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Research Article

Aumann Fuzzy Improper Integral and Its Application to Solve Fuzzy Integro-Differential Equations by Laplace Transform Method

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We introduce the Aumann fuzzy improper integral to define the convolution product of a fuzzy mapping and a crisp function in this paper. The Laplace convolution formula is proved in this case and used to solve fuzzy integro-differential equations with kernel of convolution type. Then, we report and correct an error in the article by Salahshour et al. dealing with the same topic.

1. Introduction

Integrals of set-valued functions have been studied in connection with statistical problems and have arisen in connection with economic problems. The basic theory of such integrals was developed by Aumann [1]. Ralescu and Adams defined in [2] the fuzzy integral of a positive, measurable function, with respect to a fuzzy measure, and studied some properties of this integral. Dubois and Prade [3] generalized the Riemann integral over a closed interval to fuzzy mappings. Their approach was more directly related to the works by Aumann [1] and Debreu [4] on multifunctions integration.

Puri and Ralescu [5] generalized the integral of a set-valued function to define the concepts of fuzzy random variable and its expectation. Wu proposed in [6] two types of the fuzzy Riemann integral; the first one was based on the crisp compact interval and the second one was considered on the fuzzy interval, provided a numerical method to approximate this integral by invoking the Simpson's rule, and transformed its membership function into nonlinear programming problem.

In [7], Allahviranloo et al. proposed an integral method for solving fuzzy linear differential equations, under the assumption of strongly generalized differentiability, but they omitted the proofs of their main results. Extending their method, we developed in [8] a more general integral operator method for solving some first-order fuzzy linear differential equations with variable coefficients, and we gave the general formula's solution with necessary proofs.

The notions of the fuzzy improper Riemann integral, the fuzzy random variable, and its expectation were also investigated and studied by Wu in [9] using a different approach.

This concept of improper fuzzy Riemann integral was later exploited by Allahviranloo and Ahmadi in [10] to introduce the fuzzy Laplace transform, which they used to solve some first-order fuzzy differential equations (FDEs). Salahshour and Allahviranloo gave in [11] some applications of fuzzy Laplace transform and studied sufficient conditions ensuring its existence. Recently in [12], we extended and used the fuzzy Laplace transform method to solve second-order fuzzy linear differential equations under strongly generalized Hukuhara differentiability. Then we established in [13] some important results about continuity and strongly generalized Hukuhara differentiability of functions defined via improper fuzzy Riemann integrals, and we proved some properties of fuzzy Laplace transforms for two variables functions, which we applied to solve fuzzy linear partial differential equations of first order.

In the same context, Salahshour et al. developed in [14] the fuzzy Laplace transform method to solve fuzzy convolution Volterra integral equation (FCVIE) of the second kind.

But the proof proposed for their main result, Theorem 4.1 was invalid and the arguments presented in this demonstration were incorrect. One can remark that it was literally identical to the corresponding proof in the classical case, without taking into consideration the fuzzy nature of the data.

First let us recall and enounce Theorem 4.1 in [14]; then we will show the invalid arguments presented by the authors, to prove the fuzzy convolution formula.

Theorem 1 (convolution theorem: see Theorem 4.1 in [14]). *If f and g are piecewise continuous fuzzy-valued functions on $[0,\infty[$ and of exponential order p, then*

$$L(f*g)(t) = L\{f(t)\}\cdot L\{g(t)\} = F(s)\cdot G(s), \quad s > p. \tag{1}$$

First notice that f and g are fuzzy-valued functions, so both of the improper integrals $\int_0^\infty e^{-s\tau} f(\tau)d\tau$ and $\int_0^\infty e^{-su} g(u)du$ are fuzzy numbers. Then, we cannot justify the following passage by a simple integral linearity argument:

$$\begin{aligned} &L\{f(t)\}\cdot L\{g(t)\} \\ &\quad = \left(\int_0^\infty e^{-s\tau} f(\tau)\,d\tau\right)\left(\int_0^\infty e^{-su} g(u)\,du\right) \\ &\quad = \int_0^\infty \left(\int_0^\infty e^{-s(\tau+u)} f(\tau)\,g(u)\,du\right)d\tau, \end{aligned} \tag{2}$$

without proving that for each fuzzy number $c \in E$: $\int_0^\infty c\cdot f(t)dt = c\int_0^\infty f(t)dt$.

Moreover, the authors claimed that due to the hypothesis on f and g, the fuzzy Laplace integrals of f and g converge absolutely and hence $\int_0^\infty\int_0^\infty |e^{-st} f(\tau)g(t-\tau)dt|d\tau$ converges.

It was the most important key of their proof as in the crisp case, since it allows us to reverse the order of the double integrals, but unfortunately it is also incorrect, because the notion of the absolute value of a fuzzy number is not defined at least in [14]. Furthermore, the concept of the absolute convergence of a fuzzy improper integral does not make sense in the fuzzy literature.

To overcome all of these obstacles, we propose in the actual paper the convolution product of a crisp mapping and a fuzzy function in Section 4, and we intend to investigate rigorously the case of two fuzzy functions in a future work.

The theory of fuzzy integro-differential equations has many applications and have been studied extensively in the fuzzy literature; for the reader, we refer to [15–17] and the references therein. Concerning the classical integro-differential equations, one can consult [18–20].

The aim of this work is to define the convolution product and to prove a fuzzy Laplace convolution formula, in the purpose of solving the following fuzzy integro-differential equations (FIDEs) with kernel of convolution type:

$$\begin{aligned} &y'(x) = f(x) + \int_0^x k(x-t)\,y(t)\,dt, \\ &y(0) = y_0 = \left(\underline{y}_0, \overline{y}_0\right) \in E, \end{aligned}$$

$$\begin{aligned} &y'(x) = g(x, y(x)) + \int_0^x k(x-t)\,y(t)\,dt \\ &y(0) = y_0 = \left(\underline{y}_0, \overline{y}_0\right) \in E \end{aligned} \tag{3}$$

provided that $f : [0,\infty[\to E$, $g : [0,\infty[\times E \to E$ are continuous fuzzy-valued functions and $k : [0,\infty[\to \mathbb{R}$ is a crisp continuous function verifying some assumptions to be mentioned later.

Then we give some examples to illustrate the efficiency of our method for solving FIDEs.

To achieve this goal, we first introduce the Aumann fuzzy improper integral concept, which we utilize instead of the Riemann fuzzy improper integral used in [10, 12–14].

This new definition of fuzzy generalized (improper) integral is essentially based on the notion of fuzzy integral and the expectation of a fuzzy random variable, introduced by Puri and Ralescu in [5].

The remainder of this paper is organized as follows.

