In the present communication, a parametric \((R, S)\)-norm information measure for the Pythagorean fuzzy set has been proposed with the proof of its validity. The monotonic behavior and maximality feature of the proposed information measure have been studied and presented. Further, an algorithm for solving the multicriteria decision-making problem with the help of the proposed information measure has been provided keeping in view of the different cases for weight criteria, when weights are unknown and other when weights are partially known. Numerical examples for each of the case have been successfully illustrated. Finally, the work has been concluded by providing the scope for future work.

1. Introduction

The concept of intuitionistic fuzzy set (IFS) (Atanassov) [1] has been widely studied and applied to deal with uncertainties and hesitancy inherent in practical circumstances. The prominent characteristic of an IFS is that it assigns a number from the unit interval \([0, 1]\) to every element in the domain of discourse, a degree of membership, and a degree of non-membership along with the degree of indeterminacy whose total sum equals unity. In literature, intuitionistic fuzzy sets comprehensively span applications in the field of decision-making problems, pattern recognition, sales analysis financial services, medical diagnosis, etc.

Pythagorean fuzzy set (PFS), proposed by Yager [2], is an efficient generalization of intuitionistic fuzzy set, characterized by a membership value and a nonmembership value satisfying the inequality that the squared sum of these values is less than or equal to 1. Yager and Abbasov [3] well stated that, in some practical multiple-criteria decision-making problems, it is viable that sum of the degree of the membership and the degree of nonmembership value of a particular alternative provided by a decision-maker may be in such a way that their sum is bigger than 1, where it would not be feasible to use intuitionistic fuzzy set. Therefore, PFS proves to be proficiently more capable of representing and handling vagueness, impreciseness, and uncertainties than IFS in various decision-making processes. It may be noted that PFS is more generalized than IFS as the span of membership degree of PFS is more than span of membership degree of IFS which enables wider applicability.

Various researchers theoretically developed the concept of Yager’s Pythagorean fuzzy sets [4] and applied it in the field of decision-making problems, medical diagnosis, and pattern recognition and in other real-world problem. In order to deal the decision-making problem with PFSs, Zhang and Xu [5] proposed a comparison method based on a score function to identify the Pythagorean fuzzy positive ideal solution (PIS) and the Pythagorean fuzzy negative ideal solution (NIS). Further they extended the TOPSIS method to compute the distances between each alternative with PIS and NIS, respectively. Peng and Yang [6] proposed some basic operations for PFSs and provided Pythagorean fuzzy aggregation operators along with their important properties. In continuation, they developed a Pythagorean superiority and inferiority ranking algorithm to solve group decision-making problems in view of uncertainty. Further, Peng et al. [7] established the relationship between the distance measure, similarity measure, entropy, and the inclusion measure and suggested the systematic transformation of information measures for PFSs. Yager [8] introduced some of the basic set operations for PFSs...
and established the relationship between Pythagorean membership values and complex number. In addition to this, the solutions of multicriteria decision-making with satisfactions through Pythagorean membership values have been carried out. A new method for Pythagorean fuzzy MCDM problems with the help of aggregation operators and distance measures has been developed by Zeng et al. [9]. Further, they proposed the Pythagorean fuzzy ordered weighted averaging weighted average distance (PFOWAWAD) operator and developed a hybrid TOPSIS method.

Using PFSs, Ren et al. [10] had a simulation test to study the effect of the risk attitudes of the decision makers over the solutions of decision-making problems. Zhang [11] introduced a novel closeness index-based ranking method for Pythagorean fuzzy numbers and proposed interval valued Pythagorean fuzzy set with basic operations and important properties. In addition to this, the hierarchical multicriteria decision-making problems in Pythagorean fuzzy environment have been solved by developing a closeness index-based Pythagorean fuzzy QUALIFLEX method. Liu et al. [12] developed various types of Pythagorean fuzzy aggregation operators and used them to solve decision-making problems. Zeng [13] developed a Pythagorean fuzzy multiattribute group decision-making method on the basis of a new Pythagorean fuzzy probabilistic ordered weighted averaging (OWA) operator. Though various researchers have significantly contributed in the development of the theory of PFSs as deliberated above, a seldom study on the entropy of PFSs and its applications has been found in literature. Xue et al. [14] studied the linear programming technique for multidimensional analysis of preference (LINMAP) method under Pythagorean fuzzy environment to solve multiple attribute group decision-making problem by incorporating Pythagorean fuzzy entropy along with various other applications. Vital applications of entropy and information measures based on the IFS theory have been well known in the literature. In order to deal with real-world problems more efficiently and to cater the need of the hour, generalizations of the existing approaches play an important role as they contribute more flexibility in applications; e.g., parameters may characterize various factors such as time constraint, lack of knowledge, and environmental conditions, etc.

