Research Article

On Tri-α-Open Sets in Fuzzifying Tritopological Spaces

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In this paper, we introduced and studied (1,2,3)-α-open set, (1,2,3)-α-neighborhood system, (1,2,3)-α-derived, (1,2,3)-α-closure, (1,2,3)-α-interior, (1,2,3)-α-exterior, (1,2,3)-α-boundary, (1,2,3)-α-convergence of nets, and (1,2,3)-α-convergence of filters in fuzzifying tritopological spaces.

1. Introduction

The fuzzy set is an important concept, which was introduced for the first time in 1965 by Zadeh [1]; it was then used in many studies in various fields. Here, we are interested in fuzzy with topology. The fuzzy and fuzzifying topologies are two branches of fuzzy mathematics. The basic concepts and properties of fuzzy topologies were subedited and investigated by Chang in 1968 [2] and Wong in 1974 [3]. After that, so many works of literature have appeared for different kinds of fuzzy topological spaces for, e.g., [4–8]. In 1991-1993, Ying introduced a new approach for fuzzy topology with fuzzy logic and established some properties in fuzzifying topology [9–11]. Also, we are interested in the concept of α-open set which was introduced by Njástad in 1965 [12], and the tritopological space which was first initiated by Kovar in 2000 [13]. In 2017, Tapi and Sharma introduced α-open sets in tritopological spaces [14]. In 1999, Khedir et al. presented semiopen sets and semi-continuity in fuzzifying topology [15]. In 2016, Allam and et al. studied semiopen sets in fuzzifying bitopological spaces [16]. We will use in this paper Ying’s basic fuzzy logic formulas with appropriate set theoretical notations from [9, 10].

The following are some useful definitions and results that will be used in the rest of the present work.

If X is the universe of discourse, and if τ ∈ ℳ(𝒫(𝑋)) satisfy the following three conditions:

(1) τ(𝑋) = 1 and τ(∅) = 1;

(2) for any 𝐺, ℎ, 𝜏(𝐺 ∩ ℎ) ≥ 𝜏(𝐺) ∧ 𝜏(ℎ);

(3) for any {𝐺α : ℎ ∈ Λ}, 𝜏(⋃){ GDPR就是因为 } GDPR is a fuzzifying topology and (𝑋, 𝜏) a fuzzifying topological space [9].

The family of fuzzifying closed sets is denoted by ℱ and defined as ℱ ∋ 𝐿 = 𝑋 ∩ ℎ, where 𝑋 ∩ ℎ is the complement of 𝐿 [9].

The neighborhood system of 𝑥 is denoted by 𝑁𝑥 ∈ ℳ(𝒫(𝑋)) and defined as 𝑁𝑥(𝐺) = sup {ℎ ∈ ℱ(𝑥)} [9].

The closure of a set 𝐺 ⊆ 𝑋 is denoted by cl(𝐺) ∈ ℳ(𝑋) and defined as cl(𝐺)(𝑥) = 1 − 𝑁𝑥(𝑋 ∩ 𝐺) [9].

The fuzzifying interior set of a set 𝐺 ⊆ 𝑋 is denoted by int(𝐺) ∈ ℳ(𝑋) and defined as int(𝐺)(𝑥) = 𝑁𝑥(𝐺) [10].

The family of all fuzzifying α-open sets is denoted by ατ and defined as 𝑋 ∈ ατ = ∀𝑥 (𝑥 ∈ 𝐺 → 𝑥 ∈ int(cl(𝑖𝑛𝑡(𝐺)))], i.e., ατ(𝐺) = inf 𝜆∈Γ (𝑖𝑛𝑡(𝑐𝑙(𝑖𝑛𝑡(𝐺))))(𝑥)] [17].

The family of all fuzzifying α-closed sets is denoted by αℱ and defined as 𝑋 ∈ αℱ = 𝑋 ∩ ℎ ∈ ατ [17].

The fuzzifying α-interior set of a set 𝐺 ⊆ 𝑋 is denoted by α𝑖𝑛𝑡(𝐺) ∈ ℳ(𝑋) and defined as follows: α𝑖𝑛𝑡(𝐺)(𝑥) = α𝑁𝑥(𝐺), where α𝑁𝑥 is α-neighborhood system of 𝑥 defined as α𝑁𝑥(𝐺) = sup ℱ∈Γ(𝑥) ∈ cl(𝑖𝑛𝑡(𝐺))) (𝑥) [17].