Section 2 is reserved for some preliminaries. And Section 3 is devoted to the definition of the Aumann fuzzy improper integral. In Section 4, fuzzy Laplace transform is introduced, its basic properties are studied, and a particular case of Laplace convolution is investigated. Then in Section 5, the main result about Laplace convolution is enounced and proved. The procedure for solving fuzzy integro-differential equations by fuzzy Laplace transform is proposed and some numerical examples are given in Section 6. In the last section, we present conclusion and a further research topic.

2. Preliminaries

Denote by $P_K(\mathbb{R})$ the family of all nonempty compact convex subsets of $\mathbb{R}$ and define the addition and scalar multiplication in $P_K(\mathbb{R})$ as usual. The distance between two nonempty bounded subsets A and B of $\mathbb{R}$ is defined by the Hausdorff metric

$$d(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}|a-b|, \sup_{b\in B}\inf_{a\in A}|a-b|\right\}. \tag{4}$$

Define

$$E = \{u : \mathbb{R} \longrightarrow [0,1] \mid u \text{ satisfies (i)–(iv) below}\}, \tag{5}$$

where

(i) u is normal, that is, $\exists x_0 \in \mathbb{R}$ for which $u(x_0) = 1$,

(ii) u is fuzzy convex,

(iii) u is upper semicontinuous,

(iv) $\operatorname{supp} u = \{x \in \mathbb{R} \mid u(x) > 0\}$ is the support of u, and its closure $\operatorname{cl}(\operatorname{supp} u)$ is compact.

For $0 < \alpha \le 1$, the α-cut (or level) of u is denoted

$$[u]^\alpha = \{x \in \mathbb{R} \mid u(x) \ge \alpha\}. \tag{6}$$

Then, from (i) to (iv), it follows that the α-level set $[u]^\alpha \in P_K(\mathbb{R})$ for all $0 \le \alpha \le 1$. It is well known that

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \qquad [ku]^\alpha = k\,[u]^\alpha. \tag{7}$$

Let $D : E \times E \to [0, \infty)$ be a function which is defined by the equation

$$D(u,v) = \sup_{0\le\alpha\le1} d\left([u]^\alpha, [v]^\alpha\right), \tag{8}$$

where d is the Hausdorff metric defined in $P_K(\mathbb{R})$. Then, the following properties hold true (see [5, 21]):

(1) (E, D) is a complete metric space.

(2) $D(u+w, v+w) = D(u,v)$ for $u, v, w \in E$.

(3) $D(ku, kv) = |k|\, D(u,v)$ for all $u, v \in E$ and $k \in \mathbb{R}$.

(4) $D(u+w, v+t) \le D(u,v) + D(w,t)$ for all $u, v, w, t \in E$.

Definition 2. A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$, $\overline{u}(r)$, $0 \le r \le 1$, which satisfy the following requirements:

(1) $\underline{u}(r)$ is a bounded nondecreasing left continuous function in $(0,1]$ and right continuous at 0.

(2) $\overline{u}(r)$ is a bounded nonincreasing left continuous function in $(0,1]$ and right continuous at 0.

(3) $\underline{u}(r) \le \overline{u}(r)$ for all $0 \le r \le 1$.

A crisp number k is simply represented by $\underline{u}(r) = \overline{u}(r) = k$, $0 \le r \le 1$.

The following general definition and properties were developed by Puri and Ralescu in [5], for the fuzzy Aumann integral theory in E^n. Here, we restrict their theory to $E = E^1$ instead of E^n.

Let $(\Omega, \mathscr{A}, P)$ be a probability space where the probability measure P is assumed to be nonatomic.

Definition 3 (Puri and Ralescu [5]). A mapping $F : \Omega \to E$ is strongly measurable if for all $\alpha \in [0,1]$ the set-valued function $F_\alpha : \Omega \to \mathscr{P}_K(\mathbb{R})$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is Lebesgue measurable.

A mapping $F : \Omega \to E$ is called integrably bounded if there exists an integrable function k such that $|x| \le k(t)$ for all $x \in F_0(t)$.

Definition 4 (Puri and Ralescu [5]). Let $(\Omega, \mathscr{A}, P)$ be a probability space where the probability measure P is assumed to be nonatomic. A set-valued function is a function $F : \Omega \to \mathscr{P}(\mathbb{R})$ such that $F(\omega) \ne \emptyset$ for every $\omega \in \Omega$. By $L^1(P)$ we denote the space of P-integrable functions $f : \Omega \to \mathbb{R}$. We denote by $S(F)$ the set of all $L^1(P)$ selections of F; that is,

$$S(F) = \left\{ f \in L^1(P) \mid f(\omega) \in F(\omega) \ \text{a.e} \right\}. \tag{9}$$

The Aumann integral of F, denoted by $\int_\Omega F\,dP$ or $\int F$ for short, is defined by

$$\int_\Omega F = \left\{ \int_\Omega f\,dP \mid f \in S(F) \right\}. \tag{10}$$

Definition 5. A strongly measurable and integrably bounded mapping $F : \Omega \to E$ is said to be integrable over Ω if $\int_\Omega F \in E$.

Lemma 6 (Puri and Ralescu [5]). *If $F : \Omega \to \mathscr{P}(\mathbb{R})$ is measurable and integrably bounded, then F is integrable over Ω.*

Theorem 7 (Puri and Ralescu [5]). *If $F_k : \Omega \to \mathscr{P}(\mathbb{R})$ are measurable and if there exists $h \in L^1(P, \mathbb{R})$ such that $\sup_{k\ge1} \|f_k(\omega)\| \le h(\omega)$ for every $f_k \in S(F_k)$ and if $F_k(\omega) \to F(\omega)$ (in the sense of Kuratowski), then $\int_\Omega F_k \to \int_\Omega F$.*

Remark 8 (Puri and Ralescu [5]). It is important to observe that Theorem 7 can be stated in a different form by replacing convergence in the sense of Kuratowski by convergence in the Hausdorff metric. The statement of the theorem remains unchanged provided that we assume that all functions take values in $Q(\mathbb{R})$, the set of all nonempty, compact subsets of $\mathbb{R}$.

Now, we define the Hukuhara difference and the strongly generalized differentiability.

For $u, v \in E$, if there exists $w \in E$ such that $u = v + w$, then w is the Hukuhara difference of u and v denoted by $u \ominus v$.