Bajaj et al. [15] proposed a new $R$-norm intuitionistic fuzzy entropy and a weighted $R$-norm Intuitionistic fuzzy divergence measure with their computational applications in pattern recognition and image thresholding. Gandotra et al. [16] studied multiple-criteria decision-making problem with the help of parametric entropy under $\alpha$-cut and $(\alpha, \beta)$-cut based distance measures for different possible values of parameters and provided the ranking method for the available alternatives.

In this communication, we have proposed a new $(R, S)$-norm information measure of Pythagorean fuzzy set and applied the information measure in an algorithm to solve multicriteria decision-making problem. In continuation, the implementation of the proposed algorithm by taking suitable examples has also been illustrated. The rest of this paper is organized as follows: in Section 2, we present some basic notions and preliminaries related to the proposed information measure. A new $(R, S)$-norm information measure of Pythagorean fuzzy set has been well proposed with the proof of its validity in Section 3. Further, in Section 4, the maximality and the monotonic behavior of the proposed information measure with respect to parameters $R$ and $S$ have been studied and validated empirically. In Section 5, a new multicriteria decision-making algorithm is provided on the basis of the proposed $(R, S)$-norm information measure of PFS in view of two cases of weights of criteria: one when weights are unknown and other when weights are partially known. In order to support and implement the proposed algorithm, an example for each case has been explicitly dealt in Section 6. The paper is finally concluded in Section 7.

2. Preliminaries

In this section, we recall and present some fundamental concepts in connection with Pythagorean fuzzy set, which are well known in literature.

**Definition 1** (see [1]). An intuitionistic fuzzy set (IFS) $I$ in $X$ (universe of discourse) is given by

$$I = \{ (x, \mu_I(x), \nu_I(x)) \mid x \in X \}$$  \hspace{1cm} (1)

where $\mu_I : X \rightarrow [0, 1]$ and $\nu_I : X \rightarrow [0, 1]$ denote the degree of membership and degree of nonmembership, respectively, and for every $x \in X$ satisfy the condition

$$0 \leq \mu_I(x) + \nu_I(x) \leq 1$$  \hspace{1cm} (2)

and the degree of indeterminacy for any IFS $I$ and $x \in X$ is given by $\pi_I(x) = 1 - \mu_I(x) - \nu_I(x)$.

**Definition 2** (see [2]). A Pythagorean fuzzy set (PFS) $M$ in $X$ (universe of discourse) is given by

$$M = \{ (x, \mu_M(x), \nu_M(x)) \mid x \in X \}$$  \hspace{1cm} (3)

where $\mu_M : X \rightarrow [0, 1]$ and $\nu_M : X \rightarrow [0, 1]$ denote the degree of membership and degree of nonmembership, respectively, and for every $x \in X$ satisfy the condition

$$0 \leq \mu_M^2(x) + \nu_M^2(x) \leq 1$$  \hspace{1cm} (4)

and the degree of indeterminacy for any PFS $M$ and $x \in X$ is given by

$$\pi_M(x) = \sqrt{1 - \mu_M^2(x) - \nu_M^2(x)}.$$  \hspace{1cm} (5)

In case of PFS, the restriction corresponding to the degree of membership $\mu_M(x)$ and the degree of nonmembership $\nu_M(x)$ is

$$0 \leq \mu_M^2(x) + \nu_M^2(x) \leq 1,$$  \hspace{1cm} (6)

whereas the condition in case of IFS is

$$0 \leq \mu_I(x) + \nu_I(x) \leq 1,$$  \hspace{1cm} (7)

for $\mu_M(x), \nu_M(x) \in [0, 1]$. This difference in constraint conditions gives a wider coverage for information span which can be geometrically shown in Figure 1.