The fuzzifying α-derived set of a set 𝐺 ⊆ 𝑋 is denoted by ad(𝐺) ∈ ℳ(𝑋) and defined as 𝑥 ∈ ad(𝐺) = ∀ℎ (ℎ ∈ α𝑁𝑥 𝑆 𝑋 → ℎ ∩ (𝐺 ∩ ℎ) ≠ ø), i.e., ad(𝐺)(𝑥) = inf ℱ∈Γ(𝐺 ∩ ℎ) = α(1 − α𝑁𝑥(ℎ)) [18].
The α-closure set of a set $G \subseteq X$ is denoted by $acl(G) \in \mathcal{F}(X)$ and defined as $acl(G)(x) = \inf_{x \in cl(x)}(1 - \alpha F(H))$ [17].

2. (1, 2, 3)-α-Open Sets in Fuzzifying Tri-topological Spaces

Definition 1. If $(X, \tau_1, \tau_2, \tau_3)$ is a fuzzifying tritopological space (FTTS), then we have the following:

(i) The family of all fuzzifying (1,2,3)-α-open sets is denoted by $\alpha \tau_{(1,2,3)} \in \mathcal{F}(P(X))$ and defined as $G \in \alpha \tau_{(1,2,3)} = \forall x (x \in G \rightarrow x \in int(1,2,3)cl(int_3(x)))$, i.e., $\alpha \tau_{(1,2,3)}(G) = \inf_{x \in G} int_3(x)(x)$. (ii) The family of all fuzzifying (1,2,3)-α-closed sets is denoted by $\alpha F_{(1,2,3)} \in \mathcal{F}(P(X))$ and defined as $G \in \alpha F_{(1,2,3)} = X \sim G \in \alpha \tau_{(1,2,3)}$.

Lemma 2. Let $(X, \tau_1, \tau_2, \tau_3)$ be a FTTS. If $[G \subseteq H] = 1$, then $\implies int_1(cl_2(int_3(G))) \subseteq int_1(cl_2(int_3(H)))$.

Proof. If $[G \subseteq H] = 1$, then $int_1(G) \subseteq int_1(H) \implies cl_2(int_1(G)) \subseteq cl_2(int_1(H))$.

Lemma 3. If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS and $G \subseteq X$. Then

(i) $\implies X \sim int_1(cl_1(int_3(G))) \equiv cl_1(int_1(cl_3(G) \sim X \sim G))$;

(ii) $\implies X \sim cl_1(int_1(cl_3(G))) \equiv cl_1(verbose(int_1(cl_3(x))))$.

Proof. From Theorem 2.2.-(5) in [10], we have

(i) $X \sim int_1(cl_2(int_3(G)))(x) = cl_1(X \sim cl_2(int_3(G)))(x) = cl_1(int_1(cl_3(X \sim G)))(x)$.

(ii) $X \sim cl_1(int_1(cl_3(G)))(x) = int_1(X \sim int_1(cl_3(G)))(x) = int_1(cl_1(int_3(x))))$.

Theorem 4. If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS, then

(i) $\alpha \tau_{(1,2,3)}(X) = 1$, $\alpha \tau_{(1,2,3)}(0) = 1$;

(ii) for any $\{G_1 \in \Lambda \}$, $\alpha \tau_{(1,2,3)}(\bigcup_{\lambda \in \Lambda} G_1) \geq \bigwedge_{\lambda \in \Lambda} \alpha \tau_{(1,2,3)}(G_1)$;

(iii) $\alpha F_{(1,2,3)}(X) = 1$, $\alpha F_{(1,2,3)}(0) = 1$;

(iv) for any $\{G_1 \in \Lambda \}$, $\alpha F_{(1,2,3)}(\bigcap_{\lambda \in \Lambda} G_1) \geq \bigwedge_{\lambda \in \Lambda} \alpha F_{(1,2,3)}(G_1)$.

Proof.

(i) $\alpha \tau_{(1,2,3)}(X) = \inf_{x \in X} int_1(cl_2(int_3(x)))(x) = \inf_{x \in X} int_1(cl_1(x))(x) = \inf_{x \in X}(X)(x) = 1$.

Remark 7. The following example shows that

(i) $\tau_1 \subseteq \alpha \tau_{(1,2,3)}$;

(ii) $\tau_2 \subseteq \alpha \tau_{(1,2,3)}$;

(iii) $\tau_3 \subseteq \alpha \tau_{(1,2,3)}$;

(iv) $\alpha \tau_{(1,2,3)} = \alpha \tau_{(1,2,3)}$.