Definition 9. We say that a fuzzy mapping $f : (a,b) \to E$ is strongly generalized differentiable at $x_0 \in (a,b)$, if there exists an element $f'(x_0) \in E$ such that

(i) for all $h > 0$ sufficiently small, there exist $f(x_0+h) \ominus f(x_0)$; $f(x_0) \ominus f(x_0 - h)$ and

$$\lim_{h\to0^+} \frac{f(x_0+h) \ominus f(x_0)}{h} = \lim_{h\to0^+} \frac{f(x_0) \ominus f(x_0-h)}{h} = f'(x_0) \tag{11}$$

or

(ii) for all $h > 0$ sufficiently small, there exist $f(x_0) \ominus f(x_0+h)$; $f(x_0-h) \ominus f(x_0)$ and

$$\lim_{h\to0^+} \frac{f(x_0) \ominus f(x_0+h)}{(-h)} = \lim_{h\to0^+} \frac{f(x_0-h) \ominus f(x_0)}{(-h)} = f'(x_0) \tag{12}$$

or

(iii) for all $h > 0$ sufficiently small, there exist $f(x_0+h) \ominus f(x_0)$; $f(x_0-h) \ominus f(x_0)$ and

$$\lim_{h\to0^+} \frac{f(x_0+h) \ominus f(x_0)}{h} = \lim_{h\to0^+} \frac{f(x_0-h) \ominus f(x_0)}{(-h)} = f'(x_0) \tag{13}$$

or

(iv) for all $h > 0$ sufficiently small, there exist $f(x_0) \ominus f(x_0+h)$; $f(x_0) \ominus f(x_0-h)$ and

$$\lim_{h\to 0^+} \frac{f(x_0) \ominus f(x_0+h)}{(-h)} = \lim_{h\to 0^+} \frac{f(x_0) \ominus f(x_0-h)}{h} = f'(x_0). \tag{14}$$

All the limits are taken in the metric space (E, D). At the end points of (a, b), we consider only one-sided derivatives.

The following theorem (see [22]) allows us to consider case (i) or (ii) of the previous definition almost everywhere in the domain of the functions under discussion.

Theorem 10. *Let $f : (a, b) \to E$ be strongly generalized differentiable on each point $x \in (a, b)$ in the sense of Definition 9, (iii) or (iv). Then $f'(x) \in \mathbb{R}$ for all $x \in (a, b)$.*

Theorem 11 (see, e.g., [23]). *Let $f : \mathbb{R} \to E$ be a function and denote $f(t) = (\underline{f}(t, r), \overline{f}(t, r))$, for each $r \in [0, 1]$.*

(1) *If f is (i)-differentiable, then $\underline{f}(t, r)$ and $\overline{f}(t, r)$ are differentiable functions and $f'(t) = (\underline{f}'(t, r), \overline{f}'(t, r))$.*

(2) *If f is (ii)-differentiable, then $\underline{f}(t, r)$ and $\overline{f}(t, r)$ are differentiable functions and $f'(t) = (\overline{f}'(t, r), \underline{f}'(t, r))$.*

3. Aumann Fuzzy Improper Integral

Considering the positive measure related to the exponential law on the positive real line $\Omega = [a, \infty[$, defined by $dP = \exp(-x)dx$, where dx refers to the Lebesgue measure.

We define the Aumann fuzzy improper integral $\int_a^\infty F(x)dx$ of a fuzzy function $F : [a, \infty[\to E$, by its α-levels as follows: $[\int_a^\infty F(x)dx]^\alpha = \int_\Omega F_\alpha(x) \exp(x)dP$; that is,

$$\left[\int_a^\infty F(x)\,dx\right]^\alpha = \int_a^\infty F_\alpha(x)\,dx = \left\{\int_a^\infty f(x)\,dx \mid f \in S(F_\alpha)\right\}. \tag{15}$$

Definition 12. A strongly measurable and integrably bounded mapping $F : [a, \infty[\to E$ is said to be integrable over $[a, \infty[$ if $\int_a^\infty F(x)dx \in E$.

Using Lemma 6, we deduce that if $F : [a, \infty[\to E$ is measurable and integrably bounded, then it is integrable over $[a, \infty[$ and $\int_a^\infty F_\alpha(x)dx$ is a real interval, since it is a nonempty, convex, and compact subset of $\mathbb{R}$; that is,

$$\int_a^\infty F_\alpha(x)\,dx = \left[\int_a^\infty \underline{F}(x, \alpha)\,dx, \int_a^\infty \overline{F}(x, \alpha)\,dx\right]. \tag{16}$$

In the parametric form, the fuzzy improper integral $\int_a^\infty F(x)dx$ can be written

$$\int_a^\infty F(x)\,dx = \left(\int_a^\infty \underline{F}(x, \alpha)\,dx, \int_a^\infty \overline{F}(x, \alpha)\,dx\right). \tag{17}$$

Taking $F_{k,\alpha}(x) = \chi_{[a,k]}(x) \cdot F_\alpha(x)$ in Theorem 7 and Remark 8 implies the following result.

Theorem 13. *If $F : [a, \infty[\to E$ is measurable and integrably bounded, then for all $0 \le \alpha \le 1$*

$$\int_a^T F_\alpha(x)\,dx \longrightarrow \int_a^\infty F_\alpha(x)\,dx \quad \text{as } T \longrightarrow \infty. \tag{18}$$

Since the Aumann integral over $[a, T]$ is linear (see [24]), then from Theorem 13, we deduce the linearity of the Aumann improper fuzzy integral over $[a, \infty[$.

Lemma 14. *If $F, G : [a, \infty[\to E$ are (fuzzy) integrable over $[a, \infty[$, then for all real λ the mappings $F + G$ and λF are integrable over $[a, \infty[$ and we have*

$$\int_a^\infty (F(x) + G(x))\,dx = \int_a^\infty F(x)\,dx + \int_a^\infty G(x)\,dx, \qquad \int_a^\infty \lambda F(x)\,dx = \lambda \int_a^\infty F(x)\,dx. \tag{19}$$

Remark 15. Analogously, we define the integrability and the Aumann fuzzy improper integral $\int_{-\infty}^a F(x)dx$ of a fuzzy function $F :]-\infty, a] \to E$.

Then, we said that a fuzzy mapping $F : \mathbb{R} \to E$ is integrable over $\mathbb{R}$, if it is integrable over $]-\infty, a]$ and over $[a, \infty[$, for each real a. In this case, we define

$$\int_{-\infty}^\infty F(x)\,dx = \int_{-\infty}^a F(x)\,dx + \int_a^\infty F(x)\,dx. \tag{20}$$

For more details concerning Aumann fuzzy improper integral, one can see [5].

Remark 16. The concepts of the fuzzy improper integral, the fuzzy random variable, and its expectation were defined and studied in a different way by Wu in [9]. His proposal of the improper fuzzy Riemann integral was an appropriate attempt for finding the expectations of fuzzy random variables numerically.

He stated that the developments in [5] were in measure-theoretic sense; thus, it was difficult to provide a numerical method in applications.