Some of the important binary operations on PFSs are being presented below which are available in literature.
\[ \sum_{i=1}^{\infty} \Delta \text{function from Joshi and Kumar [17] defined and studied a real valued measure of the distribution.} \]

\[ \text{Let } (R, S) \text{ be two PFSs, then the Euclidean distance between } M \text{ and } N \text{ is defined as follows:} \]

\[ d(M, N) = \sqrt{\frac{1}{2} \left[ \left( (\mu_M(x))^2 - (\mu_N(x))^2 \right)^2 + \left( (\nu_M(x))^2 - (\nu_N(x))^2 \right)^2 + \left( (\pi_M(x))^2 - (\pi_N(x))^2 \right)^2 \right].} \] (8)

**Definition 3** (see [7]). If M and N are two Pythagorean fuzzy sets in X, then the operations can be defined as follows:

(a) **Complement:** \( M' = \{ (x, \nu_M(x), \mu_M(x)) \mid x \in X \} \).

(b) **Containment:** \( M \subset N \) iff \( \forall x \in X, \mu_M(x) \leq \mu_N(x) \) and \( \nu_M(x) \geq \nu_N(x) \).

(c) **Union:** \( M \cup N = \{ (x, \mu_M(x) \vee \mu_N(x), \nu_M(x) \wedge \nu_N(x)) \mid x \in X \} \).

(d) **Intersection:** \( M \cap N = \{ (x, \mu_M(x) \wedge \mu_N(x), \nu_M(x) \vee \nu_N(x)) \mid x \in X \} \).

**Definition 4** (see [10]). Let M and N be two PFSs, then the Hamming distance between M and N is defined as follows:

\[ l(M, N) = \frac{1}{2} \left[ \left( (\mu_M(x))^2 - (\mu_N(x))^2 \right)^2 + \left( (\nu_M(x))^2 - (\nu_N(x))^2 \right)^2 + \left( (\pi_M(x))^2 - (\pi_N(x))^2 \right)^2 \right]. \] (9)

\[ d(M, N) = \sqrt{\frac{1}{2} \left[ \left( (\mu_M(x))^2 - (\mu_N(x))^2 \right)^2 + \left( (\nu_M(x))^2 - (\nu_N(x))^2 \right)^2 + \left( (\pi_M(x))^2 - (\pi_N(x))^2 \right)^2 \right]^2}. \] (10)

where \( 0 < S < 1 \) and \( 1 < R < \infty \). Or \( 0 < S < 1 \) and \( 1 < R < \infty \).

The most important property of this measure is that when \( S=1 \) or \( R=1 \), then (10) becomes the R or S-norm entropy studied by Boeke and Lubbe [18] and if \( R = 1 \) and \( S \to 1 \) or \( S = 1 \) and \( R \to 1 \), then it gives Shannon's [19] entropy.

Based on the axiomatic definition of entropy for intuitionistic fuzzy set, proposed by Hung and Yang (2006) [20], we analogously define a real valued function \( H : X \to [0, 1] \), called entropy of Pythagorean fuzzy set \( M \) if and only if the following four axioms are satisfied:

(i) **(PFS1) Sharpness:** \( H(M) = 0 \) iff \( M \) is a crisp set, i.e., \( \mu_M(x_i) = 0, \nu_M(x_i) = 1; \) or \( \mu_M(x_i) = 1, \nu_M(x_i) = 0; \forall x_i \in X \).

(ii) **(PFS2) Maximal:** \( H(M) \) is maximum iff \( \mu_M(x_i) = \nu_M(x_i) = \pi_M(x_i) = \frac{1}{\sqrt{3}} \forall x_i \in X \).

(iii) **(PFS3) Symmetry:** \( H(M) = H(M^c) \).

(iv) **(PFS4) Resolution:** \( H(M) \leq H(N) \) iff \( M \subset N \), i.e., \( \mu_M(x_i) \leq \mu_N(x_i) \) and \( \nu_M(x_i) \geq \nu_N(x_i) \) or \( \nu_M(x_i) \geq \nu_N(x_i) \) for \( \mu_M(x_i) \leq \nu_M(x_i) \).