It may not be true for all FTTS $(X, \tau_1, \tau_2, \tau_3)$.
Example 8. For $X = \{a, b, c\}$ and $B = \{a, b\}$. Let $\tau_1, \tau_2, \tau_3$ be a fuzzifying tri-topological space $X$ defined by

$$
\tau_1(A) = \begin{cases} 
1 & \text{if } A \in \{\emptyset, X, [a]\}, \\
\frac{3}{4} & \text{if } A \in \{[c], [a, c]\}, \\
0 & \text{Otherwise.}
\end{cases}
$$

$$
\tau_2(A) = \begin{cases} 
1 & \text{if } A \in \{\emptyset, X\}, \\
\frac{1}{4} & \text{if } A = [c], \\
0 & \text{Otherwise.}
\end{cases}
$$

$$
\tau_3(A) = \begin{cases} 
1 & \text{if } A \in \{\emptyset, X, [b], [a, c]\}, \\
\frac{3}{4} & \text{if } A \in \{[c], [b, c]\}, \\
0 & \text{if } A \in \{[a], [a, b]\},
\end{cases}
$$

Now, we have $\int_{\tau_1}(B)(a) = 1$, $\int_{\tau_1}(B)(b) = 0$, $\int_{\tau_2}(B)(c) = 0$, $\int_{\tau_3}(B)(c) = 0$, $\int_{\tau_1}(B)(a) = 1$, $\int_{\tau_3}(B)(b) = 0$, $\int_{\tau_1}(B)(b) = 1$, $\int_{\tau_2}(B)(c) = 0$, $\int_{\tau_3}(B)(c) = 0$, $\int_{\tau_1}(B)(a) = 0$, $\int_{\tau_3}(B)(b) = 0$, $\int_{\tau_1}(B)(b) = 1$, $\int_{\tau_2}(B)(c) = 0$, $\int_{\tau_3}(B)(c) = 0$.

Lemma 9. If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS, then $\Rightarrow \tau_1 \equiv \tau_3 \rightarrow \alpha \tau_{(1,2,3)} \equiv \alpha \tau_{(3,2,1)}$.

Proof. From Theorem (2.2)-(3) in [10], we have $[(G \in \tau_1) \land (G \in \tau_2)] = [(G \equiv \int_{\tau_1}(G)) \land (G \equiv \int_{\tau_2}(G))]$. Hence $\Rightarrow \Rightarrow (\alpha \tau_{(1,2,3)} \equiv \alpha \tau_{(3,2,1)}).

Theorem 10. If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS, then $\forall \tau \in \alpha \mathcal{T}_{(1,2,3)}$.

Proof. From Lemma 3 -(ii), we have $[\forall x (x \in \int_{\tau_2}(\int_{\tau_3}(G)) \rightarrow x \in G)] = [\forall x (x \in X \sim G \Rightarrow x \in \int_{\tau_2}(\int_{\tau_3}(G))].

Lemma 11. If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS, then

(i) $\tau \in \alpha \mathcal{T}_{(1,2,3)} \Rightarrow \tau \in \int_{\tau_1}(H)$;

(ii) $\tau \in \alpha \mathcal{T}_{(1,2,3)} \Rightarrow \tau \in \int_{\tau_2}(G)$.

Proof.

(i) $[(\tau \in \int_{\tau_3}(\int_{\tau_2}(G)) \land (\tau \in \alpha \mathcal{T}_{(1,2,3)})] = [(\tau \in \int_{\tau_3}(\int_{\tau_2}(G)) \land (\tau \in \int_{\tau_1}(\int_{\tau_2}(G)))]$.

(ii) From Theorem 2.2-(5) in [10], we have $[(\tau \in \alpha \mathcal{T}_{(1,2,3)}) \land (\tau \in \int_{\tau_2}(G)) \land (\tau \in \int_{\tau_3}(\int_{\tau_2}(G)))]$.

Theorem 12. If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS, then

(i) $\forall \tau \in \tau_3 \land H \subseteq G \implies \tau \in \int_{\tau_2}(H)$;

(ii) $\forall K \in \mathcal{F}_3 \land \int_{\tau_3}(K) \subseteq G \subseteq K \implies \tau \in \alpha \mathcal{T}_{(1,2,3)}$.

Proof.