However, this statement seems to be false because of the approach developed in our present article and precisely by the identities (16) and (17); the Aumann fuzzy improper integral (and the integral over a compact subset of $\mathbb{R}$) has the same properties and qualities as well as the improper fuzzy Riemann integral.

4. Fuzzy Laplace Convolution

Definition 17 (see [10]). Let $f(x)$ be continuous fuzzy-valued function. Suppose that $e^{-p_0 x} f(x)$ is integrable on $[0, \infty[$, for some $p_0 > 0$, then for all $p \ge p_0$ the improper integral $\int_0^\infty e^{-px} f(x)dx$, which is well defined, is called fuzzy Laplace transform of f and is denoted as

$$\mathbf{L}[f(x)] = \int_0^\infty e^{-px} f(x)\,dx, \quad p \ge p_0. \tag{21}$$

If $\mathscr{L}(g(x))$ denotes the classical Laplace transform of a crisp function $g(x)$, then since

$$\int_0^\infty e^{-px} f(x)\,dx = \left(\int_0^\infty e^{-px} \underline{f}(x,r)\,dx, \int_0^\infty e^{-px} \overline{f}(x,r)\,dx\right), \tag{22}$$

we have

$$\mathbf{L}[f(x)] = \left(\mathscr{L}\left(\underline{f}(x,r)\right), \mathscr{L}\left(\overline{f}(x,r)\right)\right). \tag{23}$$

Theorem 18. *Let f be a differentiable fuzzy-valued function such that $e^{-px} f(x)$ and $e^{-px} f'(x)$ are integrable on $[0,\infty[$.*

(a) *If f is (i)-differentiable, then*

$$\mathbf{L}\left[f'(x)\right] = p\mathbf{L}[f(x)] \ominus f(0). \tag{24}$$

(b) *If f is (ii)-differentiable, then*

$$\mathbf{L}\left[f'(x)\right] = (-f(0)) \ominus (-p)\,\mathbf{L}[f(x)]. \tag{25}$$

Proof. To prove Theorem 18, one can adopt the proof in [10] using Aumann fuzzy improper integral instead of Riemann fuzzy improper integral. □

Theorem 19. *Let $f(x)$, $g(x)$ be continuous fuzzy-valued functions such that $e^{-px} f(x)$ and $e^{-px} g(x)$ are integrable on $[0,\infty[$ and c_1, c_2 two real constants; then*

$$\mathbf{L}[c_1 f(x) + c_2 g(x)] = c_1\mathbf{L}[f(x)] + c_2\mathbf{L}[g(x)]. \tag{26}$$

Theorem 19 is an obvious consequence of linearity of the Aumann fuzzy improper integral.

Definition 20. Let $k : [0,\infty[\to \mathbb{R}$ be a crisp continuous function and $f : [0,\infty[\to E$ a fuzzy-valued continuous mapping. We define the convolution product of k and f on $[0,\infty[$ as follows:

$$(k * f)(x) = \int_0^x k(x-t) f(t)\,dt, \quad x \geq 0. \tag{27}$$

Remark 21. Suppose that $e^{-px} f(x)$ and $e^{-px} k(x)$ are integrable on $[0,\infty[$. We examine the two following alternatives:

(a) If the function k is nonnegative on $[0,\infty[$, then

$$(k * f)(x) = \left(\int_0^x k(x-t)\,\underline{f}(t)\,dt, \int_0^x k(x-t)\,\overline{f}(t)\,dt\right). \tag{28}$$

Therefore,

$$(k * f)(x) = \left(\left(k * \underline{f}\right)(x), \left(k * \overline{f}\right)(x)\right). \tag{29}$$

If f_1 and f_2 are two crisp functions defined from $[0,\infty[$ into $\mathbb{R}$, then, we recall the well-known classical convolution Laplace formula:

$$\mathscr{L}[(f_1 * f_2)(x)] = \mathscr{L}[f_1(x)] \cdot \mathscr{L}[f_2(x)]. \tag{30}$$

Then using (29)-(30) and the fact that $\mathscr{L}[k(x)] \geq 0$, we get

$$\begin{aligned} &\mathbf{L}[(k * f)(x)] \\ &= \left(\mathscr{L}\left[\left(k * \underline{f}\right)(x)\right], \mathscr{L}\left[\left(k * \overline{f}\right)(x)\right]\right) \\ &= \left(\mathscr{L}[k(x)] \cdot \mathscr{L}\left[\underline{f}(x)\right], \mathscr{L}[k(x)] \cdot \mathscr{L}\left[\overline{f}(x)\right]\right) \\ &= \mathscr{L}[k(x)] \cdot \left(\mathscr{L}\left[\underline{f}(x)\right], \mathscr{L}\left[\overline{f}(x)\right]\right) \\ &= \mathscr{L}[k(x)] \cdot \mathbf{L}[f(x)]. \end{aligned} \tag{31}$$

(b) If the function k is nonpositive on $[0,\infty[$, then

$$(k * f)(x) = \left(\int_0^x k(x-t)\,\overline{f}(t)\,dt, \int_0^x k(x-t)\,\underline{f}(t)\,dt\right). \tag{32}$$

Therefore,

$$(k * f)(x) = \left(\left(k * \overline{f}\right)(x), \left(k * \underline{f}\right)(x)\right). \tag{33}$$

Then from (30)-(33) and since $\mathscr{L}[k(x)] \leq 0$, we deduce

$$\begin{aligned} &\mathbf{L}[(k * f)(x)] \\ &= \left(\mathscr{L}[k(x)] \cdot \mathscr{L}\left[\overline{f}(x)\right], \mathscr{L}[k(x)] \cdot \mathscr{L}\left[\underline{f}(x)\right]\right) \\ &= \mathscr{L}[k(x)] \cdot \left(\mathscr{L}\left[\underline{f}(x)\right], \mathscr{L}\left[\overline{f}(x)\right]\right) \\ &= \mathscr{L}[k(x)] \cdot \mathbf{L}[f(x)]. \end{aligned} \tag{34}$$

In both cases, we have

$$\mathbf{L}[(k * f)(x)] = \mathscr{L}[k(x)] \cdot \mathbf{L}[f(x)]. \tag{35}$$

Remark 22. Now let us recall the error in [25] Example 1. The authors studied the following fuzzy integro-differential equation using fuzzy differential transform method (DTM):

$$\begin{aligned} u'(x) &= (1+x)(r+1, r-2) + \int_0^x u(t)\,dt, \\ u(0) &= (0,0), \\ u'(0) &= (r+1, r-2). \end{aligned} \tag{36}$$

But $\sigma = (r+1, r-2)$ is not a fuzzy number in the parametric form, since the function $\overline{\sigma} = r-2$ is not decreasing.

