mu_1)$ and $\lambda_2 = \varphi_L^{\leftarrow}(\mu_2)$ such that

$$\langle \varphi_1^{*\Leftarrow}(\mathscr{B}^*)\rangle(\lambda_1 \odot \lambda_2) \nleq \mathscr{B}^*(\mu_1) \tilde{\oplus} \mathscr{B}^*(\mu_2). \tag{36}$$

Since $(\mathscr{B}, \mathscr{B}^*)$ is an (L, M)-dffb, the following is valid:

$$\langle \mathscr{B}^*\rangle(\mu_1 \odot \mu_2) \leq \mathscr{B}^*(\mu_1) \tilde{\oplus} \mathscr{B}^*(\mu_2). \tag{37}$$

Then,

$$\langle \varphi_1^{*\Leftarrow}(\mathscr{B}^*)\rangle(\lambda_1 \odot \lambda_2) \nleq \langle \mathscr{B}^*\rangle(\mu_1 \odot \mu_2). \tag{38}$$

By the definition of $\langle \mathscr{B}^*\rangle$, there exists $\nu \in L^Y$ with $\nu \leq \mu_1 \odot \mu_2$ such that

$$\langle \varphi_1^{*\Leftarrow}(\mathscr{B}^*)\rangle(\lambda_1 \odot \lambda_2) \nleq \mathscr{B}^*(\nu). \tag{39}$$

On the other hand, since

$$\begin{aligned} \lambda_1 \odot \lambda_2 &= \varphi_L^{\leftarrow}(\mu_1) \odot \varphi_L^{\leftarrow}(\mu_2) = \varphi_L^{\leftarrow}(\mu_1 \odot \mu_2) \\ &\geq \varphi_L^{\leftarrow}(\nu), \end{aligned} \tag{40}$$

then $\langle \varphi_1^{*\Leftarrow}(\mathscr{B}^*)\rangle(\lambda_1 \odot \lambda_2) \leq \varphi_1^{*\Leftarrow}(\mathscr{B}^*)(\varphi_L^{\leftarrow}(\nu)) \leq \mathscr{B}^*(\nu)$.

This contradicts the assumption. Thus,

$$\begin{aligned} &\langle \varphi_1^{*\Leftarrow}(\mathscr{B}^*)\rangle(\lambda_1 \odot \lambda_2) \\ &\quad \leq \varphi_1^{*\Leftarrow}(\mathscr{B}^*)(\lambda_1) \tilde{\oplus} \varphi_1^{*\Leftarrow}(\mathscr{B}^*)(\lambda_2), \\ &\qquad\qquad \text{for each } \lambda_1, \lambda_2 \in L^X. \end{aligned} \tag{41}$$

Similarly, $\langle \varphi_1^{\Leftarrow}(\mathscr{B})\rangle(\lambda_1 \odot \lambda_2) \geq \varphi_1^{\Leftarrow}(\mathscr{B})(\lambda_1) \tilde{\odot} \varphi_1^{\Leftarrow}(\mathscr{B})(\lambda_2)$, for each $\lambda_1, \lambda_2 \in L^X$.

Hence, $(\varphi_1^{\Leftarrow}(\mathscr{B}), \varphi_1^{*\Leftarrow}(\mathscr{B}^*))$ is an (L, M)-dffb on X.

Let $(\mathscr{F}, \mathscr{F}^*)$ be another (L, M)-dff on X such that $\varphi : (X, \mathscr{F}, \mathscr{F}^*) \to (Y, \langle \mathscr{B}\rangle, \langle \mathscr{B}^*\rangle)$ is a filter map. Then, for each $\lambda \in L^X$, the following inequalities are valid:

$$\begin{aligned} \langle \varphi_1^{\Leftarrow}(\mathscr{B})\rangle(\lambda) &= \bigvee\{\mathscr{B}(\mu) : \varphi_L^{\leftarrow}(\mu) \leq \lambda\} \\ &\leq \bigvee\{\mathscr{F}(\varphi_L^{\leftarrow}(\mu)) : \varphi_L^{\leftarrow}(\mu) \leq \lambda\} \\ &\leq \mathscr{F}(\lambda), \\ \langle \varphi_1^{*\Leftarrow}(\mathscr{B}^*)\rangle(\lambda) &= \bigwedge\{\mathscr{B}^*(\mu) : \varphi_L^{\leftarrow}(\mu) \leq \lambda\} \\ &\geq \bigwedge\{\mathscr{F}^*(\varphi_L^{\leftarrow}(\mu)) : \varphi_L^{\leftarrow}(\mu) \leq \lambda\} \\ &\geq \mathscr{F}^*(\lambda). \end{aligned} \tag{42}$$

(ii) Since φ is surjective, $\varphi_L^{\leftarrow}(\mu) = \underline{0}$ implies $\mu = \underline{0}$. So, $\mathscr{B}(\mu) = 0_M$ and $\mathscr{B}^*(\mu) = 1_M$. Then, by (i), $(\varphi_1^{\Leftarrow}(\mathscr{B}), \varphi_1^{*\Leftarrow}(\mathscr{B}^*))$ is an (L, M)-dffb on X.

(iii) (DFF1)–(DFF3) are obvious.

(DFF4) Let $\lambda_1 \leq \lambda_2$, for $\lambda_1, \lambda_2 \in L^X$. Since φ is injective, there exists $\nu \in L^Y$ with $\varphi_L^{\leftarrow}(\nu) = \lambda_1$ and $\lambda_2 = \varphi_L^{\leftarrow}(\nu \vee \varphi_L^{\rightarrow}(\lambda_2))$. It implies

$$\begin{aligned} \varphi_1^{\Leftarrow}(\mathscr{B})(\lambda_2) &\geq \mathscr{B}(\nu \vee \varphi_L^{\rightarrow}(\lambda_2)) \geq \mathscr{B}(\nu), \\ \varphi_1^{*\Leftarrow}(\mathscr{B}^*)(\lambda_2) &\leq \mathscr{B}^*(\nu \vee \varphi_L^{\rightarrow}(\lambda_2)) \leq \mathscr{B}^*(\nu). \end{aligned} \tag{43}$$

If supremum and infimum are taken over $\{\lambda_1 \mid \varphi_L^{\leftarrow}(\nu) = \lambda_1\}$, respectively, then it is clear that $\varphi_1^{\Leftarrow}(\mathscr{B})(\lambda_2) \geq \varphi_1^{\Leftarrow}(\mathscr{B})(\lambda_1)$ and $\varphi_1^{*\Leftarrow}(\mathscr{B}^*)(\lambda_2) \leq \varphi_1^{*\Leftarrow}(\mathscr{B}^*)(\lambda_1)$. □

Theorem 26. *Let $\{\varphi_i : X \to X_i\}_{i\in\Gamma}$ be a family of functions and $\{(\mathscr{B}_i, \mathscr{B}_i^*)\}_{i\in\Gamma}$ be a family of (L, M)-dffb's on X_i satisfying the following condition:*

(C) For each finite subset K of Γ, if $\bigodot_{i\in K}(\lambda_i \circ \varphi_i) = \underline{0}$, then $\bigodot_{i\in K}\mathscr{B}_i(\lambda_i) = 0_M$ and $\bigoplus_{i\in K}\mathscr{B}_i^*(\lambda_i) = 1_M$.

We define the maps $\bigsqcup_{i\in\Gamma}(\varphi_i)_1^{\Leftarrow}(\mathscr{B}_i), \bigsqcap_{i\in\Gamma}(\varphi_i^*)_1^{\Leftarrow}(\mathscr{B}_i^*) : L^X \to M$ as

$$\begin{aligned} &\bigsqcup_{i\in\Gamma}(\varphi_i)_1^{\Leftarrow}(\mathscr{B}_i)(\lambda) \\ &\quad = \bigvee\left\{\widetilde{\bigodot_{i\in K}}\mathscr{B}_i(\lambda_i) : \lambda = \bigodot_{i\in K}(\lambda_i \circ \varphi_i)\right\}, \\ &\bigsqcap_{i\in\Gamma}(\varphi_i^*)_1^{\Leftarrow}(\mathscr{B}_i^*)(\lambda) \\ &\quad = \bigwedge\left\{\widetilde{\bigoplus_{i\in K}}\mathscr{B}_i^*(\lambda_i) : \lambda = \bigodot_{i\in K}(\lambda_i \circ \varphi_i)\right\}, \end{aligned} \tag{44}$$

where $\bigvee$ and $\bigwedge$ are taken for every finite index subset K of Γ such that $\lambda = \bigodot_{i\in K}(\lambda_i \circ \varphi_i)$.

Let $\mathscr{B} = \bigsqcup_{i\in\Gamma}(\varphi_i)_1^{\Leftarrow}(\mathscr{B}_i)$ and $\mathscr{B}^* = \bigsqcap_{i\in\Gamma}(\varphi_i^*)_1^{\Leftarrow}(\mathscr{B}_i^*)$ be given. Then, the following properties are satisfied.

(i) $(\mathscr{B}, \mathscr{B}^*)$ is an (L, M)-dffb on X and $(\langle \mathscr{B}\rangle, \langle \mathscr{B}^*\rangle)$ is the coarsest (L, M)-dff for which $\varphi_i : (X, \langle \mathscr{B}\rangle, \langle \mathscr{B}^*\rangle) \to (X_i, \langle \mathscr{B}_i\rangle, \langle \mathscr{B}_i^*\rangle)$ is a filter map for each $i \in \Gamma$.