In context with Pythagorean fuzzy information, we propose the following Pythagorean fuzzy entropy analogous to measure (10):

\[ H_S^S(M) = \frac{R \times S}{(R-S)^R} \left[ \sum_{i=1}^{n} \left( \left( \mu_M(x_i)^{2S} + \nu_M(x_i)^{2S} + \pi_M(x_i)^{2S} \right)^{1/S} - \left( \mu_N(x_i)^{2S} + \nu_N(x_i)^{2S} + \pi_N(x_i)^{2S} \right)^{1/S} \right)^{1/R} \right]. \] (12)

where \( R, S > 0 \), either \( 0 < S < 1 \) and \( 1 < R < \infty \) or \( 0 < S < 1 \) and \( 1 < R < \infty \),

where \( S = 1 \), \( R > 0 \), \( R \neq 1 \),

where \( S = 1 \) and \( R \to 1 \) or \( S = 1 \) and \( R \to 1 \).
Theorem 6. The proposed entropy measure $H_R^S(M)$ is a valid Pythagorean fuzzy information measure.

Proof. To prove this, we shall show that it satisfies all the axioms PFS1 to PFS4.

(i) (PFS1) (Sharpness): If $H_R^S(M) = 0$, then
\[
\left(\mu_M(x_i) - \frac{1}{\sqrt{3}}\right)^2 + \left(\nu_M(x_i) - \frac{1}{\sqrt{3}}\right)^2 + \left(\pi_M(x_i) - \frac{1}{\sqrt{3}}\right)^2 = 0.
\]

(ii) (PFS2) (Maximality):
In Section 4, we have empirically proved that $H_R^S(M)$ is maximum iff
\[
\mu_M(x_i) = \nu_M(x_i) = \pi_M(x_i) = \frac{1}{\sqrt{3}}.
\]
Analytically, we prove the concavity of the $H_R^S(M)$ by calculating its hessian at the critical point, i.e., $1/\sqrt{3}$ with particular values of $R$ and $S$. The Hessian of $H_R^S(M)$ is as $[R > 1 (= 3)$ and $S < 1 (= 0.3)]$:
\[
H_R^S(M) = \frac{2}{n} \begin{bmatrix}
-10.4589 & 2.232816 & 2.232816 \\
2.232816 & -10.4589 & 2.232816 \\
2.232816 & 2.232816 & -10.4589
\end{bmatrix}.
\]

We may be noted that $H_R^S(M)$ is a negative semidefinite matrix for different possible values of $R$ and $S$ which shows that it is a concave function. Hence, the concavity of the function establishes the maximality property.

(iii) (PFS3) (Symmetry): It is obvious from the definition that
\[
H_R^S(M) = H_R^S(M^c).
\]

(iv) (PFS4) (Resolution): We have
\[
\left|\left(\mu_M(x_i) - \frac{1}{\sqrt{3}}\right)\right| + \left|\left(\nu_M(x_i) - \frac{1}{\sqrt{3}}\right)\right| + \left|\left(\pi_M(x_i) - \frac{1}{\sqrt{3}}\right)\right| \geq \left(\mu_N(x_i) - \frac{1}{\sqrt{3}}\right) + \left(\nu_N(x_i) - \frac{1}{\sqrt{3}}\right) + \left(\pi_N(x_i) - \frac{1}{\sqrt{3}}\right).
\]

Now, since $H_R^S(M)$ is a concave function on the Pythagorean fuzzy set $M$, therefore, if max$[\mu_M(x_i), \nu_M(x_i)] \leq 1/\sqrt{3}$ then $\mu_M(x_i) \leq \mu_N(x_i)$ and $\nu_M(x_i) \leq \nu_N(x_i)$ imply $\pi_M(x_i) = \pi_N(x_i)$.

Therefore, by the above explained result, we conclude that $H_R^S(M)$ satisfies condition of resolution PFS4.

Similarly, if $\min[\mu_M(x_i), \nu_M(x_i)] \geq 1/\sqrt{3}$, then $\mu_M(x_i) \geq \mu_N(x_i)$ and $\nu_M(x_i) \geq \nu_N(x_i)$. By using the above proved result, we conclude that $H_R^S(M)$ satisfies the condition PFS4.

Hence, $H_R^S(M)$ satisfies all the axioms of Pythagorean fuzzy entropy and, therefore, $H_R^S(M)$ is a valid measure of Pythagorean fuzzy information.