Note that the second initial data $u'(0)$ can be obviously deduced by taking $x = 0$ in the equation.

Example 23. We correct the previous fuzzy Volterra integro-differential equation as follows:

$$\begin{aligned} y'(x) &= (1+x)\sigma + \int_0^x y(t)\,dt, \\ y(0,\alpha) &= (0,0), \end{aligned} \tag{37}$$

where $f(x) = (1+x)\sigma$, $\sigma = (\alpha-1, 1-\alpha)$ and $k(x) = 1$ is nonnegative.

Case 1. If $y(x)$ is (i)-differentiable, then from (35) we have

$$\mathscr{L}\left[\underline{y}(x,\alpha)\right] = \frac{\alpha-1}{p(p-1)},$$
$$\mathscr{L}\left[\overline{y}(x,\alpha)\right] = \frac{1-\alpha}{p(p-1)}. \tag{38}$$

By the inverse Laplace transform, we get the lower and upper functions of solution of (37) for $x \geq 0$

$$\underline{y}(x,\alpha) = (\alpha-1)(\exp(x)-1),$$
$$\overline{y}(x,\alpha) = (1-\alpha)(\exp(x)-1). \tag{39}$$

In this case, since $y(x)$ is (i)-differentiable, the solution is valid.

Case 2. If $y(x)$ is (ii)-differentiable, then from (35) we obtain

$$\mathscr{L}\left[\underline{y}(x,\alpha)\right] = (1-\alpha)\frac{p+1}{p(p^2+1)},$$
$$\mathscr{L}\left[\overline{y}(x,\alpha)\right] = (\alpha-1)\frac{p+1}{p(p^2+1)}. \tag{40}$$

Then by the inverse Laplace transform the lower and upper functions of solution of (37) are given for $x \in [3\pi/2, 2\pi]$ as follows:

$$\underline{y}(x,\alpha) = (\alpha-1)(\cos(x)-\sin(x)-1),$$
$$\overline{y}(x,\alpha) = (1-\alpha)(\cos(x)-\sin(x)-1). \tag{41}$$

In this case, $y(x)$ is (ii)-differentiable only for $x \in [7\pi/4, 2\pi]$ and the solution is acceptable only over this interval.

Example 24. We consider the following fuzzy Volterra integro-differential equation:

$$y'(x) = (\alpha, 2-\alpha) + \int_0^x (-1)\, y(t)\, dt,$$
$$y(0,\alpha) = (0,0), \tag{42}$$

where $f(x) = (\alpha, 2-\alpha)$ and $k(x) = -1$ is nonpositive.

Case 1. If $y(x)$ is (i)-differentiable, then from (35) we have

$$\mathscr{L}\left[\underline{y}(x,\alpha)\right] = \frac{\alpha p^2+\alpha-2}{p^4-1},$$
$$\mathscr{L}\left[\overline{y}(x,\alpha)\right] = \frac{(2-\alpha)p^2-\alpha}{p^4-1}. \tag{43}$$

By the inverse Laplace transform we get the lower and upper functions of solution of (42) for $x \geq 0$

$$\underline{y}(x,\alpha) = (\alpha-1)\sinh(x) + \sin(x),$$
$$\overline{y}(x,\alpha) = (1-\alpha)\sinh(x) + \sin(x). \tag{44}$$

In this case, the solution is acceptable since $y(x)$ is (i)-differentiable.

Case 2. If $y(x)$ is (ii)-differentiable, then from (35) we get

$$\mathscr{L}\left[\underline{y}(x,\alpha)\right] = (1-\alpha)\frac{p+1}{p(p^2+1)},$$
$$\mathscr{L}\left[\overline{y}(x,\alpha)\right] = (\alpha-1)\frac{p+1}{p(p^2+1)}. \tag{45}$$

Using the inverse Laplace transform, we obtain the solution of (42) for $x \in [\pi, 2\pi]$:

$$\underline{y}(x,\alpha) = (2-\alpha)\sin(x),$$
$$\overline{y}(x,\alpha) = \alpha\sin(x). \tag{46}$$

In this case, $y(x)$ is (ii)-differentiable only for $x \in [3\pi/2, 2\pi]$, so the solution is valid only over this interval.

5. Main Result

To overcome all the obstacles and to avoid the error in [14], we propose in this paper the convolution product of crisp and fuzzy functions, and we intend to investigate rigorously the case of two fuzzy functions in a future work. Now, we enounce our main result giving the convolution Laplace formula generalizing the result in Section 4.

Theorem 25. *Let $F : [0,\infty[\to E$ be a fuzzy-valued continuous mapping and let $k : [0,\infty[\to \mathbb{R}$ be a crisp continuous function. Assume that the mappings $e^{-px}k(x)$, $e^{-px}F(x)$, and $e^{-px}(k * F)(x)$ are integrable over $[0,\infty[$ for all $p > 0$; then*

$$\mathbf{L}[(k*F)(x)] = \mathscr{L}[k(x)] \cdot \mathbf{L}[F(x)]. \tag{47}$$

Proof. Let $x \geq 0$ and $p > 0$. It is obvious that $[(k*F)(x)]^\alpha = (k * F_\alpha)(x)$.

Step 1. We claim that

$$(k * F_\alpha)(x) = \{(k*f)(x) \mid f \in S(F_\alpha)\}. \tag{48}$$

Let $y \in (k * F_\alpha)(x) = \int_0^x k(x-t)F_\alpha(t)dt$. So, there exists a measurable selection g of $t \mapsto k(x-t)F_\alpha(t)$ such that $y = \int_0^x g(t)dt$. It is clear that the function f defined by

$$f(t) = \begin{cases} \dfrac{g(t)}{k(x-t)} & \text{if } k(x-t) \neq 0 \\ \underline{F}_\alpha(t) & \text{if } k(x-t) = 0 \end{cases} \tag{49}$$

is a measurable selection of F_α verifying $g(t) = k(x-t)f(t)$.

Hence, $y = (k*f)(x)$, which implies that $(k*F_\alpha)(x) \subset \{(k*f)(x) \mid f \in S(F_\alpha)\}$.

Let f be a measurable selection of F_α. It is clear that $t \mapsto k(x-t)f(t)$ is a measurable selection of $t \mapsto k(x-t)F_\alpha(t)$ and

$$(k*f)(x) \in (k*F_\alpha)(x), \tag{50}$$

because

$$\int_0^x k(x-t) f(t)\, dt \in \int_0^x k(x-t) F_\alpha(t)\, dt. \tag{51}$$

Therefore, (48) is proved.