(ii) A function $\varphi : (Y, \mathscr{F}, \mathscr{F}^*) \to (X, \langle \mathscr{B}\rangle, \langle \mathscr{B}^*\rangle)$ is a filter map if and only if, for each $i \in \Gamma$, $\varphi_i \circ \varphi : (Y, \mathscr{F}, \mathscr{F}^*) \to (X_i, \langle \mathscr{B}_i\rangle, \langle \mathscr{B}_i^*\rangle)$ is a filter map.

(iii) $\langle \bigsqcup_{i\in\Gamma}(\varphi_i)_1^{\Leftarrow}(\mathscr{B}_i)\rangle = \langle \bigsqcup_{i\in\Gamma}(\varphi_i)_1^{\Leftarrow}(\langle \mathscr{B}_i\rangle)\rangle$ and $\langle \bigsqcap_{i\in\Gamma}(\varphi_i^*)_1^{\Leftarrow}(\mathscr{B}_i^*)\rangle = \langle \bigsqcap_{i\in\Gamma}(\varphi_i^*)_1^{\Leftarrow}(\langle \mathscr{B}_i^*\rangle)\rangle$.

Proof. (i) (DFFB1) Let $\lambda \in L^X$ with $\lambda = \bigodot_{i\in K}(\lambda_i \circ \varphi_i)$. Then, the following inequality is valid:

$$\begin{aligned} \mathscr{B}(\lambda) &= \bigvee\left\{\widetilde{\bigodot_{i\in K}}\mathscr{B}_i(\lambda_i) : \lambda = \bigodot_{i\in K}(\lambda_i \circ \varphi_i)\right\} \\ &\leq \bigvee\left\{\widetilde{\bigodot_{i\in K}}(\mathscr{B}_i^*(\lambda_i))^\star : \lambda = \bigodot_{i\in K}(\lambda_i \circ \varphi_i)\right\} \\ &= \left(\bigwedge\left\{\widetilde{\bigoplus_{i\in K}}\mathscr{B}_i^*(\lambda_i) : \lambda = \bigodot_{i\in K}(\lambda_i \circ \varphi_i)\right\}\right)^\star \\ &= (\mathscr{B}^*(\lambda))^\star. \end{aligned} \tag{45}$$

(DFFB2) By condition (C), $\mathscr{B}(\underline{0}) = 0_M$ and $\mathscr{B}^*(\underline{0}) = 1_M$. Since, $\underline{1} = \underline{1} \circ \varphi_i$, then $\mathscr{B}(\underline{1}) = 1_M$ and $\mathscr{B}^*(\underline{1}) = 0_M$.

(DFFB3) Suppose that there exist $\lambda, \mu \in L^X$ such that $\langle \mathscr{B}^*\rangle(\lambda \odot \mu) \nleq \mathscr{B}^*(\lambda) \tilde{\oplus} \mathscr{B}^*(\mu)$.

By the definition of $\mathscr{B}^*(\lambda)$, and (L4′), there exists a finite subset K of Γ with $\lambda = \bigodot_{k\in K}(\lambda_k \circ \varphi_k)$ such that $\langle \mathscr{B}^*\rangle(\lambda \odot \mu) \nleq (\widetilde{\bigoplus_{k\in K}}\mathscr{B}_k^*(\lambda_k)) \tilde{\oplus} \mathscr{B}(\mu)$.

Again, by the definition of $\mathscr{B}^*(\mu)$ and (L4′), there exists a finite subset J of Γ with $\mu = \bigodot_{j\in J}(\mu_j \circ \varphi_j)$ such that

$$\langle \mathscr{B}^* \rangle (\lambda \odot \mu) \nleq \left(\widetilde{\bigoplus_{k\in K}} \mathscr{B}_k^* (\lambda_k) \right) \widetilde{\oplus} \left(\widetilde{\bigoplus_{j\in J}} \mathscr{B}_j^* (\mu_j) \right). \tag{46}$$

Put $m \in (K \cup J)$ such that

$$\rho_m = \begin{cases} \lambda_m, & \text{if } m \in K \setminus (K \cap J) \\ \mu_m, & \text{if } m \in J \setminus (K \cap J) \\ \lambda_m \odot \mu_m, & \text{if } m \in K \cap J. \end{cases} \tag{47}$$

Since, for each $m \in K \cap J$, $\langle \mathscr{B}_m^* \rangle (\lambda_m \odot \mu_m) \leq \mathscr{B}_m^*(\lambda_m) \widetilde{\oplus} \mathscr{B}_m^*(\mu_m)$, we have

$$\langle \mathscr{B}^* \rangle (\lambda \odot \mu) \nleq \left(\widetilde{\bigoplus_{m\in(K\cup J)-(K\cap J)}} \mathscr{B}_m^* (\rho_m) \right) \widetilde{\oplus} \left(\widetilde{\bigoplus_{m\in(K\cap J)}} \langle \mathscr{B}_m^* \rangle (\lambda_m \odot \mu_m) \right). \tag{48}$$

From the definition of $\langle \mathscr{B}_m^* \rangle$, there exists $\nu_m \in L^{X_m}$ with $\nu_m \leq \lambda_m \odot \mu_m$ such that

$$\langle \mathscr{B}^* \rangle (\lambda \odot \mu) \nleq \left(\widetilde{\bigoplus_{m\in(K\cup J)-(K\cap J)}} \mathscr{B}_m^* (\rho_m) \right) \widetilde{\oplus} \left(\widetilde{\bigoplus_{m\in(K\cap J)}} \mathscr{B}_m^* (\nu_m) \right). \tag{49}$$

On the other hand, since

$$\begin{aligned} \lambda \odot \mu &= \left(\bigodot_{k\in K} (\lambda_k \circ \varphi_k) \right) \odot \left(\bigodot_{j\in J} (\mu_j \circ \varphi_j) \right) \\ &\geq \left(\bigodot_{m\in(K\cup J)-(K\cap J)} (\rho_m \circ \varphi_m) \right) \\ &\quad \odot \left(\bigodot_{m\in(K\cap J)} (\nu_m \circ \phi_m) \right) \end{aligned} \tag{50}$$

and since $K \cup J$ is finite, we have

$$\langle \mathscr{B}^* \rangle (\lambda \odot \mu) \leq \left(\widetilde{\bigoplus_{m\in(K\cup J)-(K\cap J)}} \mathscr{B}_m^* (\rho_m) \right) \widetilde{\oplus} \left(\widetilde{\bigoplus_{m\in(K\cap J)}} \mathscr{B}_m^* (\nu_m) \right). \tag{51}$$

This contradicts the assumption. Then, $\langle \mathscr{B}^* \rangle (\lambda \odot \mu) \leq \mathscr{B}^*(\lambda) \widetilde{\oplus} \mathscr{B}^*(\mu)$, for each $\lambda, \mu \in L^X$. Similarly, $\langle \mathscr{B} \rangle (\lambda \odot \mu) \geq \mathscr{B}(\lambda) \widetilde{\odot} \mathscr{B}(\mu)$, for each, $\lambda, \mu \in L^X$. Hence, $(\mathscr{B}, \mathscr{B}^*)$ is an (L, M)-dffb on X.

Since $\mathscr{B}(\lambda_i \circ \varphi_i) \geq \mathscr{B}_i(\lambda_i)$ and $\mathscr{B}^*(\lambda_i \circ \varphi_i) \leq \mathscr{B}_i^*(\lambda_i)$, for each $i \in \Gamma$, by Theorem 23(iii), $\varphi_i : (X, \langle \mathscr{B} \rangle, \langle \mathscr{B}^* \rangle) \to (X_i, \langle \mathscr{B}_i \rangle, \langle \mathscr{B}_i^* \rangle)$ is a filter map.

Let $(\mathscr{F}, \mathscr{F}^*)$ be an (L, M)-dff on X such that, for each $i \in \Gamma$, the map $\varphi_i : (X, \mathscr{F}, \mathscr{F}^*) \to (X_i, \langle \mathscr{B}_i \rangle, \langle \mathscr{B}_i^* \rangle)$ is a filter map. Then,

$$\begin{aligned} &\mathscr{F}(\nu_i \circ \varphi_i) \geq \langle B_i \rangle (\nu_i), \\ &\mathscr{F}^*(\nu_i \circ \varphi_i) \leq \langle B_i^* \rangle (\nu_i), \\ &\qquad \text{for each } i \in \Gamma,\ \nu_i \in L^{X_i}. \end{aligned} \tag{52}$$

For any finite subset K of Γ with $\nu \geq \bigodot_{k\in K}(\nu_k \circ \varphi_k)$, since $\mathscr{F}(\nu_k \circ \varphi_k) \geq \langle B_k \rangle (\nu_k)$ and $\mathscr{F}^*(\nu_k \circ \varphi_k) \leq \langle B_k^* \rangle (\nu_k)$, for each $k \in K$, we have

$$\begin{aligned} \mathscr{F}(\lambda) &\geq \mathscr{F}\left(\bigodot_{k\in K} (\nu_k \circ \varphi_k) \right) \geq \widetilde{\bigodot_{k\in K}} \mathscr{F}(\nu_k \circ \varphi_k) \\ &\geq \widetilde{\bigodot_{k\in K}} \langle B_k \rangle (\nu_k) \geq \widetilde{\bigodot_{k\in K}} \mathscr{B}_k (\nu_k), \\ \mathscr{F}^*(\lambda) &\leq \mathscr{F}^*\left(\bigodot_{k\in K} (\nu_k \circ \varphi_k) \right) \leq \widetilde{\bigoplus_{k\in K}} \mathscr{F}^*(\nu_k \circ \varphi_k) \\ &\leq \widetilde{\bigoplus_{k\in K}} \langle B_k \rangle (\nu_k) \leq \widetilde{\bigoplus_{k\in K}} \mathscr{B}_k (\nu_k). \end{aligned} \tag{53}$$

Hence, by the definition of $\langle \mathscr{B} \rangle$ and $\langle \mathscr{B}^* \rangle$, it is obvious that $(\langle \mathscr{B} \rangle, \langle \mathscr{B}^* \rangle) \leq (\mathscr{F}, \mathscr{F}^*)$.