$\square$

Theorem 7. Let $M$ and $N$ be two PFSs defined in $X = \{x_1, x_2, \ldots, x_n\}$ where $M = \{M(x_i), \nu_M(x_i), \pi_M(x_i)\} \mid x_i \in X$ and $N = \{M(x_i), \nu_N(x_i), \pi_N(x_i)\} \mid x_i \in X$ such that $\forall x_i \in X$ either $M \subseteq N$ or $N \subseteq M$. Then
\[
H_R^S(M \cup N) + H_R^S(M \cap N) = H_R^S(M) + H_R^S(N).
\]

Proof. Divide $X$ into two parts $X_1$ and $X_2$ such that $X_1 = \{x_i \in X \mid M \subseteq N\}$, i.e., $\mu_M(x_i) \leq \mu_N(x_i), \nu_M(x_i) \leq \nu_N(x_i) \forall x_i \in X_1; X_2 = \{x_i \in X \mid N \subseteq M\}$, i.e., $\mu_M(x_i) \geq \mu_N(x_i), \nu_M(x_i) \leq \nu_N(x_i) \forall x_i \in X_2$. Now
\[
H_R^S(M \cup N) = \frac{R \times S}{(R - S)} \sum_{i=1}^{n} \left(\mu_{M \cup N}(x_i) - \frac{1}{\sqrt{3}}\right)^2 + \nu_{M \cup N}(x_i) - \frac{1}{\sqrt{3}}\right)^2 + \pi_{M \cup N}(x_i) - \frac{1}{\sqrt{3}}\right)^2
\]
\[
+ \left(\mu_{M \cup N}(x_i) - \frac{1}{\sqrt{3}}\right) + \left(\nu_{M \cup N}(x_i) - \frac{1}{\sqrt{3}}\right) + \left(\pi_{M \cup N}(x_i) - \frac{1}{\sqrt{3}}\right);
\]

and
\[
H_R^S(M) + H_R^S(N) = \frac{R \times S}{(R - S)} \sum_{i=1}^{n} \left(\mu_M(x_i) - \frac{1}{\sqrt{3}}\right)^2 + \nu_M(x_i) - \frac{1}{\sqrt{3}}\right)^2 + \pi_M(x_i) - \frac{1}{\sqrt{3}}\right)^2
\]
\[
+ \left(\mu_N(x_i) - \frac{1}{\sqrt{3}}\right) + \left(\nu_N(x_i) - \frac{1}{\sqrt{3}}\right) + \left(\pi_N(x_i) - \frac{1}{\sqrt{3}}\right);
\]

because if $\mu_M(x_i) \leq \mu_N(x_i)$ and $\nu_M(x_i) \leq \nu_N(x_i)$ with max$[\mu_M(x_i), \nu_M(x_i)] \leq 1/\sqrt{3}$, then $\mu_M(x_i) \leq \mu_N(x_i)$ and $\nu_M(x_i) \leq \nu_N(x_i)$ and $\pi_M(x_i) \geq \pi_N(x_i)$ which implies that the above result holds. Similarly, if $\mu_M(x_i) \geq \mu_N(x_i)$ and $\nu_M(x_i) \geq \nu_N(x_i)$ with max$[\mu_M(x_i), \nu_M(x_i)] \geq 1/\sqrt{3}$, then also the above result holds.

$\square$
Proof. By definition, the proof is obvious.

which implies

\[ H^S_R(M \cup N) = \frac{R \times S}{(R - S)} \]

\[ + \sum_{x \in X} \left[ \left( \mu_N(x_i) \right)^2S + v_N(x_i)^2S + \pi_N(x_i)^2S \right]^{1/3} \]

\[ - \left( \mu_N(x_i)^2R + \pi_N(x_i)^2R \right]^{1/3} \]

4. Monotonicity of \((R, S)\)-Norm Information Measure of PFS

In this section, we carry an empirical study for investigating the maximality and monotonic nature of the proposed \((R, S)\)-norm information measure of PFS. For this, we consider the following four Pythagorean fuzzy sets \(M_1, M_2, M_3, \) and \(M_4 \) over the universe of discourse \(X = \{x_1, x_2, x_3\} \):

\[ M_1 = \left\{ \left(x_1, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right), \left(x_2, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \right\} \]

\[ M_2 = \left\{ \left(x_1, 0.6, 0.6 \right), \left(x_2, 0.7, 0.7 \right), \left(x_3, 0.55, 0.55 \right) \right\} \]

\[ M_3 = \left\{ \left(x_1, 0.5, 0.6 \right), \left(x_2, 0.2, 0.9 \right), \left(x_3, 0.9, 0.3 \right) \right\} \]

\[ M_4 = \left\{ \left(x_1, 0.4, 0.8 \right), \left(x_2, 0.9, 0.4 \right), \left(x_3, 0.7, 0.6 \right) \right\} \]

For various values of \(R\) and \(S\) and using equation (12), we compute and tabulate all the values of \(H^S_R(M)\). On the basis of the tabulated data and the plots given in Table 1 and Figure 2, it is quite clear that \(H^S_R(M)\) takes maximum value when \(\mu_M(x_i) = \gamma_M(x_i) = \pi_M(x_i) = 1/\sqrt{3}; \forall x_i \in X\) and is a monotonically decreasing function of \(R\) and \(S\).