Step 2. Now we show that

$$\mathbf{L}\left[(k * F_\alpha)(x)\right] = \mathscr{L}[k(x)] \cdot \mathbf{L}\left[F_\alpha(x)\right]. \tag{52}$$

If we denote $k_1(t) = \exp(-pt)k(t)$ and $H_\alpha(t) = \exp(-pt)F_\alpha(t)$, then using (48) we can write

$$\begin{aligned}
\mathbf{L}\left[(k * F_\alpha)(x)\right] &= \int_0^\infty \exp(-px)(k * F_\alpha)(x)\, dx \\
&= \int_0^\infty (k_1 * H_\alpha)(x)\, dx \\
&= \left\{\int_0^\infty (k_1 * h)(x)\, dx \mid h \in S(H_\alpha)\right\} \\
&= \left\{\int_0^\infty (k_1 * h)(x)\, dx \mid h : t \right. \\
&\quad \left. \mapsto e^{-pt} f(t);\ \text{with } f \in S(F_\alpha)\right\} \\
&= \left\{\int_0^\infty e^{-px}(k * f)(x)\, dx \mid f \in S(F_\alpha)\right\} \\
&= \left\{\mathscr{L}\left[(k * f)(x)\right] \mid f \in S(F_\alpha)\right\}.
\end{aligned} \tag{53}$$

Since $\mathscr{L}[k(x)]$ is a real number, then from (30) it follows that

$$\begin{aligned}
&\mathbf{L}\left[(k * F_\alpha)(x)\right] \\
&\quad = \left\{\mathscr{L}[k(x)] \cdot \mathscr{L}\left[f(x)\right] \mid f \in S(F_\alpha)\right\} \\
&\quad = \mathscr{L}[k(x)] \cdot \left\{\mathscr{L}\left[f(x)\right] \mid f \in S(F_\alpha)\right\} \\
&\quad = \mathscr{L}[k(x)] \cdot L\left[F_\alpha(x)\right]
\end{aligned} \tag{54}$$

□

6. Fuzzy Laplace Transform Algorithm for Solving Fuzzy Integro-Differential Equations

Our aim now is to solve the following fuzzy integro-differential equation using fuzzy Laplace transform method under strongly generalized differentiability:

$$\begin{aligned}
y'(x) &= f(x) + \int_0^x k(x-t)\, y(t)\, dt, \\
y(0) &= y_0 = \left(\underline{y}_0, \overline{y}_0\right) \in E,
\end{aligned} \tag{55}$$

where the unknown function $y(x) = (\underline{y}(x,\alpha), \overline{y}(x,\alpha))$ is a fuzzy function of $x \geq 0$, provided that $f : [0,\infty[\to E$ is a continuous fuzzy-valued function and $k : [0,\infty[\to \mathbb{R}$ is a crisp continuous function.

Please notice that Theorem 10 allows us to use only (i) or (ii) type of strongly generalized differentiability.

Assume in a first time that $\mathscr{L}[k(x)] \geq 0$.

By using the fuzzy Laplace transform and Theorem 25, we have

$$\mathbf{L}\left[y'(x)\right] = \mathbf{L}\left[f(x)\right] + \mathscr{L}[k(x)] \cdot \mathbf{L}\left[y(x)\right]. \tag{56}$$

Then, we have the following alternatives for solving (56).

Case 1. If y is (i)-differentiable, then

$$\begin{aligned}
y'(x) &= \left(\underline{y}'(x,\alpha), \overline{y}'(x,\alpha)\right), \\
\mathbf{L}\left[y'(x)\right] &= p\mathbf{L}\left[y(x)\right] \ominus y(0).
\end{aligned} \tag{57}$$

Then from (56), it follows that

$$p\mathbf{L}\left[y(x)\right] = y(0) + \mathbf{L}\left[f(x)\right] + \mathscr{L}[k(x)] \cdot \mathbf{L}\left[y(x)\right]. \tag{58}$$

Using $\mathscr{L}[k(x)] \geq 0$, we deduce

$$\begin{aligned}
p\mathscr{L}\left[\underline{y}(x,\alpha)\right] &= \underline{y}_0(\alpha) + \mathscr{L}\left[\underline{f}(x,\alpha)\right] + \mathscr{L}[k(x)] \\
&\quad \cdot \mathscr{L}\left[\underline{y}(x,\alpha)\right], \\
p\mathscr{L}\left[\overline{y}(x,\alpha)\right] &= \overline{y}_0(\alpha) + \mathscr{L}\left[\overline{f}(x,\alpha)\right] + \mathscr{L}[k(x)] \\
&\quad \cdot \mathscr{L}\left[\overline{y}(x,\alpha)\right].
\end{aligned} \tag{59}$$

Therefore,

$$\begin{aligned}
\mathscr{L}\left[\underline{y}(x,\alpha)\right] &= \frac{\underline{y}_0(\alpha) + \mathscr{L}\left[\underline{f}(x,\alpha)\right]}{p - \mathscr{L}[k(x)]} = H_1(p,\alpha), \\
\mathscr{L}\left[\overline{y}(x,\alpha)\right] &= \frac{\overline{y}_0(\alpha) + \mathscr{L}\left[\overline{f}(x,\alpha)\right]}{p - \mathscr{L}[k(x)]} = K_1(p,\alpha).
\end{aligned} \tag{60}$$

By using the inverse Laplace transform, we get

$$\begin{aligned}
\underline{y}(x,\alpha) &= \mathscr{L}^{-1}\left[H_1(p,\alpha)\right], \\
\overline{y}(x,\alpha) &= \mathscr{L}^{-1}\left[K_1(p,\alpha)\right].
\end{aligned} \tag{61}$$

Case 2. If y is (ii)-differentiable, then

$$\begin{aligned}
y'(x) &= \left(\overline{y}'(x,\alpha), \underline{y}'(x,\alpha)\right), \\
\mathbf{L}\left[y'(x)\right] &= -y(0) \ominus \left(-p\mathbf{L}\left[y(x)\right]\right).
\end{aligned} \tag{62}$$

Then from (56), it follows that

$$\begin{aligned}
&-y(0) \ominus \left(-p\mathbf{L}\left[y(x)\right]\right) \\
&\quad = \mathbf{L}\left[f(x)\right] + \mathscr{L}[k(x)] \cdot \mathbf{L}\left[y(x)\right].
\end{aligned} \tag{63}$$