(i) From Theorem (2.2)-(3) in [10], we have $[(\exists H \in \tau_2 \land H \subseteq G \implies \tau \in \int_{\tau_2}(H))] = \sup_{\mathcal{H}(X)}([\tau \equiv \int_{\tau_2}(H) \land H \subseteq G] \land [G \subseteq \int_{\tau_2}(H)]).$
Theorem 16. If $\alpha N^{(1,2,3)} : X \rightarrow \mathcal{N}(P(X))$, $x \rightarrow \alpha N^{(1,2,3)}(x)$, then the set of all normal fuzzy subset of $P(X)$, has the following properties:

(i) $\exists G \in \alpha N^{(1,2,3)} \iff x \in G$,

(ii) $\exists G \subseteq H \implies (G \in \alpha N^{(1,2,3)} \implies H \subseteq \alpha N^{(1,2,3)})$,

(iii) $\exists G \in \alpha N^{(1,2,3)} \implies \exists K \in \alpha N^{(1,2,3)} \land H \subseteq K \implies H \subseteq K$.

Proof.

\begin{itemize}
  \item[(i)] From the definition of $\alpha N^{(1,2,3)}$, we have $G \in \alpha N^{(1,2,3)} \iff \forall x \in G \implies H \subseteq \alpha N^{(1,2,3)}(H)$.
  \item[(ii)] If $G \in \alpha N^{(1,2,3)}$, then $H \subseteq \alpha N^{(1,2,3)} \implies H \subseteq \alpha N^{(1,2,3)}$.
  \item[(iii)] From Theorem 14, $G \in \alpha N^{(1,2,3)} \implies \exists K \in \alpha N^{(1,2,3)} \land H \subseteq K \implies H \subseteq K$.
\end{itemize}

\[\Box\]

4. \((1,2,3)\)-Derived Set and \((1,2,3)\)-Closure Operator in Fuzzifying Tritopological Space

Definition 17. If $\alpha d^{(1,2,3)}(G)(x) = 1 - \alpha N^{(1,2,3)}(x)$, then $G \in \alpha d^{(1,2,3)}(G)$ defines the "(1,2,3)-derived set" of $G$ and defined as follows: $x \in \alpha d^{(1,2,3)}(G) \iff \forall y( y \in K \implies G \subseteq K \implies K \subseteq \alpha N^{(1,2,3)}(K))$. 

Lemma 18. $\alpha d^{(1,2,3)}(G)(x) = 1 - \alpha N^{(1,2,3)}(x \cup \{x\})$.

Proof. $\alpha d^{(1,2,3)}(G)(x) = 1 - \sup_{x \neq K \in \mathcal{N}(X)} \alpha N^{(1,2,3)}(K) = 1 - \sup_{x \neq K \in \mathcal{N}(X)} \alpha N^{(1,2,3)}(K) = 1 - \sup_{x \neq K \in \mathcal{N}(X \cup \{x\})} \alpha N^{(1,2,3)}(K) = 1 - \alpha N^{(1,2,3)}(x) \cup \{x\}$.

\[\Box\]

Theorem 19. If $\alpha d^{(1,2,3)}(G)(x) = 1 - \sup_{x \neq K \in \mathcal{N}(X \cup \{x\})} \alpha N^{(1,2,3)}(K) = 1 - \sup_{x \neq K \in \mathcal{N}(X \cup \{x\})} \alpha N^{(1,2,3)}(K) = 1 - \alpha N^{(1,2,3)}(x \cup \{x\})$

Proof.

\begin{itemize}
  \item[(i)] From Lemma 18, we have $\alpha d^{(1,2,3)}(G)(x) = 1 - \sup_{x \neq K \in \mathcal{N}(X \cup \{x\})} \alpha N^{(1,2,3)}(K) = 1 - \sup_{x \neq K \in \mathcal{N}(X \cup \{x\})} \alpha N^{(1,2,3)}(K) = 1 - \alpha N^{(1,2,3)}(x \cup \{x\})$
\end{itemize}
Theorem 21. If $\alpha_{cl}$ indicates the $(/one.fitted,/two.fitted,/three.fitted)-\text{Definition 20.}$

\begin{itemize}
  \item[(ii)] Let $G \subseteq H$, then from Lemma 18 and Theorem 16 -(ii) we get
  \[ a_{cl}(1,2,3)\{G(x) = 1 - \alpha N_{x}^{(1,2,3)}(X \sim G \cup \{x\}) \leq 1 - \alpha N_{x}^{(1,2,3)}((X \sim H) \cup \{x\}) = a_{cl}(1,2,3)\{x(x). \]