5. MCDM Algorithm with \((R, S)\)-Norm Entropy

Suppose that there is a set of \(m\) feasible alternatives, i.e., \(Z = \{z_1, z_2, \ldots, z_m\}\) and a set of \(n\) criteria \(O = \{o_1, o_2, \ldots, o_n\}\). The decision-making problem is to select the most suitable alternative out of these \(m\) alternatives. The appraisal values of an alternative \(z_i (i = 1, 2, 3, \ldots, m)\) with respect to the criteria \(o_j (j = 1, 2, 3, \ldots, n)\) are given by \(p_{ij} = (p_{ij}, q_{ij})\), where \(p_{ij}\) is the degree to which the alternative \(z_i\) satisfies the criteria \(o_j\) and \(q_{ij}\) is the degree to which the alternative \(z_i\) does not satisfy the criteria \(o_j\), satisfying \(0 \leq p_{ij} \leq 1, 0 \leq q_{ij} \leq 1\) and \(0 \leq p_{ij} + q_{ij} \leq 1\) with \(i = 1, 2, 3, \ldots, m\) and \(j = 1, 2, 3, \ldots, n\). This problem can be modeled by representing it through the following Pythagorean fuzzy decision matrix:

\[ R = (p_{ij}, q_{ij})_{m \times n} = (z_{ij}) \]

\[ o_1 \quad o_2 \quad \cdots \quad o_n \]

\[ z_1 \quad (p_{11}, q_{11}) \quad (p_{12}, q_{12}) \quad \cdots \quad (p_{1m}, q_{1n}) \]

\[ z_2 \quad (p_{21}, q_{21}) \quad (p_{22}, q_{22}) \quad \cdots \quad (p_{2n}, q_{2n}) \]

\[ \vdots \quad \vdots \quad \ddots \quad \vdots \]

\[ z_m \quad (p_{m1}, q_{m1}) \quad (p_{m2}, q_{m2}) \quad \cdots \quad (p_{mn}, q_{mn}) \]

Let \(w = (w_1, w_2, \ldots, w_n)^T\) be the weight vector of all the criteria where \(0 \leq w_j \leq 1\) and \(\sum_{j=1}^{n} w_j\) is the degree of importance of the \(j\)th criteria. Sometimes this criteria weight is completely unknown and sometimes it is partially known because of the lack of knowledge, time, data, and the limited expertise of the problem domain. In this section, we discuss and devise two methods to determine the weights of criteria by using the proposed entropy (12).
Table 1: Values of entropy for different values of \( R \) and \( S \).

<table>
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<tr>
<th>Sl. No.</th>
<th>( R )</th>
<th>( S = 0.15 )</th>
<th>( S = 0.25 )</th>
<th>( S = 1 )</th>
<th>( S = 4 )</th>
<th>( S = 15 )</th>
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<td>19.000</td>
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<td>0.3200</td>
<td>0.2900</td>
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<td>0.2050</td>
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<td>0.1720</td>
<td>0.1900</td>
<td>0.1700</td>
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<tr>
<td>13</td>
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<tr>
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<td>0.0170</td>
<td>0.0200</td>
<td>0.0150</td>
<td>0.0130</td>
</tr>
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</table>
Case 1 (unknown weights). When the criteria weights are completely unknown, then we calculate the weights by using the proposed PFS entropy as

\[ w_j = \frac{1 - e_j}{n - \sum_{j=1}^{n} e_j}, \quad j = 1, 2, \ldots, n; \quad (27) \]

where \( e_j = \frac{1}{m} \sum_{i=1}^{m} H_R^S(z_{ij}) \), and

\[
H_R^S(z_{ij}) = \frac{R \times S}{(R - S)} \left[ \frac{1}{m} \sum_{i=1}^{m} \left( \mu_M(x_i)^2S + \nu_M(x_i)^2S + \pi_M(x_i)^2S \right)^{1/S} \right]
- \left( \mu_M(x_i)^2R + \nu_M(x_i)^2R + \pi_M(x_i)^2R \right)^{1/R} \quad (28)
\]
is the proposed Pythagorean fuzzy entropy for \( z_{ij} = (p_{ij}, q_{ij}) \).
Case 2 (partially known weights). In case the weights are partially known for a multiple-criteria decision-making problem, we use the minimum entropy principle (Wang and Wang [21]) to determine the weight vector of the criteria by constructing the programming model as follows.