Using $\mathscr{L}[k(x)] \geq 0$, we deduce

$$\begin{aligned}
&-\overline{y}_0(\alpha) + p\mathscr{L}\left[\overline{y}(x,\alpha)\right] \\
&\quad = \mathscr{L}\left[\underline{f}(x,\alpha)\right] + \mathscr{L}[k(x)] \cdot \mathscr{L}\left[\underline{y}(x,\alpha)\right], \\
&-\underline{y}_0(\alpha) + p\mathscr{L}\left[\underline{y}(x,\alpha)\right] \\
&\quad = \mathscr{L}\left[\overline{f}(x,\alpha)\right] + \mathscr{L}[k(x)] \cdot \mathscr{L}\left[\overline{y}(x,\alpha)\right].
\end{aligned} \tag{64}$$

That is,

$$\begin{aligned} &\mathscr{L}[k(x)]\cdot\mathscr{L}\left[\underline{y}(x,\alpha)\right]-p\mathscr{L}[\overline{y}(x,\alpha)]=A(p,\alpha),\\ &-p\mathscr{L}\left[\underline{y}(x,\alpha)\right]+\mathscr{L}[k(x)]\cdot\mathscr{L}[\overline{y}(x,\alpha)]=B(p,\alpha), \end{aligned} \tag{65}$$

where $A(p,\alpha)=-\overline{y}_0(\alpha)-\mathscr{L}[\underline{f}(x,\alpha)]$ and $B(p,\alpha)=-\underline{y}_0(\alpha)-\mathscr{L}[\overline{f}(x,\alpha)]$.

Then by solving the linear system (65), we have

$$\begin{aligned} \mathscr{L}\left[\underline{y}(x,\alpha)\right]&=\frac{\mathscr{L}[k(x)]\cdot A(p,\alpha)+pB(p,\alpha)}{(\mathscr{L}[k(x)])^2-p^2}\\ &=H_2(p,\alpha),\\ \mathscr{L}[\overline{y}(x,\alpha)]&=\frac{\mathscr{L}[k(x)]\cdot B(p,\alpha)+pA(p,\alpha)}{(\mathscr{L}[k(x)])^2-p^2}\\ &=K_2(p,\alpha). \end{aligned} \tag{66}$$

By using the inverse Laplace transform, we get

$$\begin{aligned} \underline{y}(x,\alpha)&=\mathscr{L}^{-1}[H_2(p,\alpha)],\\ \overline{y}(x,\alpha)&=\mathscr{L}^{-1}[K_2(p,\alpha)]. \end{aligned} \tag{67}$$

Remark 26. Similarly, if we assume that $\mathscr{L}[k(x)]<0$, we obtain the following results.

(1) If y is (i)-differentiable, then

$$\begin{aligned} \mathscr{L}\left[\underline{y}(x,\alpha)\right]&=\frac{\mathscr{L}[k(x)]\cdot D(p,\alpha)+pC(p,\alpha)}{p^2-(\mathscr{L}[k(x)])^2}\\ &=H_3(p,\alpha),\\ \mathscr{L}[\overline{y}(x,\alpha)]&=\frac{\mathscr{L}[k(x)]\cdot C(p,\alpha)+pD(p,\alpha)}{p^2-(\mathscr{L}[k(x)])^2}\\ &=K_3(p,\alpha), \end{aligned} \tag{68}$$

where $C(p,\alpha)=\underline{y}_0(\alpha)+\mathscr{L}[\underline{f}(x,\alpha)]$ and $D(p,\alpha)=\overline{y}_0(\alpha)+\mathscr{L}[\overline{f}(x,\alpha)]$.

By using the inverse Laplace transform, we get

$$\begin{aligned} \underline{y}(x,\alpha)&=\mathscr{L}^{-1}[H_3(p,\alpha)],\\ \overline{y}(x,\alpha)&=\mathscr{L}^{-1}[K_3(p,\alpha)]. \end{aligned} \tag{69}$$

(2) If y is (ii)-differentiable, then

$$\begin{aligned} \mathscr{L}\left[\underline{y}(x,\alpha)\right]&=\frac{\underline{y}_0(\alpha)+\mathscr{L}\left[\overline{f}(x,\alpha)\right]}{p-\mathscr{L}[k(x)]}=H_4(p,\alpha),\\ \mathscr{L}[\overline{y}(x,\alpha)]&=\frac{\overline{y}_0(\alpha)+\mathscr{L}\left[\underline{f}(x,\alpha)\right]}{p-\mathscr{L}[k(x)]}=K_4(p,\alpha). \end{aligned} \tag{70}$$

By using the inverse Laplace transform, we obtain

$$\begin{aligned} \underline{y}(x,\alpha)&=\mathscr{L}^{-1}[H_4(p,\alpha)],\\ \overline{y}(x,\alpha)&=\mathscr{L}^{-1}[K_4(p,\alpha)]. \end{aligned} \tag{71}$$

Example 27. We consider the following fuzzy integro-differential equation:

$$\begin{aligned} y'(x)+y(x)&=\int_0^x\sin(x-t)\,y(t)\,dt,\\ y(0,\alpha)&=(\alpha-1,1-\alpha). \end{aligned} \tag{72}$$

Case 1. If $y(x)$ is (i)-differentiable, then from Theorems 18 and 25 we have

$$\begin{aligned} \mathscr{L}\left[\underline{y}(x,\alpha)\right]&=(\alpha-1)\frac{(p^2+1)}{p^3+p^2+p},\\ \mathscr{L}[\overline{y}(x,\alpha)]&=(1-\alpha)\frac{(p^2+1)}{p^3+p^2+p}. \end{aligned} \tag{73}$$

By the inverse Laplace transform we get the lower and upper functions of solution of (72) for $x\geq 0$

$$\begin{aligned} &\underline{y}(x,\alpha)\\ &\quad=(\alpha-1)\left[1-\frac{2\sqrt{3}}{3}\exp\left(-\frac{x}{2}\right)\sin\left(\frac{\sqrt{3}x}{2}\right)\right],\\ &\overline{y}(x,\alpha)\\ &\quad=(1-\alpha)\left[1-\frac{2\sqrt{3}}{3}\exp\left(-\frac{x}{2}\right)\sin\left(\frac{\sqrt{3}x}{2}\right)\right]. \end{aligned} \tag{74}$$

In this case, the solution is invalid over $[0,\infty[$, since $y(x)$ is not (i)-differentiable.