(ii) Necessity of the composition condition is obvious.

Conversely, for every finite subset K of Γ with $\nu \geq \bigodot_{k\in K}(\nu_k \circ \varphi_k)$, since, for each $k \in K$, $\varphi_k \circ \varphi : (Y, \mathscr{F}, \mathscr{F}^*) \to (X_k, \langle \mathscr{B}_k \rangle, \langle \mathscr{B}_k^* \rangle)$ is a filter map, that is, $\langle \mathscr{B}_k \rangle (\nu_k) \leq \mathscr{F}(\nu_k \circ (\varphi_k \circ \varphi))$ and $\langle \mathscr{B}_k^* \rangle (\nu_k) \geq \mathscr{F}^*(\nu_k \circ (\varphi_k \circ \varphi))$. Since $\nu \circ \varphi \geq \bigodot_{k\in K}((\nu_k \circ \varphi_k) \circ \varphi)$, we have

$$\begin{aligned} \mathscr{F}(\nu \circ \varphi) &\geq \widetilde{\bigodot_{k\in K}} \mathscr{F}(\nu_k \circ (\varphi_k \circ \varphi)) \\ &\geq \widetilde{\bigodot_{k\in K}} \langle \mathscr{B}_k \rangle (\nu_k) \geq \widetilde{\bigodot_{k\in K}} \mathscr{B}_k (\nu_k), \\ \mathscr{F}^*(\nu \circ \varphi) &\leq \widetilde{\bigoplus_{k\in K}} \mathscr{F}^*(\nu_k \circ (\varphi_k \circ \varphi)) \\ &\leq \widetilde{\bigoplus_{k\in K}} \langle \mathscr{B}_k^* \rangle (\nu_k) \leq \widetilde{\bigoplus_{k\in K}} \mathscr{B}_k^* (\nu_k). \end{aligned} \tag{54}$$

By the definition of $\langle \mathscr{B} \rangle$ and $\langle \mathscr{B}^* \rangle$, we have $\langle \mathscr{B} \rangle (\nu) \leq \mathscr{F}(\nu \circ \varphi)$ and $\langle \mathscr{B}^* \rangle (\nu) \geq \mathscr{F}^*(\nu \circ \varphi)$.

(iii) Put $\mathscr{F} = \bigsqcup_{i\in\Gamma} (\varphi_i)_1^{\leftarrow}(\langle \mathscr{B}_i \rangle)$ and $\mathscr{F}^* = \bigsqcap_{i\in\Gamma} (\varphi_i^*)_1^{\leftarrow}(\langle \mathscr{B}_i^* \rangle)$; by applying (i) to both $(\langle \mathscr{B} \rangle, \langle \mathscr{B}^* \rangle)$ and $(\langle \mathscr{F} \rangle, \langle \mathscr{F}^* \rangle)$, the desired equality is obtained. □

The following corollaries are the direct results of Theorem 26.

Corollary 27. *Let $\{(\mathscr{B}_i, \mathscr{B}_i^*)\}_{i\in\Gamma}$ be a family of (L, M)-dffb's on X satisfying the following condition:*

(C) For any finite subset K of Γ, if $\bigodot_{i\in K}\lambda_i = \underline{0}$, then $\widetilde{\bigodot}_{i\in K}\mathscr{B}_i(\lambda_i) = 0_M$ and $\widetilde{\bigoplus}_{i\in K}\mathscr{B}_i^*(\lambda_i) = 1_M$.

We define the maps $\bigsqcup_{i\in\Gamma}\mathscr{B}_i$, $\bigsqcap_{i\in\Gamma}\mathscr{B}_i^* : L^X \to M$ as

$$\begin{aligned}\bigsqcup_{i\in\Gamma}\mathscr{B}_i(\lambda) &= \bigvee\left\{\widetilde{\bigodot_{i\in K}}\mathscr{B}_i(\lambda_i) : \lambda = \bigodot_{i\in K}\lambda_i\right\},\\ \bigsqcap_{i\in\Gamma}\mathscr{B}_i^*(\lambda) &= \bigwedge\left\{\widetilde{\bigoplus_{i\in K}}\mathscr{B}_i^*(\lambda_i) : \lambda = \bigodot_{i\in K}\lambda_i\right\},\end{aligned} \tag{55}$$

where $\bigvee$ and $\bigwedge$ are taken for every finite index subset K of Γ such that $\lambda = \bigodot_{i\in K}\lambda_i$. Then, $(\bigsqcup_{i\in\Gamma}\mathscr{B}_i, \bigsqcap_{i\in\Gamma}\mathscr{B}_i^*)$ is an (L, M)-dffb on X and $(\langle\bigsqcup_{i\in\Gamma}\mathscr{B}_i\rangle, \langle\bigsqcap_{i\in\Gamma}\mathscr{B}_i^*\rangle)$ is the coarsest (L, M)-dff which is finer than $(\langle\mathscr{B}_i\rangle, \langle\mathscr{B}_i^*\rangle)$ for each $i \in \Gamma$.

Example 28. Let $X = \{x, y\}$ be a set and $L = M = [0, 1]$ be an stsc-quantale with $\odot$ (Lukasiewicz t-norm). We define maps $\mathscr{B}_i, \mathscr{B}_i^* : L^X \to M$ as follows $(i = 1, 2)$:

$$\begin{aligned}\mathscr{B}_1(\lambda) &= \begin{cases}1, & \text{if } \lambda = \underline{1};\\ 0.4, & \text{if } \lambda = 0.6\cdot\underline{1};\\ 0.5, & \text{if } \lambda = 0.1\cdot\underline{1};\\ 0, & \text{otherwise},\end{cases}\\ \mathscr{B}_1^*(\lambda) &= \begin{cases}0, & \text{if } \lambda = \underline{1};\\ 0.6, & \text{if } \lambda = 0.6\cdot\underline{1};\\ 0.5, & \text{if } \lambda = 0.1\cdot\underline{1};\\ 1, & \text{otherwise},\end{cases}\\ \mathscr{B}_2(\lambda) &= \begin{cases}1, & \text{if } \lambda = \underline{1};\\ 0.6, & \text{if } \lambda = 0.7\cdot\underline{1};\\ 0.3, & \text{if } \lambda = 0.3\cdot\underline{1};\\ 0, & \text{otherwise},\end{cases}\\ \mathscr{B}_2^*(\lambda) &= \begin{cases}0, & \text{if } \lambda = \underline{1};\\ 0.4, & \text{if } \lambda = 0.7\cdot\underline{1};\\ 0.7, & \text{if } \lambda = 0.3\cdot\underline{1};\\ 1, & \text{otherwise}.\end{cases}\end{aligned} \tag{56}$$

Each $(\mathscr{B}_i, \mathscr{B}_i^*)$ for $i = 1, 2$ is an (L, M)-double fuzzy filter base but $(\mathscr{B}_1 \sqcup \mathscr{B}_2, \mathscr{B}_1^* \sqcap \mathscr{B}_2^*)$ is not.

Corollary 29. *Let $\pi_i : X \to X_i$ be projection maps, for all $i \in \Gamma$, where $X = \Pi_{i\in\Gamma}X_i$ is the product set. Let $\{(\mathscr{B}_i, \mathscr{B}_i^*)\}_{i\in\Gamma}$ be a family of (L, M)-dffb's on X_i satisfying the following condition:*

(C) For any finite subset K of Γ, if $\bigodot_{i\in K}(\lambda_i \circ \pi_i) = \underline{0}$, then $\widetilde{\bigodot}_{i\in K}\mathscr{B}_i(\lambda_i) = 0_M$ and $\widetilde{\bigoplus}_{i\in K}\mathscr{B}_i^*(\lambda_i) = 1_M$.

We define the maps $\bigsqcup_{i\in\Gamma}(\pi_i)_1^{\Leftarrow}(\mathscr{B}_i), \bigsqcap_{i\in\Gamma}(\pi_i^*)_1^{\Leftarrow}(\mathscr{B}_i^*) : L^X \to M$ as

$$\begin{aligned}&\bigsqcup_{i\in\Gamma}(\pi_i)_1^{\Leftarrow}(\mathscr{B}_i)(\lambda)\\ &\quad= \bigvee\left\{\widetilde{\bigodot_{i\in K}}\mathscr{B}_i(\lambda_i) : \lambda = \bigodot_{i\in K}(\lambda_i\circ\pi_i)\right\},\\ &\bigsqcap_{i\in\Gamma}(\pi_i^*)_1^{\Leftarrow}(\mathscr{B}_i^*)(\lambda)\\ &\quad= \bigwedge\left\{\widetilde{\bigoplus_{i\in K}}\mathscr{B}_i^*(\lambda_i) : \lambda = \bigodot_{i\in K}(\lambda_i\circ\pi_i)\right\},\end{aligned} \tag{57}$$

where $\bigvee$ and $\bigwedge$ are taken for every finite subset K of Γ such that $\lambda = \bigodot_{i\in K}(\lambda_i \circ \pi_i)$.