The overall entropy of the alternative \( z_i \) is

\[
E[z_i] = \sum_{j=1}^{n} H^S_R(z_{ij})
\]

\[
= \frac{R \times S}{(R - S)} \sum_{j=1}^{n} \left\{ \sum_{i=1}^{m} \frac{1}{m} \left\{ \frac{1}{m} \left( (\mu_M(x_i)^{2S} + \nu_M(x_i)^{2S} + \pi_M(x_i)^{2S})^{1/S} - (\mu_M(x_i)^{2R} + \nu_M(x_i)^{2R} + \pi_M(x_i)^{2R})^{1/R} \right) \right\} \right\};
\] (29)

where \( R, S > 0; R > 1, S < 1 \) or \( R < 1, S > 1 \).

Step 4. By using Definition 5, the distance measures of \( z_i^- \) from \( z^+ \) and \( z^- \) will be evaluated as follows:

\[
l(z_i, z^+) = \frac{1}{2} \sum_{j=1}^{n} w_j \left( \left| (\alpha_{ij} - (\alpha_j^+))^2 \right| + \left| (\beta_{ij} - (\beta_j^+))^2 \right| + \left| (\pi_{ij} - (\pi_j^+))^2 \right| \right);
\] (33)

and

\[
l(z_i, z^-) = \frac{1}{2} \sum_{j=1}^{n} w_j \left( \left| (\alpha_{ij} - (\alpha_j^-))^2 \right| + \left| (\beta_{ij} - (\beta_j^-))^2 \right| + \left| (\pi_{ij} - (\pi_j^-))^2 \right| \right);
\] (34)

Step 5. Determine the relative degrees of closeness \( l_i^- \) as follows:

\[
l_i = \frac{l(z_i, z^-)}{l(z_i, z^+) + l(z_i, z^-)}.
\] (35)

Step 6. On the basis of the relative degree of closeness obtained in Step 5, we determine the optimal ranking order of the alternatives. The alternative with the maximal degree of closeness \( l(z_i) \) is supposed to be the best alternative.

6. Numerical Examples

Based on two different cases considered in the proposed algorithm, we present two different examples as follows.

Example 1 (unknown weights). Suppose an automobile company produces four different cars, say, \( z_1, z_2, z_3, \) and \( z_4 \). Suppose a customer wants to buy a car based on the four given criteria, say, comfort \( o_1 \), good mileage \( o_2 \), safety \( o_3 \), and interiors \( o_4 \). Assume the appraisal values of the alternatives with respect to each criterion provided by the expert are represented by PFS as

\[
o_1 = (0.9, 0.3, 0.7, 0.6) \quad o_2 = (0.5, 0.8) \quad o_3 = (0.6, 0.3)
\]

\[
z_1 = (0.9, 0.3, 0.7, 0.6) \quad z_2 = (0.5, 0.8)
\]

\[
z_3 = (0.8, 0.4) \quad z_4 = (0.7, 0.5)
\]

\[
z_1 = (0.9, 0.3, 0.7, 0.6) \quad z_2 = (0.5, 0.8)
\]

\[
z_3 = (0.8, 0.4) \quad z_4 = (0.7, 0.5)
\]
Then the calculations for the ranking procedure are as follows:

1. Calculate the criteria weight vector using (27):
\[ w = (w_1, w_2, w_3, w_4)^T = (0.272107, 0.263037, 0.34878, 0.116077)^T. \] (37)

2. The most preferred solution \((z^+)\) and the least preferred solution \((z^-)\) are given by
\[ z^+ = \{(0.9, 0.3), (0.9, 0.2), (0.8, 0.1), (0.7, 0.4)\} \] (38)
and
\[ z^- = \{(0.4, 0.7), (0.7, 0.6), (0.5, 0.8), (0.6, 0.6)\} \] (39)
respectively.