Case 2. If $y(x)$ is (ii)-differentiable, then Theorems 18 and 25 yield

$$\begin{aligned} &p\mathscr{L}\left[\underline{y}(x,\alpha)\right]+(p^2+1)\,p\mathscr{L}[\overline{y}(x,\alpha)]\\ &\quad=(1-\alpha)\frac{p^2+1}{p},\\ &(p^2+1)\,\mathscr{L}\left[\underline{y}(x,\alpha)\right]+p\mathscr{L}[\overline{y}(x,\alpha)]\\ &\quad=(\alpha-1)\frac{p^2+1}{p}. \end{aligned} \tag{75}$$

By solving the linear system (75) and using the inverse Laplace transform, we get

$$\begin{aligned} \underline{y}(x,\alpha)&=(\alpha-1)\left[1+\frac{2\sqrt{3}}{3}\exp\left(\frac{x}{2}\right)\sin\left(\frac{\sqrt{3}x}{2}\right)\right],\\ \overline{y}(x,\alpha)&=(1-\alpha)\left[1+\frac{2\sqrt{3}}{3}\exp\left(\frac{x}{2}\right)\sin\left(\frac{\sqrt{3}x}{2}\right)\right]. \end{aligned} \tag{76}$$

One can verify that in this case the solution is acceptable over a closed interval $[c, d]$ such that $[2.45, 3.75] \subset [c, d] \subset [2.4, 3.8]$.

Remark 28. Analogously, we can solve the following generalized fuzzy integro-differential equation, with kernel of convolution type via Laplace transform method:

$$\begin{aligned} y'(x) &= g(x, y(x)) + \int_0^x k(x-t)\, y(t)\, dt, \\ y(0) &= y_0 = \left(\underline{y}_0, \overline{y}_0\right) \in E \end{aligned} \tag{77}$$

provided that $g : [0, \infty[\times E \to E$ is a continuous fuzzy-valued function, which is linear with respect to its second argument, and $k : [0, \infty[\to \mathbb{R}$ is a crisp continuous function over $[0, \infty[$.

Example 29. We consider the following known fuzzy integro-differential equation:

$$\begin{aligned} &y'(x) + 3y(x) + \int_0^x e^{-(x-t)} y(t)\, dt \\ &\quad = (1+x)(\alpha - 1, 1 - \alpha), \\ &y(0, \alpha) = (1, 1). \end{aligned} \tag{78}$$

Its corresponding crisp problem, studied in [20], is as follows:

$$\begin{aligned} y'(x) &= -3y(x) - \int_0^x \exp(-(x-t))\, y(t)\, dt, \\ y(0) &= 1. \end{aligned} \tag{79}$$

Case 1. If $y(x)$ is (i)-differentiable, then from Theorems 18 and 25 we have

$$\begin{aligned} \mathscr{L}\left[\underline{y}(x, \alpha)\right] &= (\alpha - 1) \frac{(p+1)^2}{p^2 (p+2)^2} + \frac{p+1}{(p+2)^2}, \\ \mathscr{L}\left[\overline{y}(x, \alpha)\right] &= (1 - \alpha) \frac{(p+1)^2}{p^2 (p+2)^2} + \frac{p+1}{(p+2)^2}. \end{aligned} \tag{80}$$

By the inverse Laplace transform we get the lower and upper functions of solution of (78) for $x \geq 0$

$$\begin{aligned} \underline{y}(x, \alpha) &= \frac{(\alpha - 1)}{2} [x \cosh(x) + \sinh(x)]\, e^{-x} \\ &\quad + (1 - x)\, e^{-2x}, \\ \overline{y}(x, \alpha) &= \frac{(1 - \alpha)}{2} [x \cosh(x) + \sinh(x)]\, e^{-x} \\ &\quad + (1 - x)\, e^{-2x}. \end{aligned} \tag{81}$$

In this case, the solution is valid over $[0, \infty[$, since $y(x)$ is (i)-differentiable.

Case 2. If $y(x)$ is (ii)-differentiable, then Theorems 18 and 25 yield

$$\begin{aligned} &p(p+1)\, \mathscr{L}\left[\underline{y}(x, \alpha)\right] + (3p+4)\, \mathscr{L}\left[\overline{y}(x, \alpha)\right] \\ &\quad = (1 - \alpha) \frac{p+1}{p^2} + p + 1, \\ &(3p+4)\, \mathscr{L}\left[\underline{y}(x, \alpha)\right] + p(p+1)\, \mathscr{L}\left[\overline{y}(x, \alpha)\right] \\ &\quad = (\alpha - 1) \frac{p+1}{p^2} + p + 1. \end{aligned} \tag{82}$$

By solving the linear system (82) and using the inverse Laplace transform, we get the lower and upper functions of solution of (78) for $x \geq 0$ as follows:

$$\begin{aligned} \underline{y}(x, \alpha) &= (1 - x) \exp(-2x) + (\alpha - 1) \frac{-2x - 3}{8} \\ &\quad + (\alpha - 1)\, e^x \left(\frac{3}{8} \cosh\left(\sqrt{5}x\right) + \frac{7\sqrt{5}}{40} \sinh\left(\sqrt{5}x\right) \right), \\ \overline{y}(x, \alpha) &= (1 - x) \exp(-2x) + (1 - \alpha) \frac{-2x - 3}{8} \\ &\quad + (1 - \alpha)\, e^x \left(\frac{3}{8} \cosh\left(\sqrt{5}x\right) + \frac{7\sqrt{5}}{40} \sinh\left(\sqrt{5}x\right) \right). \end{aligned} \tag{83}$$

Notice that the length of $y'(x, \alpha)$

$$\begin{aligned} \operatorname{len}\left(y'(x, \alpha)\right) &= \underline{y}'(x, \alpha) - \overline{y}'(x, \alpha) \\ &= -\frac{(\alpha - 1)}{2} + (\alpha - 1) \frac{5}{2} e^x \cosh\left(\sqrt{5}x\right) \\ &\quad + \frac{11\sqrt{5}}{10} (\alpha - 1)\, e^x \sinh\left(\sqrt{5}x\right) \end{aligned} \tag{84}$$

is a nonnegative increasing function over $[0, \infty[$; then $y(x)$ is (ii)-differentiable. So, in this case the solution is acceptable for all $x \geq 0$.

Taking $\alpha = 1$ in formulas (81) and (83) yields the crisp solution,

$y(x) = (1 - x) \exp(-2x)$, of the classic problem (79) (see [20] page 8 Example 1.2.1).

7. Conclusion

In this paper, we have introduced the Aumann fuzzy improper integral, and also we have applied Laplace transform method for solving FIDEs, with kernel of convolution type, under the assumption of strongly generalized differentiability. Clearly, the suggested formula allows us to solve more difficult FIDEs by Laplace method compared to the previously reported works.

Indeed, in the most fuzzy examples studied before, the considered kernels $k(x)$ were real and nonnegative constants.

But in this paper, we treated various cases for this kernel $k(x)$: positive or negative in the first and second examples, respectively; $k(x) = \sin(x)$ and $k(x) = \exp(-x)$ were nonconstant functions of x in the third and fourth ones.

For future research, we will apply Laplace transform method to solve FIDEs with a fuzzy kernel.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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