  \item[(iii)] From Lemma 18 and Theorem 15 -(ii). We have
  \[ a_{cl}(1,2,3)\{G(x) = 1 - \alpha N_{x}^{(1,2,3)}((X \sim G) \cup \{x\}) = \alpha_{cl}(1,2,3)\{G(x). \]

  \item[(iv)] From Theorem 15 -(ii) and Lemma 5.1 in [9] we have
  \[ a_{cl}(1,2,3)\{G(x) = 1 - \alpha N_{x}^{(1,2,3)}((X \sim G) \cup \{x\}) \leq 1 - \alpha N_{x}^{(1,2,3)}((X \sim H) \cup \{x\}) = d_{1}(G(x). \]
\end{itemize}

\[ \square \]

**Definition 20.** If $(X, \tau_{1}, \tau_{2}, \tau_{3})$ is a FTTS, then $a_{cl}(1,2,3)(G)$ indicates the "$(1,2,3)$-\alpha\-closure set of $G^{\prime}$ and defined as $x \in a_{cl}(1,2,3)(G) = \forall H (H \supseteq G \land (H \in a_{cl}(1,2,3))) \rightarrow x \in H)$, i.e., $a_{cl}(1,2,3)(G)(x) = \inf_{x \in H \in a_{cl}(1,2,3)}(1 - a_{cl}(1,2,3)(H)).$

**Theorem 21.** If $(X, \tau_{1}, \tau_{2}, \tau_{3})$ is a FTTS, $G, H \in P(X)$ and $x \in X, then

\begin{itemize}
  \item[(i)] $a_{cl}(1,2,3)(G)(x) = 1 - \alpha N_{x}^{(1,2,3)}(X \sim G);$
  \item[(ii)] $\forall H \in a_{cl}(1,2,3)(G) \subseteq G;$
  \item[(iii)] $\exists H \in a_{cl}(1,2,3)(G) \subseteq G;$
  \item[(iv)] $\forall x \in a_{cl}(1,2,3)(G) \rightarrow H \subseteq a_{cl}(1,2,3)(G);$\]

\end{itemize}

\[ \square \]

**Proof.**

\begin{itemize}
  \item[(i)] $a_{cl}(1,2,3)(G)(x) = \inf_{x \in X \cap G}(1 - \alpha a_{cl}(1,2,3)(H)) = \inf_{x \in a_{cl}(1,2,3)}(1 - \alpha a_{cl}(1,2,3)(X \sim H)) = 1 - \sup_{x \in X \cap G}(1 - \alpha a_{cl}(1,2,3)(X \sim H)) = 1 - \alpha N_{x}^{(1,2,3)}((X \sim H) \cup \{x\}).$

  \item[(ii)] $\alpha_{cl}(1,2,3)(G)(x) = 1 - \alpha N_{x}^{(1,2,3)}((X \sim G) \cup \{x\}) = 1 - \alpha N_{x}^{(1,2,3)}((X \sim H) \cup \{x\}) = 1 - 0 = 1.$

  \item[(iii)] If $G \subseteq P(X)$ and for any $x \in X$ and if $x \notin G$, then $[x \in G] = 0$. If $x \in G$, then $a_{cl}(1,2,3)(G)(x) = 1 - \alpha N_{x}^{(1,2,3)}(X \sim G) = 1 - 0 = 0.$

  \item[(iv)] From Lemma 18 and (iii) above, for any $x \in X$ we have

\end{itemize}

[x \in (a_{cl}(1,2,3)(G)) = max((1 - \alpha N_{x}^{(1,2,3)}(X \sim G) \cup \{x\}), G(x))]. If $x \in G$, then $[x \in (a_{cl}(1,2,3)(G) \cup G) = G(x) = 1 = [x \in a_{cl}(1,2,3)(G)]$. If $x \notin G$, then $[x \in (a_{cl}(1,2,3)(G) \cup G)] = 1 - \alpha N_{x}^{(1,2,3)}(X \sim G) = [x \in a_{cl}(1,2,3)]$. 