Let $\mathscr{B} = \bigsqcup_{i\in\Gamma}(\pi_i)_1^{\Leftarrow}(\mathscr{B}_i)$ and $\mathscr{B}^* = \bigsqcap_{i\in\Gamma}(\pi_i^*)_1^{\Leftarrow}(\mathscr{B}_i^*)$ be given. Then, the following properties are satisfied:

(i) $(\mathscr{B}, \mathscr{B}^*)$ is an (L, M)-dffb on X and $(\langle\mathscr{B}\rangle, \langle\mathscr{B}^*\rangle)$ is the coarsest (L, M)-dff on X for which $\pi_i : (X, \langle\mathscr{B}\rangle, \langle\mathscr{B}^*\rangle) \to (X_i, \langle\mathscr{B}_i\rangle, \langle\mathscr{B}_i^*\rangle)$ is a filter map.

(ii) A map $\varphi : (Y, \mathscr{F}, \mathscr{F}^*) \to (X, \langle\mathscr{B}\rangle, \langle\mathscr{B}^*\rangle)$ is a filter map if and only if, for each $i \in \Gamma$, $\pi_i \circ \varphi : (Y, \mathscr{F}, \mathscr{F}^*) \to (X_i, \langle\mathscr{B}_i\rangle, \langle\mathscr{B}_i^*\rangle)$ is a filter map.

In Corollary 29, the structure $(\langle\bigsqcup_{i\in\Gamma}(\pi_i)_1^{\Leftarrow}(\mathscr{B}_i)\rangle, \langle\bigsqcap_{i\in\Gamma}(\pi_i^*)_1^{\Leftarrow}(\mathscr{B}_i^*)\rangle)$ is called a product of (L, M)-dffs's on X.

Theorem 30. *Let $\varphi : X \to Y$ be an injective function and $(\mathscr{B}, \mathscr{B}^*)$ be an (L, M)-dffb on X. Then, the following properties are satisfied:*

(i) $(\varphi_2^{\Rightarrow}(\mathscr{B}), \varphi_2^{\Rightarrow}(\mathscr{B}^*))$ is an (L, M)-dffb on Y, and $(\langle\varphi_2^{\Rightarrow}(\mathscr{B})\rangle, \langle\varphi_2^{\Rightarrow}(\mathscr{B}^*)\rangle)$ is the coarsest (L, M)-dff, for which the function $\varphi : (X, \langle\mathscr{B}\rangle, \langle\mathscr{B}^*\rangle) \to (Y, \langle\varphi_2^{\Rightarrow}(\mathscr{B})\rangle, \langle\varphi_2^{\Rightarrow}(\mathscr{B}^*)\rangle)$ is a filter preserving map.

(ii) $(\varphi_1^{\Leftarrow}(\varphi_2^{\Rightarrow}(\mathscr{B})), \varphi_1^{*\Leftarrow}(\varphi_2^{\Rightarrow}(\mathscr{B}^*)))$ is an (L, M)-dffb on X with $\varphi_1^{\Leftarrow}(\varphi_2^{\Rightarrow}(\mathscr{B})) = \mathscr{B}$ and $\varphi_1^{*\Leftarrow}(\varphi_2^{\Rightarrow}(\mathscr{B}^*)) = \mathscr{B}^*$.

Proof. (i) (DFFB1) For each $\nu \in L^Y$, we have

$$\begin{aligned}\varphi_2^{\Rightarrow}(\mathscr{B})(\nu) &= \mathscr{B}(\varphi_L^{\leftarrow}(\nu)) \le (\mathscr{B}^*(\varphi_L^{\leftarrow}(\nu)))^{\star}\\ &= (\varphi_2^{\Rightarrow}(\mathscr{B}^*)(\nu))^{\star}.\end{aligned} \tag{58}$$

(DFFB2) It is straightforward from the definition.

(DFFB3) Suppose that there exist $\nu_1, \nu_2 \in L^Y$ such that

$$\begin{aligned}\langle\varphi_2^{\Rightarrow}(\mathscr{B}^*)\rangle(\nu_1\odot\nu_2) &\nleq \varphi_2^{\Rightarrow}(\mathscr{B}^*)(\nu_1)\\ &\quad\tilde{\oplus}\,\varphi_2^{\Rightarrow}(\mathscr{B}^*)(\nu_2).\end{aligned} \tag{59}$$

By the definition of $\varphi_2^{\Rightarrow}(\mathscr{B}^*)$, we have

$$\langle\varphi_2^{\Rightarrow}(\mathscr{B}^*)\rangle(\nu_1\odot\nu_2) \nleq \mathscr{B}^*(\nu_1\circ\varphi)\,\tilde{\oplus}\,\mathscr{B}^*(\nu_2\circ\varphi). \tag{60}$$

Since $(\mathscr{B}, \mathscr{B}^*)$ is an (L, M)-dffb, the following is obtained:

$$\begin{aligned}\langle\mathscr{B}^*\rangle((\nu_1\odot\nu_2)\circ\varphi) &= \langle\mathscr{B}^*\rangle((\nu_1\circ\varphi)\odot(\nu_2\circ\varphi))\\ &\le \mathscr{B}^*(\nu_1\circ\varphi)\,\tilde{\oplus}\,\mathscr{B}^*(\nu_2\circ\varphi).\end{aligned} \tag{61}$$

Thus,

$$\langle \varphi_2^{\rightrightarrows}(\mathscr{B}^*)\rangle(\nu_1\odot\nu_2)\nleq\langle\mathscr{B}^*\rangle((\nu_1\odot\nu_2)\circ\varphi). \tag{62}$$

By the definition of $\langle\mathscr{B}^*\rangle$, there exists $\lambda\in L^X$ with $\lambda\le(\nu_1\odot\nu_2)\circ\varphi$ such that

$$\langle \varphi_2^{\rightrightarrows}(\mathscr{B}^*)\rangle(\nu_1\odot\nu_2)\nleq\mathscr{B}^*(\lambda). \tag{63}$$

Since $\varphi_L^{\rightarrow}(\lambda)\le\varphi_L^{\rightarrow}(\varphi_L^{\leftarrow}(\nu_1\odot\nu_2))\le\nu_1\odot\nu_2$ and φ is injective,

$$\begin{aligned}\langle \varphi_2^{\rightrightarrows}(\mathscr{B}^*)\rangle(\nu_1\odot\nu_2)&\le\varphi_2^{\rightrightarrows}(\mathscr{B}^*)(\varphi_L^{\rightarrow}(\lambda))\\&=\mathscr{B}^*(\varphi_L^{\leftarrow}(\varphi_L^{\rightarrow}(\lambda)))\\&=\mathscr{B}^*(\lambda).\end{aligned} \tag{64}$$

This contradicts the assumption. Then,

$$\begin{aligned}\langle \varphi_2^{\rightrightarrows}(\mathscr{B}^*)\rangle(\nu_1\odot\nu_2)&\le\varphi_2^{\rightrightarrows}(\mathscr{B}^*)(\nu_1)\\&\quad\tilde{\oplus}\,\varphi_2^{\rightrightarrows}(\mathscr{B}^*(\nu_2)),\\&\qquad\text{for each }\nu_1,\nu_2\in L^Y.\end{aligned} \tag{65}$$

Similarly, it can be verified that

$$\begin{aligned}\langle \varphi_2^{\rightrightarrows}(\mathscr{B})\rangle(\nu_1\odot\nu_2)&\ge\varphi_2^{\rightrightarrows}(\mathscr{B})(\nu_1)\,\tilde{\odot}\,\varphi_2^{\rightrightarrows}(\mathscr{B}(\nu_2)),\\&\qquad\text{for each }\nu_1,\nu_2\in L^Y.\end{aligned} \tag{66}$$

Hence, $(\varphi_2^{\rightrightarrows}(\mathscr{B}),\varphi_2^{\rightrightarrows}(\mathscr{B}^*))$ is an (L,M)-dffb on Y.

For each $\lambda\in L^X$, we have

$$\begin{aligned}\langle \varphi_2^{\rightrightarrows}(\mathscr{B})\rangle(\varphi_L^{\rightarrow}(\lambda))&\ge\varphi_2^{\rightrightarrows}(\mathscr{B})(\varphi_L^{\rightarrow}(\lambda))\\&=\mathscr{B}(\varphi_L^{\leftarrow}(\varphi_L^{\rightarrow}(\lambda)))=\mathscr{B}(\lambda),\\\langle \varphi_2^{\rightrightarrows}(\mathscr{B}^*)\rangle(\varphi_L^{\rightarrow}(\lambda))&\le\varphi_2^{\rightrightarrows}(\mathscr{B}^*)(\varphi_L^{\rightarrow}(\lambda))\\&=\mathscr{B}^*(\varphi_L^{\leftarrow}(\varphi_L^{\rightarrow}(\lambda)))\\&=\mathscr{B}^*(\lambda).\end{aligned} \tag{67}$$

Hence, by Theorem 23(ii), $\varphi:(X,\langle\mathscr{B}\rangle,\langle\mathscr{B}^*\rangle)\to(Y,\langle\varphi_2^{\rightrightarrows}(\mathscr{B})\rangle,\langle\varphi_2^{\rightrightarrows}(\mathscr{B}^*)\rangle)$ is a filter preserving map.