3. The distances between each of \(z_i\)'s from \(z^+\) and \(z^-\) are given by
   \[ I(z_1, z^+) = 0.040622, \]
   \[ I(z_2, z^+) = 0.186515, \]
   \[ I(z_3, z^+) = 0.006623, \]
   \[ I(z_4, z^+) = 0.048795, \]
   \[ I(z_1, z^-) = 0.209804, \]
   \[ I(z_2, z^-) = 0.13179, \]
   \[ I(z_3, z^-) = 0.177491, \]
   \[ I(z_4, z^-) = 0.116968. \] (40)

4. The values of relative degree of closeness are as follows:
   \[ l_1 = 0.837788, \]
   \[ l_2 = 0.414036, \]
   \[ l_3 = 0.728256, \]
   \[ l_4 = 0.705633. \] (41)

5. The ranking of the alternatives as per the relative degree of closeness is \(z_1 > z_3 > z_4 > z_2\) and \(z_1\) is the best available alternative. It may be noted that the above ranking is with respect to the specific values of \(R = 3\) and \(S = 0.3\).

The consistency of the ranking procedure for different values of parameters \(R\) and \(S\) may also be observed and studied by making a simulation study over the varying values of the parameters depending on the requirement.

Example 2 (partially known weights). Suppose there are 1000 students in a college. On the basis of three selected criteria, say, \(o_1\) (personality), \(o_2\) (intelligence), and \(o_3\) (communication skills), the administration wants to select a college representative. Let there be three candidates, say, \(z_1, z_2,\) and \(z_3\). The PFS decision matrix for the above problem is
\[
\begin{array}{ccc}
o_1 & o_2 & o_3 \\
(z_1) & (0.8,0.5) & (0.6,0.6) & (0.8,0.2) \\
(z_2) & (0.6,0.5) & (0.7,0.4) & (0.8,0.4) \\
(z_3) & (0.5,0.7) & (0.7,0.6) & (0.9,0.3)
\end{array}
\] (42)
Let the information about the criteria weight be partially given in the following form \[0.10 \leq w_1 \leq 0.30, 0.35 \leq w_2 \leq 0.60, 0.25 \leq w_3 \leq 0.70\]. The calculation for the ranking procedure for the above decision-making problem is presented as follows:

1. Using (30), we determine the criteria weights by following linear programming model:

   \[
   \begin{align*}
   \min E &= 0.609037w_1 + 0.641365w_2 + 0.590874w_3 \\
   \text{subject to} & \quad 0.10 \leq w_1 \leq 0.30, \\
   & \quad 0.35 \leq w_2 \leq 0.60, \\
   & \quad 0.25 \leq w_3 \leq 0.70, \\
   & \quad w_1 + w_2 + w_3 = 1.
   \end{align*}
   \] (43)

   By solving this linear programming problem using MATLAB software, we obtained the criteria weight vector as follows:

   \[
   w = (0.10, 0.35, 0.55)^T. \] (44)

2. The most preferred solution \(z^+\) and the least preferred solution \(z^-\) are given by \(z^+ = \{(0.8, 0.5), (0.7, 0.4), (0.9, 0.2)\}\) and \(z^- = \{(0.5, 0.7), (0.6, 0.6), (0.8, 0.4)\}\), respectively.

3. The distances between each of \(z'_i\) from \(z^+\) and \(z^-\) are given by

   \[
   \begin{align*}
   l(z_1, z^+) &= 0.013843, \\
   l(z_2, z^+) &= 0.015888, \\
   l(z_3, z^+) &= 0.068163, \\
   l(z_1, z^-) &= 0.052213, \\
   l(z_2, z^-) &= 0.026855, \\
   l(z_3, z^-) &= 0.049273
   \end{align*}
   \] (45)

4. The values of relative degree of closeness are

   \[
   \begin{align*}
   l_1 &= 0.79044, \\
   l_2 &= 0.628297, \\
   l_3 &= 0.419573.
   \end{align*}
   \] (46)

5. The ranking of the alternatives as per the relative degree of closeness is \(z_1 > z_2 > z_3\) and \(z_1\) is the best available alternative. It may be noted that the above ranking is with respect to the specific values of \(R = 3\) and \(S = 0.3\).

Data Availability

The data for the implementation of the proposed algorithm in the numerical example are hypothetical data and have no connection with any particular agency’s data.

Disclosure

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References