Thus $[a_{cl}(1,2,3)(G)] = [a_{cl}(1,2,3)(G) \cup G].$

\begin{itemize}
  \item[(v)] $\forall H (H \subseteq G \land (H \in a_{cl}(1,2,3))) \rightarrow x \in H)$, then from Lemma (ii) above, since

[\begin{enumerate}[a].]
\end{enumerate}]

\end{itemize}

\[ \square \]
5. (1, 2, 3)-α–Interior, (1, 2, 3)-α–Exterior, and (1, 2, 3)-α–Boundary Operators in Fuzzifying Tritopological Space

Definition 22. If \((X, \tau_1, \tau_2, \tau_3)\) is a FTTS and \(G \in P(X)\), then 
\[\operatorname{aint}_{(1,2,3)}(G) \] 
indicates the "(1,2,3)-α-interior set of \(G\)" defined as 
\[\operatorname{aint}_{(1,2,3)}(G)(x) = \alpha N_x^{(1,2,3)}(G)\]

Theorem 23. If \((X, \tau_1, \tau_2, \tau_3)\) is a FTTS, \(G, H \in P(X)\) and \(x \in X\), then
\[(i) \Downarrow \operatorname{aint}_{(1,2,3)}(X) = X;\]
\[(ii) \Downarrow \operatorname{aint}_{(1,2,3)}(G) \subseteq G;\]
\[(iii) \Downarrow \tau_1 \equiv \tau_3 \rightarrow \operatorname{int}_1(G) \subseteq \operatorname{aint}_{(1,2,3)}(G);\]
\[(iv) \Downarrow H \in \tau_{(1,2,3)} \land H \subseteq G \rightarrow H \subseteq \operatorname{aint}_{(1,2,3)}(G);\]
\[(v) \Downarrow G = \operatorname{aint}_{(1,2,3)}(G) \rightarrow G \in \tau_{(1,2,3)};\]
\[(vi) \Downarrow G \in H \rightarrow \operatorname{aint}_{(1,2,3)}(G) \subseteq \operatorname{aint}_{(1,2,3)}(H);\]
\[(vii) \Downarrow \operatorname{aint}_{(1,2,3)}(G) \equiv X \sim \alpha cl_{(1,2,3)}(X \sim G);\]
\[(viii) \Downarrow \operatorname{aint}_{(1,2,3)}(G) \equiv G \cap (X \sim \alpha cl_{(1,2,3)}(X \sim G));\]
\[(ix) \Downarrow H \equiv \operatorname{aint}_{(1,2,3)}(G) \land H \in \tau_{(1,2,3)}.\]

Proof. 
\[(i) \Downarrow \operatorname{aint}_{(1,2,3)}(X)(x) = \alpha N_x^{(1,2,3)}(X) = 1 \Rightarrow \operatorname{aint}_{(1,2,3)}(X) = X;\]
\[(ii) \Downarrow \operatorname{aint}_{(1,2,3)}(G) \subseteq G;\]
\[(iii) \Downarrow \tau_1 \equiv \tau_3 \rightarrow \operatorname{int}_1(G) \subseteq \operatorname{aint}_{(1,2,3)}(G);\]
\[(iv) \Downarrow H \in \tau_{(1,2,3)} \land H \subseteq G \rightarrow H \subseteq \operatorname{aint}_{(1,2,3)}(G);\]
\[(v) \Downarrow G = \operatorname{aint}_{(1,2,3)}(G) \rightarrow G \in \tau_{(1,2,3)};\]
\[(vi) \Downarrow G \in H \rightarrow \operatorname{aint}_{(1,2,3)}(G) \subseteq \operatorname{aint}_{(1,2,3)}(H);\]
\[(vii) \Downarrow \operatorname{aint}_{(1,2,3)}(G) \equiv X \sim \alpha cl_{(1,2,3)}(X \sim G);\]
\[(viii) \Downarrow \operatorname{aint}_{(1,2,3)}(G) \equiv G \cap (X \sim \alpha cl_{(1,2,3)}(X \sim G));\]
\[(ix) \Downarrow H \equiv \operatorname{aint}_{(1,2,3)}(G) \land H \in \tau_{(1,2,3)}.\]

\[\{G \cap (X \sim \alpha cl_{(1,2,3)}(X \sim G))\} = \alpha N_x^{(1,2,3)}(G) = \operatorname{aint}_{(1,2,3)}(G)(x).\] 
Therefore 
\[\operatorname{aint}_{(1,2,3)}(G) = G \cap (X \sim \alpha cl_{(1,2,3)}(X \sim G)).\]