Let $(\mathscr{F},\mathscr{F}^*)$ be another (L,M)-dff such that $\varphi:(X,\langle\mathscr{B}\rangle,\langle\mathscr{B}^*\rangle)\to(Y,\mathscr{F},\mathscr{F}^*)$ is a filter preserving map. So, for each $\mu\in L^Y$, the following is valid:

$$\begin{aligned}\langle \varphi_2^{\rightrightarrows}(\mathscr{B}^*)\rangle(\mu)&=\bigwedge\{\varphi_2^{\rightrightarrows}(\mathscr{B}^*)(\nu):\nu\le\mu\}\\&=\bigwedge\{\mathscr{B}^*(\varphi_L^{\leftarrow}(\nu)):\nu\le\mu\}\\&\ge\bigwedge\{\mathscr{F}^*(\varphi_L^{\rightarrow}(\varphi_L^{\leftarrow}(\nu))):\nu\le\mu\}\\&\ge\bigwedge\{\mathscr{F}^*(\nu):\nu\le\mu\}\ge\mathscr{F}^*(\mu).\end{aligned} \tag{68}$$

Hence, $\langle\varphi_2^{\rightrightarrows}(\mathscr{B}^*)\rangle\ge\mathscr{F}^*$. Similarly, it can be proved that $\langle\varphi_2^{\rightrightarrows}(\mathscr{B})\rangle\le\mathscr{F}$.

(ii) If $\varphi_L^{\leftarrow}(\mu)=\underline{0}$, then $\varphi_2^{\rightrightarrows}(\mathscr{B})(\mu)=\mathscr{B}(\varphi_L^{\leftarrow}(\mu))=\mathscr{B}(\underline{0})=0_M$, and $\varphi_2^{\rightrightarrows}(\mathscr{B}^*)(\mu)=\mathscr{B}^*(\varphi_L^{\leftarrow}(\mu))=\mathscr{B}^*(\underline{0})=1_M$. By Theorem 25(i), $(\varphi_1^{\leftleftarrows}(\varphi_2^{\rightrightarrows}(\mathscr{B})),\varphi_1^{*\leftleftarrows}(\varphi_2^{\rightrightarrows}(\mathscr{B}^*)))$ is an (L,M)-dffb on X. For each $\nu\in L^X$, following equalities are obtained:

$$\begin{aligned}\varphi_1^{\leftleftarrows}(\varphi_2^{\rightrightarrows}(\mathscr{B}))(\nu)&=\bigvee\{\varphi_2^{\rightrightarrows}(\mathscr{B})(\mu):\nu=\varphi_L^{\leftarrow}(\mu)\}\\&=\bigvee\{\mathscr{B}(\varphi_L^{\leftarrow}(\mu)):\nu=\varphi_L^{\leftarrow}(\mu)\}=\mathscr{B}(\nu),\\\varphi_1^{*\leftleftarrows}(\varphi_2^{\rightrightarrows}(\mathscr{B}^*))(\nu)&\\&=\bigwedge\{\varphi_2^{\rightrightarrows}(\mathscr{B}^*)(\mu):\nu=\varphi_L^{\leftarrow}(\mu)\}\\&=\bigwedge\{\mathscr{B}^*(\varphi_L^{\leftarrow}(\mu)):\nu=\varphi_L^{\leftarrow}(\mu)\}=\mathscr{B}^*(\nu).\end{aligned} \tag{69}$$

□

Example 31. Let $X=\{a,b\},Y=\{x,y\}$ be sets and $L=M=[0,1]$ be the stsc-quantale with Lukasiewicz t-norm $\odot$. Let $\varphi:X\to Y$ be a function defined by $\varphi(a)=\varphi(b)=x$ and $\mu_1,\mu_2\in[0,1]^X$ be defined by $\mu_1(a)=\mu_1(b)=0.6$, $\mu_2(a)=0.1$, $\mu_2(b)=0$. We define maps $\mathscr{B},\mathscr{B}^*:[0,1]^X\to[0,1]$ as follows:

$$\begin{aligned}\mathscr{B}(\lambda)&=\begin{cases}1, & \text{if }\lambda=\underline{1};\\0.6, & \text{if }\lambda=\mu_1;\\0.3, & \text{if }\lambda=\mu_2;\\0, & \text{otherwise,}\end{cases}\\\mathscr{B}^*(\lambda)&=\begin{cases}0, & \text{if }\lambda=\underline{1};\\0.4, & \text{if }\lambda=\mu_1;\\0.7, & \text{if }\lambda=\mu_2;\\1, & \text{otherwise.}\end{cases}\end{aligned} \tag{70}$$

Then, $(\mathscr{B},\mathscr{B}^*)$ is an (L,M)-double fuzzy filter base but $(\varphi_2^{\rightrightarrows}(\mathscr{B}),\varphi_2^{\rightrightarrows}(\mathscr{B}^*))$ is not an (L,M)-double fuzzy filter base.

Theorem 32. *Let $\{\varphi_i:X_i\to X\}_{i\in\Gamma}$ be a family of injective functions and $\{(\mathscr{B}_i,\mathscr{B}_i^*)\}_{i\in\Gamma}$ be a family of (L,M)-dffb's on X_i satisfying the following condition:*

(C) For any finite subset K of Γ, if $\bigodot_{i\in K}\lambda_i=\underline{0}$, then $\widetilde{\bigodot}_{i\in K}(\varphi_i)_2^{\rightrightarrows}(\mathscr{B}_i)(\lambda_i)=0_M$, and $\widetilde{\bigoplus}_{i\in K}(\varphi_i)_2^{\rightrightarrows}(\mathscr{B}_i^*)(\lambda_i)=1_M$.

We define the maps $\mathscr{B},\mathscr{B}^*:L^X\to M$ as

$$\begin{aligned}\mathscr{B}(\lambda)&=\bigvee\left\{\widetilde{\bigodot_{i\in K}}(\varphi_i)_2^{\rightrightarrows}(\mathscr{B}_i)(\lambda_i):\lambda=\bigodot_{i\in K}\lambda_i\right\},\\\mathscr{B}^*(\lambda)&=\bigwedge\left\{\widetilde{\bigoplus_{i\in K}}(\varphi_i)_2^{\rightrightarrows}(\mathscr{B}_i^*)(\lambda_i):\lambda=\bigodot_{i\in K}\lambda_i\right\},\end{aligned} \tag{71}$$

where $\bigvee$ and $\bigwedge$ are taken for every finite index subset K of Γ. Then, the following properties are satisfied:

(i) $(\mathscr{B},\mathscr{B}^*)$ is an (L,M)-dffb on X and $(\langle\mathscr{B}\rangle,\langle\mathscr{B}^*\rangle)$ is the coarsest (L,M)-dff for which $\varphi_i:(X_i,\langle\mathscr{B}_i\rangle,\langle\mathscr{B}_i^*\rangle)\to(X,\langle\mathscr{B}\rangle,\langle\mathscr{B}^*\rangle)$ is a filter preserving map.

(ii) A map $\varphi : (X, \langle\mathscr{B}\rangle, \langle\mathscr{B}^*\rangle) \to (Y, \mathscr{F}, \mathscr{F}^*)$ is a filter preserving map if and only if, for each $i \in \Gamma$, $\varphi \circ \varphi_i : (X_i, \langle\mathscr{B}_i\rangle, \langle\mathscr{B}_i^*\rangle) \to (Y, \mathscr{F}, \mathscr{F}^*)$ is a filter preserving map.

Proof. (i) By Corollary 27 and Theorem 30, $(\mathscr{B}, \mathscr{B}^*)$ is an (L, M)-dffb on X.

Since φ_i is injective, for each $i \in \Gamma$,

$$\begin{aligned}
\mathscr{B}\left((\varphi_i)_L^{\rightarrow}(\lambda_i)\right) &\geq (\varphi_i)_2^{\rightrightarrows}\mathscr{B}_i\left((\varphi_i)_L^{\rightarrow}(\lambda_i)\right) \\
&\geq \mathscr{B}_i\left((\varphi_i)_L^{\leftarrow}\left((\varphi_i)_L^{\rightarrow}(\lambda_i)\right)\right) \\
&= \mathscr{B}_i(\lambda_i), \\
\mathscr{B}^*\left((\varphi_i)_L^{\rightarrow}(\lambda_i)\right) &\leq (\varphi_i)_2^{\rightrightarrows}\mathscr{B}_i^*\left((\varphi_i)_L^{\rightarrow}(\lambda_i)\right) \\
&\leq \mathscr{B}_i^*\left((\varphi_i)_L^{\leftarrow}\left((\varphi_i)_L^{\rightarrow}(\lambda_i)\right)\right) \\
&= \mathscr{B}_i^*(\lambda_i).
\end{aligned} \tag{72}$$

Hence, φ_i is a filter preserving map, for each $i \in \Gamma$.

According to Theorem 26(i), other cases are similarly proved.

(ii) It is proved in the same way as Theorem 26(ii). □

Definition 33 (see [43]). (a) Let $(\mathbf{A}, U)$ be a concrete category over $\mathbf{X}$. $(\mathbf{A}, U)$ is said to be amnestic provided that its fibres are partially ordered classes; that is, no two different $\mathbf{A}$-objects are equivalent.

(b) Let $\mathbf{A}$ and $\mathbf{B}$ be categories. A functor $G : \mathbf{A} \to \mathbf{B}$ is called topological provided that every G-structured source $(f_i : B \to GA_i)_{i\in\Gamma}$ has a unique G-initial lift $(\overline{f_i} : A \to A_i)_{i\in\Gamma}$.

Proposition 34 (see [43]). *If $G : \mathbf{A} \to \mathbf{B}$ is a functor such that every G-structured source has a G-initial lift, then the following conditions are equivalent:*

(1) *G is topological.*

(2) *$(\mathbf{A}, G)$ is uniquely transportable.*

(3) *$(\mathbf{A}, G)$ is amnestic.*

Theorem 35. *The forgetful functor $V : (L, M)$-**DFIL** $\to$ **SET** defined by $V(X, \mathscr{F}, \mathscr{F}^*) = X$ and $V(\varphi) = \varphi$ is topological.*

Proof. The proof follows from Definition 33, Proposition 34, and Theorem 26. □

5. The Types $(\varphi_1^{\rightrightarrows}, \varphi_1^{*\rightrightarrows})$, $(\varphi_2^{\leftleftarrows}, \varphi_2^{\leftleftarrows})$ of Images and Preimages of (L,M)-Double Fuzzy Filter Bases

Theorem 36. *Let $\varphi : X \to Y$ be a surjective function and $(\mathscr{B}, \mathscr{B}^*)$ be an (L, M)-dffb on X. Then, the following properties are satisfied:*

(i) $(\varphi_1^{\rightrightarrows}(\mathscr{B}), \varphi_1^{*\rightrightarrows}(\mathscr{B}^*))$ is an (L, M)-dffb on Y.

(ii) $(\langle\varphi_1^{\rightrightarrows}(\mathscr{B})\rangle, \langle\varphi_1^{*\rightrightarrows}(\mathscr{B}^*)\rangle)$ is the coarsest (L, M)-dff on Y for which $\varphi : (X, \langle\mathscr{B}\rangle, \langle\mathscr{B}^*\rangle) \to (Y, \langle\varphi_1^{\rightrightarrows}(\mathscr{B})\rangle, \langle\varphi_1^{*\rightrightarrows}(\mathscr{B}^*)\rangle)$ is a filter preserving map.

(iii) If $(\mathscr{B}, \mathscr{B}^*)$ is an (L, M)-dff, then $\langle\varphi_1^{\rightrightarrows}(\mathscr{B})\rangle = \varphi_2^{\rightrightarrows}(\mathscr{B})$ and $\langle\varphi_1^{*\rightrightarrows}(\mathscr{B}^*))\rangle = \varphi_2^{\rightrightarrows}(\mathscr{B}^*)$.

Proof. (i) and (ii) are proved in the same manner as Theorem 25(i).

(iii) Let $(\mathscr{B}, \mathscr{B}^*)$ be an (L, M)-dff. Since φ is surjective, $\varphi_L^{\rightarrow}(\lambda) \leq \nu$ is equivalent to $\lambda \leq \nu \circ \varphi$. Then, for each $\nu \in L^Y$, it is clear that

$$\begin{aligned}
\langle\varphi_1^{*\rightrightarrows}(\mathscr{B}^*)\rangle(\nu) &= \bigwedge\{\varphi_1^{*\rightrightarrows}(\mathscr{B}^*)(\mu) : \mu \leq \nu\} \\
&= \bigwedge\{\mathscr{B}^*(\lambda) : \varphi_L^{\rightarrow}(\lambda) = \mu \leq \nu\} \\
&= \bigwedge\{\mathscr{B}^*(\lambda) : \lambda \leq \nu \circ \varphi\} \\
&= \mathscr{B}^*(\nu \circ \phi) = \varphi_2^{\rightrightarrows}(\mathscr{B}^*)(\nu).
\end{aligned} \tag{73}$$

Hence, $\langle\varphi_1^{*\rightrightarrows}(\mathscr{B}^*)\rangle = \varphi_2^{\rightrightarrows}(\mathscr{B}^*)$. Similarly, $\langle\varphi_1^{\rightrightarrows}(\mathscr{B})\rangle = \varphi_2^{\rightrightarrows}(\mathscr{B})$ is obtained. □

Remark 37. Let $\varphi : X \to Y$ be a bijective function, $(\mathscr{K}, \mathscr{K}^*)$ be an (L, M)-dffb on X, and $(\mathscr{B}, \mathscr{B}^*)$ be an (L, M)-dffb on Y. Then, the following equalities are clear.

(i) $\varphi_1^{\rightrightarrows}(\mathscr{K}) = \varphi_2^{\rightrightarrows}(\mathscr{K})$ and $\varphi_1^{*\rightrightarrows}(\mathscr{K}^*) = \varphi_2^{\rightrightarrows}(\mathscr{K}^*)$.

(ii) $\varphi_1^{\leftleftarrows}(\mathscr{B}) = \varphi_2^{\leftleftarrows}(\mathscr{B})$ and $\varphi_1^{*\leftleftarrows}(\mathscr{B}^*) = \varphi_2^{\leftleftarrows}(\mathscr{B}^*)$.

Remark 38. Let $\varphi : X \to Y$ be a bijective function and $(\mathscr{B}, \mathscr{B}^*)$ be an (L, M)-dffb on Y. Then, it follows from Remark 37(ii) and Theorem 25 that $(\varphi_2^{\leftleftarrows}(\mathscr{B}), \varphi_2^{\leftleftarrows}(\mathscr{B}^*))$ is an (L, M)-dffb on X and $(\langle\varphi_2^{\leftleftarrows}(\mathscr{B})\rangle, \langle\varphi_2^{\leftleftarrows}(\mathscr{B}^*)\rangle)$ is the coarsest (L, M)-dff on X for which $\varphi : (X, \langle\varphi_2^{\leftleftarrows}(\mathscr{B})\rangle, \langle\varphi_2^{\leftleftarrows}(\mathscr{B}^*)\rangle) \to (Y, \mathscr{B}, \mathscr{B}^*)$ is a filter map.

Theorem 39. *Let $\varphi : X \to Y$ be a function and $\{(\mathscr{B}_i, \mathscr{B}_i^*)\}_{i\in\Gamma}$ be a family of (L, M)-dffb's on X satisfying the following condition:*

(C) For every finite subset K of Γ, if $\bigodot_{i\in K}\lambda_i = \underline{0}$, then $\widetilde{\bigodot}_{i\in K}\mathscr{B}_i(\lambda_i) = 0_M$ and $\widetilde{\bigoplus}_{i\in K}\mathscr{B}_i^*(\lambda_i) = 1_M$. Then, the following properties are satisfied:

(i) If $\varphi : X \to Y$ is bijective, then $\varphi_1^{\rightrightarrows}(\bigsqcup_{i\in\Gamma}\mathscr{B}_i) = \bigsqcup_{i\in\Gamma}\varphi_1^{\rightrightarrows}(\mathscr{B}_i)$ and $\varphi_1^{*\rightrightarrows}(\bigsqcap_{i\in\Gamma}\mathscr{B}_i^*) = \bigsqcap_{i\in\Gamma}\varphi_1^{*\rightrightarrows}(\mathscr{B}_i^*)$.

(ii) If $\varphi : X \to Y$ is injective, then $\langle\varphi_1^{\rightrightarrows}(\bigsqcup_{i\in\Gamma}\mathscr{B}_i)\rangle = \bigsqcup_{i\in\Gamma}\langle\varphi_1^{\rightrightarrows}(\mathscr{B}_i)\rangle$ and $\langle\varphi_1^{*\rightrightarrows}(\bigsqcap_{i\in\Gamma}\mathscr{B}_i^*)\rangle = \bigsqcap_{i\in\Gamma}\langle\varphi_1^{*\rightrightarrows}(\mathscr{B}_i^*)\rangle$.

Proof. (i) Let us consider the following condition:

(C1) For every finite subset K of Γ, if $\bigodot_{i\in K}\nu_i = \underline{0}$, then $\widetilde{\bigodot}_{i\in K}\varphi_1^{\rightrightarrows}(\mathscr{B}_i)(\nu_i) = 0_M$ and $\widetilde{\bigoplus}_{i\in K}\varphi_1^{*\rightrightarrows}(\mathscr{B}_i^*)(\nu_i) = 1_M$.

For the proof, it is enough to show that (C1) $\Leftrightarrow$ (C).

(C1) $\Rightarrow$ (C): For any finite subset K of Γ with $\bigodot_{i\in K}\lambda_i = \underline{0}$, since φ is injective, by Lemma 6(!),

$$\varphi_L^{\rightarrow}\left(\bigodot_{i\in K}\lambda_i\right) = \bigodot_{i\in K}\varphi_L^{\rightarrow}(\lambda_i) = \underline{0}. \tag{74}$$

By (C1), we have

$$\begin{aligned}
0_M &= \widetilde{\bigodot_{i\in K}}\varphi_1^{\rightrightarrows}(\mathscr{B}_i)(\varphi_L^{\rightarrow}(\lambda_i)) \geq \widetilde{\bigodot_{i\in K}}\mathscr{B}_i(\lambda_i), \\
1_M &= \widetilde{\bigoplus_{i\in K}}\varphi_1^{*\rightrightarrows}(\mathscr{B}_i^*)(\varphi_L^{\rightarrow}(\lambda_i)) \leq \widetilde{\bigoplus_{i\in K}}\mathscr{B}_i^*(\lambda_i),
\end{aligned} \tag{75}$$

and thus $\widetilde{\bigodot}_{i\in K}\mathscr{B}_i(\lambda_i) = 0_M$ and $\widetilde{\bigoplus}_{i\in K}\mathscr{B}_i^*(\lambda_i) = 1_M$ is satisfied.

(C) $\Rightarrow$ (C1): Suppose that, for every finite subset K of Γ with $\bigodot_{i\in K}\nu_i = \underline{0}$, $\widetilde{\bigoplus}_{i\in K}\varphi_1^{*\rightrightarrows}(\mathscr{B}_i^*)(\nu_i) \neq 1_M$. Then, for each $i \in K$, there exists $\lambda_i \in L^X$ with $\nu_i = \varphi_L^{\rightarrow}(\lambda_i)$ such that

$$\widetilde{\bigoplus_{i\in K}} \varphi_1^{*\rightrightarrows}(\mathscr{B}_i^*)(\nu_i) \leq \widetilde{\bigoplus_{i\in K}} \mathscr{B}_i^*(\lambda_i) \neq 1_M. \tag{76}$$

By (C), $\bigodot_{i\in K}\lambda_i \neq \underline{0}$. By Lemma 6(!),

$$\varphi_L^{\rightarrow}\left(\bigodot_{i\in K}\lambda_i\right) = \bigodot_{i\in K}\varphi_L^{\rightarrow}(\lambda_i) = \bigodot_{i\in K}\nu_i \neq \underline{0}. \tag{77}$$

This contradicts the assumption. Thus, $\widetilde{\bigoplus}_{i\in K}\varphi_1^{*\rightrightarrows}(\mathscr{B}_i^*)(\nu_i) = 1_M$. Similarly, for every finite subset K of Γ, if $\bigodot_{i\in K}\nu_i = \underline{0}$, then $\widetilde{\bigodot}_{i\in K}\varphi_1^{\rightrightarrows}(\mathscr{B}_i)(\nu_i) = 0_M$.

Since φ is surjective, by Theorem 36, $(\varphi_1^{\rightrightarrows}(\mathscr{B}_i), \varphi_1^{*\rightrightarrows}(\mathscr{B}_i^*))$ exists for each $i \in \Gamma$. By Corollary 27 and (C1), $(\bigsqcup_{i\in\Gamma}\varphi_1^{\rightrightarrows}(\mathscr{B}_i), \bigsqcap_{i\in\Gamma}\varphi_1^{*\rightrightarrows}(\mathscr{B}_i^*))$ exists.

For each finite subset K of Γ such that $\lambda = \bigodot_{k\in K}\lambda_k$ with $\varphi_L^{\rightarrow}(\lambda) = \nu$, the following inequalities are satisfied:

$$\begin{aligned}
\bigsqcup_{i\in\Gamma}\varphi_1^{\rightrightarrows}(\mathscr{B}_i)(\nu) &\geq \widetilde{\bigodot_{k\in K}} \varphi_1^{\rightrightarrows}(\mathscr{B}_k)(\varphi_L^{\rightarrow}(\lambda_k)) \\
&\geq \widetilde{\bigodot_{k\in K}} \mathscr{B}_k(\lambda_k), \\
\bigsqcap_{i\in\Gamma}\varphi_1^{*\rightrightarrows}(\mathscr{B}_i^*)(\nu) &\leq \widetilde{\bigoplus_{k\in K}} \varphi_1^{*\rightrightarrows}(\mathscr{B}_k^*)(\varphi_L^{\rightarrow}(\lambda_k)) \\
&\leq \widetilde{\bigoplus_{k\in K}} \mathscr{B}_k^*(\lambda_k).
\end{aligned} \tag{78}$$

This implies that

$$\begin{aligned}
\bigsqcup_{i\in\Gamma}\mathscr{B}_i(\lambda) &\leq \bigsqcup_{i\in\Gamma}\varphi_1^{\rightrightarrows}(\mathscr{B}_i)(\nu), \\
\bigsqcap_{i\in\Gamma}\mathscr{B}_i^*(\lambda) &\geq \bigsqcap_{i\in\Gamma}\varphi_1^{*\rightrightarrows}(\mathscr{B}_i^*)(\nu).
\end{aligned} \tag{79}$$

So, the following are clear:

$$\begin{aligned}
\varphi_1^{\rightrightarrows}\left(\bigsqcup_{i\in\Gamma}\mathscr{B}_i\right) &\leq \bigsqcup_{i\in\Gamma}\varphi_1^{\rightrightarrows}(\mathscr{B}_i), \\
\varphi_1^{*\rightrightarrows}\left(\bigsqcap_{i\in\Gamma}\mathscr{B}_i^*\right) &\geq \bigsqcap_{i\in\Gamma}\varphi_1^{*\rightrightarrows}(\mathscr{B}_i^*).
\end{aligned} \tag{80}$$

For any finite subset J of Γ with $\mu = \bigodot_{j\in J}\mu_j$, there exist $\eta_j \in L^X$ with $\varphi_L^{\rightarrow}(\eta_j) = \mu_j$. Thus,

$$\begin{aligned}
\varphi_1^{\rightrightarrows}\left(\bigsqcup_{i\in\Gamma}\mathscr{B}_i\right)(\mu) &\geq \left(\bigsqcup_{i\in\Gamma}\mathscr{B}_i\right)\left(\bigodot_{j\in J}\eta_j\right) \\
&\geq \widetilde{\bigodot_{j\in J}} \mathscr{B}_j(\eta_j), \\
\varphi_1^{*\rightrightarrows}\left(\bigsqcap_{i\in\Gamma}\mathscr{B}_i^*\right)(\mu) &\leq \left(\bigsqcap_{i\in\Gamma}\mathscr{B}_i^*\right)\left(\bigodot_{j\in J}\eta_j\right) \\
&\leq \widetilde{\bigoplus_{j\in J}} \mathscr{B}_j^*(\eta_j).
\end{aligned} \tag{81}$$

This implies that

$$\begin{aligned}
\varphi_1^{\rightrightarrows}\left(\bigsqcup_{i\in\Gamma}\mathscr{B}_i\right) &\geq \bigsqcup_{i\in\Gamma}\varphi_1^{\rightrightarrows}(\mathscr{B}_i), \\
\varphi_1^{*\rightrightarrows}\left(\bigsqcap_{i\in\Gamma}\mathscr{B}_i^*\right) &\leq \bigsqcap_{i\in\Gamma}\varphi_1^{*\rightrightarrows}(\mathscr{B}_i^*).
\end{aligned} \tag{82}$$

From the above inequalities, we have

$$\begin{aligned}
\varphi_1^{\rightrightarrows}\left(\bigsqcup_{i\in\Gamma}\mathscr{B}_i\right) &= \bigsqcup_{i\in\Gamma}\varphi_1^{\rightrightarrows}(\mathscr{B}_i), \\
\varphi_1^{*\rightrightarrows}\left(\bigsqcap_{i\in\Gamma}\mathscr{B}_i^*\right) &= \bigsqcap_{i\in\Gamma}\varphi_1^{*\rightrightarrows}(\mathscr{B}_i^*).
\end{aligned} \tag{83}$$

(ii) It is proved by the same method as in (i) and Theorem 36(ii). □

Theorem 40. *Let $\{\varphi_i : X_i \to X : i \in \Gamma\}$ be a family of functions and $\{(\mathscr{B}_i, \mathscr{B}_i^*)\}_{i\in\Gamma}$ be a family of (L, M)-dffbs' on X_i satisfying the following condition:*

(C) For any finite subset K of Γ, if $\bigodot_{i\in K}(\varphi_i)_L^{\rightarrow}(\lambda_i) = \underline{0}$, then $\widetilde{\bigodot}_{i\in K}\mathscr{B}_i(\lambda_i) = 0_M$ and $\widetilde{\bigoplus}_{i\in K}\mathscr{B}_i^*(\lambda_i) = 1_M$.

We define the maps $\bigcup_{i\in\Gamma}(\varphi_i)_1^{\rightrightarrows}(\mathscr{B}_i)$, $\bigcap_{i\in\Gamma}(\varphi_i)_1^{*\rightrightarrows}(\mathscr{B}_i^*) : L^X \to M$ as

$$\begin{aligned}
&\bigcup_{i\in\Gamma}(\varphi_i)_1^{\rightrightarrows}(\mathscr{B}_i)(\nu) \\
&\quad = \bigvee\left\{\widetilde{\bigodot_{i\in K}} \mathscr{B}_i(\lambda_i) : \nu = \bigodot_{i\in K}(\varphi_i)_L^{\rightarrow}(\lambda_i)\right\}, \\
&\bigcap_{i\in\Gamma}(\varphi_i)_1^{*\rightrightarrows}(\mathscr{B}_i^*)(\nu) \\
&\quad = \bigwedge\left\{\widetilde{\bigoplus_{i\in K}} \mathscr{B}_i^*(\lambda_i) : \nu = \bigodot_{i\in K}(\varphi_i)_L^{\rightarrow}(\lambda_i)\right\},
\end{aligned} \tag{84}$$

where $\bigvee$ and $\bigwedge$ are taken for every finite subset K of Γ. Let $\mathscr{B} = \bigcup_{i\in\Gamma}(\varphi_i)_1^{\rightrightarrows}(\mathscr{B}_i)$ and $\mathscr{B}^* = \bigcap_{i\in\Gamma}(\varphi_i)_1^{*\rightrightarrows}(\mathscr{B}_i^*)$. Then, the following properties are satisfied:

(i) If φ_j is surjective for some $j \in \Gamma$, then $(\mathscr{B}, \mathscr{B}^*)$ is an (L, M)-dffb on X and $(\langle\mathscr{B}\rangle, \langle\mathscr{B}^*\rangle)$ is the coarsest (L, M)-dff for which the map $\varphi_i : (X_i, \langle\mathscr{B}_i\rangle, \langle\mathscr{B}_i^*\rangle) \to (X, \langle\mathscr{B}\rangle, \langle\mathscr{B}^*\rangle)$ is a filter preserving map.

(ii) A function $\varphi : (X, \langle\mathscr{B}\rangle, \langle\mathscr{B}^*\rangle) \to (Y, \mathscr{F}, \mathscr{F}^*)$ is a filter preserving map if and only if, for each $i \in \Gamma$, $\varphi \circ \varphi_i : (X_i, \langle\mathscr{B}_i\rangle, \langle\mathscr{B}_i^*\rangle) \to (Y, \mathscr{F}, \mathscr{F}^*)$ is a filter preserving map.

(iii) If φ_i are surjective for all $i \in \Gamma$, then

$$\left\langle \bigcup_{i\in\Gamma} (\varphi_i)_1^{\rightrightarrows} (\mathscr{B}_i) \right\rangle = \left\langle \bigsqcup_{i\in\Gamma} (\varphi_i)_1^{\rightrightarrows} (\mathscr{B}_i) \right\rangle,$$
$$\left\langle \bigcap_{i\in\Gamma} (\varphi_i)_1^{*\rightrightarrows} (\mathscr{B}_i^*) \right\rangle = \left\langle \bigsqcap_{i\in\Gamma} (\varphi_i)_1^{*\rightrightarrows} (\mathscr{B}_i^*) \right\rangle. \tag{85}$$

Proof. (i) (DFFB2) Since φ_j is surjective for some $j \in \Gamma$ and (C), $\mathscr{B}(\underline{1}) = 1_M$, $\mathscr{B}^*(\underline{1}) = 0_M$ and $\mathscr{B}(\underline{0}) = 0_M$, $\mathscr{B}^*(\underline{0}) = 1_M$.

According to Theorems 26(i) and 32(i), other cases are similarly proved.

(ii) The proof is similar to Theorem 26(ii).

(iii) Let us consider the following condition:

(C1) For any finite subset K of Γ, if $\bigodot_{i\in K} \nu_i = \underline{0}$, then $\widetilde{\bigodot}_{i\in K} (\varphi_i)_1^{\rightrightarrows} (\mathscr{B}_i)(\nu_i) = 0_M$ and $\widetilde{\bigoplus}_{i\in K} (\varphi_i)_1^{*\rightrightarrows} (\mathscr{B}_i^*)(\nu_i) = 1_M$.

For the proof, it is enough to show that (C1) $\Leftrightarrow$ (C).

(C1) $\Rightarrow$ (C): For any finite subset K of Γ, if $\bigodot_{i\in K} (\varphi_i)_L^{\rightarrow} (\lambda_i) = \underline{0}$, by (C1), we have

$$\begin{aligned} 0_M &= \widetilde{\bigodot_{i\in K}} (\varphi_i)_1^{\rightrightarrows} (\mathscr{B}_i) \left((\varphi_i)_L^{\rightarrow} (\lambda_i) \right) \\ &\geq \widetilde{\bigodot_{i\in K}} \mathscr{B}_i (\lambda_i), \\ 1_M &= \widetilde{\bigoplus_{i\in K}} (\varphi_i)_1^{*\rightrightarrows} (\mathscr{B}_i^*) \left((\varphi_i)_L^{\rightarrow} (\lambda_i) \right) \\ &\leq \widetilde{\bigoplus_{i\in K}} \mathscr{B}_i^* (\lambda_i). \end{aligned} \tag{86}$$

(C) $\Rightarrow$ (C1): Suppose that, for any finite subset K of Γ with $\bigodot_{i\in K} \nu_i = \underline{0}$, we have $\widetilde{\bigoplus}_{i\in K} (\varphi_i)_1^{*\rightrightarrows} (\mathscr{B}_i^*)(\nu_i) \neq 1_M$. Then, for each $i \in K$, there exists $\lambda_i \in L^{X_i}$ with $\nu_i = (\varphi_i)_L^{\rightarrow} (\lambda_i)$ such that

$$\widetilde{\bigoplus_{i\in K}} (\varphi_i)_1^{*\rightrightarrows} (\mathscr{B}_i^*) (\nu_i) \leq \widetilde{\bigoplus_{i\in K}} \mathscr{B}_i^* (\lambda_i) \neq 1_M. \tag{87}$$

By (C),

$$\bigodot_{i\in K} (\varphi_i)_L^{\rightarrow} (\lambda_i) = \bigodot_{i\in K} \nu_i \neq \underline{0}. \tag{88}$$

This is a contradiction. Thus, $\widetilde{\bigoplus}_{i\in K} (\varphi_i)_1^{*\rightrightarrows} (\mathscr{B}_i^*)(\nu_i) = 1_M$. Similarly, $\widetilde{\bigodot}_{i\in K} (\phi_i)_1^{\rightrightarrows} (\mathscr{B}_i)(\nu_i) = 0_M$.

For any finite index set K with $\{\lambda_i : \bigodot_{i\in K} (\varphi_i)_L^{\rightarrow} (\lambda_i) \leq \nu\}$, by the definition of

$\langle \bigsqcup_{i\in\Gamma} (\varphi_i)_1^{\rightrightarrows} (\mathscr{B}_i) \rangle$ and $\langle \bigsqcap_{i\in\Gamma} (\varphi_i)_1^{*\rightrightarrows} (\mathscr{B}_i^*) \rangle$, the following inequalities are obtained:

$$\begin{aligned} &\left\langle \bigsqcup_{i\in\Gamma} (\varphi_i)_1^{\rightrightarrows} (\mathscr{B}_i) \right\rangle (\nu) \\ &\quad \geq \bigvee_{i\in\Gamma} (\varphi_i)_1^{\rightrightarrows} (\mathscr{B}_i) \left(\bigodot_{i\in K} (\varphi_i)_L^{\rightarrow} (\lambda_i) \right) \\ &\quad \geq \widetilde{\bigodot_{i\in K}} (\varphi_i)_1^{\rightrightarrows} (\mathscr{B}_i) \left((\varphi_i)_L^{\rightarrow} (\lambda_i) \right) \\ &\qquad\qquad \text{(by Corollary 26)} \\ &\quad \geq \widetilde{\bigodot_{i\in K}} \mathscr{B}_i (\lambda_i), \\ &\left\langle \bigsqcap_{i\in\Gamma} (\varphi_i)_1^{*\rightrightarrows} (\mathscr{B}_i^*) \right\rangle (\nu) \\ &\quad \leq \bigwedge_{i\in\Gamma} (\varphi_i)_1^{*\rightrightarrows} (\mathscr{B}_i^*) \left(\bigodot_{i\in K} (\varphi_i)_L^{\rightarrow} (\lambda_i) \right) \\ &\quad \leq \widetilde{\bigoplus_{i\in K}} (\varphi_i)_1^{*\rightrightarrows} (\mathscr{B}_i^*) \left((\varphi_i)_L^{\rightarrow} (\lambda_i) \right) \\ &\qquad\qquad \text{(by Corollary 26)} \\ &\quad \leq \widetilde{\bigoplus_{i\in K}} \mathscr{B}_i^* (\lambda_i). \end{aligned} \tag{89}$$

Hence,

$$\left\langle \bigcup_{i\in\Gamma} (\varphi_i)_1^{\rightrightarrows} (\mathscr{B}_i) \right\rangle \leq \left\langle \bigsqcup_{i\in\Gamma} (\varphi_i)_1^{\rightrightarrows} (\mathscr{B}_i) \right\rangle,$$
$$\left\langle \bigcap_{i\in\Gamma} (\varphi_i)_1^{*\rightrightarrows} (\mathscr{B}_i^*) \right\rangle \geq \left\langle \bigsqcap_{i\in\Gamma} (\varphi_i)_1^{*\rightrightarrows} (\mathscr{B}_i^*) \right\rangle. \tag{90}$$

For any finite index set J with $\{\nu_i : \bigodot_{i\in J} \nu_i \leq \mu\}$, since φ_i is surjective, for each $i \in J$, there exists $\lambda_i \in L^{X_i}$ with $(\varphi_i)_L^{\rightarrow} (\lambda_i) = \nu_i$ such that $\mu \geq \bigodot_{i\in J} \nu_i = \bigodot_{i\in J} (\varphi_i)_L^{\rightarrow} (\lambda_i)$.

Thus,

$$\begin{aligned} \left\langle \bigcup_{i\in\Gamma} (\varphi_i)_1^{\rightrightarrows} (\mathscr{B}_i) \right\rangle (\mu) &\geq \widetilde{\bigodot_{i\in J}} (\varphi_i)_1^{\rightrightarrows} (\mathscr{B}_i) (\nu_i) \\ &\geq \widetilde{\bigodot_{i\in J}} \mathscr{B}_i (\lambda_i), \\ \left\langle \bigcap_{i\in\Gamma} (\varphi_i)_1^{*\rightrightarrows} (\mathscr{B}_i^*) \right\rangle (\mu) &\leq \widetilde{\bigoplus_{i\in J}} (\varphi_i)_1^{*\rightrightarrows} (\mathscr{B}_i^*) (\nu_i) \\ &\leq \widetilde{\bigoplus_{i\in J}} \mathscr{B}_i^* (\lambda_i). \end{aligned} \tag{91}$$

Hence,

$$\left\langle \bigcup_{i\in\Gamma} (\varphi_i)_1^{\rightrightarrows} (\mathscr{B}_i) \right\rangle \geq \left\langle \bigsqcup_{i\in\Gamma} (\varphi_i)_1^{\rightrightarrows} (\mathscr{B}_i) \right\rangle,$$
$$\left\langle \bigcap_{i\in\Gamma} (\varphi_i)_1^{*\rightrightarrows} (\mathscr{B}_i^*) \right\rangle \leq \left\langle \bigsqcap_{i\in\Gamma} (\varphi_i)_1^{*\rightrightarrows} (\mathscr{B}_i^*) \right\rangle. \tag{92}$$

□

6. Conclusion

In this study, we introduced the notions of (L, M)-double fuzzy filter space and (L, M)-double fuzzy filter base where L and M are stsc-quantales as an extension of frames. We showed the existence of initial and also final (L, M)-double fuzzy filter structures. We also proved that the category (L, M)-**DFIL** is a topological category over **SET**. By giving illustrative examples, we considered two types of second-order Zadeh image and preimage operators of (L, M)-double fuzzy filter.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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