(ix) From Theorem 21 -(ix) and (vii) above we get 
\[\exists H \equiv \operatorname{aint}_{(1,2,3)}(G) \equiv [X \sim H \equiv \alpha cl_{(1,2,3)}(X \sim G)] \subseteq [X \sim H \equiv \alpha cl_{(1,2,3)}(X \sim G)];\]

\[\square\]

Definition 24. If \((X, \tau_1, \tau_2, \tau_3)\) is a FTTS and \(G \subseteq X\). Then 
\[\operatorname{ext}_{(1,2,3)}(G)\] 
indicates the "(1,2,3)-α-exterior set of \(G\)" and defined as \(x \in \operatorname{ext}_{(1,2,3)}(X \sim G), i.e., \operatorname{ext}_{(1,2,3)}(G)(x) = \operatorname{aint}_{(1,2,3)}(X \sim G)(x).\)

Theorem 25. If \((X, \tau_1, \tau_2, \tau_3)\) is a FTTS and \(G \subseteq X\). Then
\[(i) \Downarrow \operatorname{ext}_{(1,2,3)}(G) \equiv X;\]
\[(ii) \Downarrow \operatorname{ext}_{(1,2,3)}(G) \subseteq X \sim G;\]
\[(iii) \Downarrow \tau_1 \equiv \tau_3 \rightarrow \operatorname{ext}_1(G) \subseteq \operatorname{ext}_{(1,2,3)}(G);\]
\[(iv) \Downarrow G \in \alpha F(1,2,3) \rightarrow \operatorname{ext}_{(1,2,3)}(G) \equiv X \sim G;\]
\[(v) \Downarrow H \in \alpha F(1,2,3) \land G \subseteq H \rightarrow \alpha F(1,2,3) \subseteq \operatorname{ext}_{(1,2,3)}(G);\]
\[(vi) \Downarrow G \subseteq G \rightarrow \operatorname{ext}_{(1,2,3)}(G) \subseteq \operatorname{ext}_{(1,2,3)}(G);\]
\[(vii) \Downarrow \operatorname{ext}_{(1,2,3)}(G) \equiv (X \sim G) \cap (X \sim \alpha cl_{(1,2,3)}(G));\]
\[(viii) \Downarrow \operatorname{ext}_{(1,2,3)}(G) \equiv X \sim \alpha cl_{(1,2,3)}(G);\]
\[(ix) \Downarrow x \in \operatorname{ext}_{(1,2,3)}(G) \leftrightarrow \exists H (x \in H \in \tau_{(1,2,3)} \land H \subseteq G = \emptyset).\]

Proof. The proofs of (i) - (vii) follow from Theorem 23.

\[(ix) \Downarrow \exists H : (x \in H \in \tau_{(1,2,3)} \land H \subseteq G = \emptyset) = \sup_{x \in X}(G \cap \alpha cl_{(1,2,3)}(X \sim G)(x) = \operatorname{ext}_{(1,2,3)}(X \sim G)(x). By Definition 24.\]

\[\square\]

Definition 26. If \((X, \tau_1, \tau_2, \tau_3)\) is a FTTS and \(G \subseteq X\), then 
\[\operatorname{ab}_{(1,2,3)}(G)\] 
indicates the "(1,2,3)-α-boundary of a set \(G\)" and defined as \(x \in \operatorname{ab}_{(1,2,3)}(G) = (x \notin \operatorname{aint}_{(1,2,3)}(G) \land (x \notin \operatorname{aint}_{(1,2,3)}(X \sim G)), \) i.e., \(x \in \operatorname{ab}_{(1,2,3)}(G)(x) = \min(1 \land \operatorname{aint}_{(1,2,3)}(X \sim G)(x)) \land (1 \land \operatorname{aint}_{(1,2,3)}(X \sim G)(x)).\)

Lemma 27. If \((X, \tau_1, \tau_2, \tau_3)\) is a FTTS, \(G \in P(X)\) and \(x \in X\), then 
\[\Downarrow x \in \operatorname{ab}_{(1,2,3)}(A) \rightarrow \forall H (H \in \alpha N_{(1,2,3)} \rightarrow (H \cap G) \neq \emptyset) \land \exists H (x \notin (X \sim G) \neq \emptyset).\]

Proof. The \(\forall H \in \alpha N_{(1,2,3)} \rightarrow \forall H \neq \emptyset \land \exists H (x \notin (X \sim G) \neq \emptyset).\)

\[\square\]

Theorem 28. If \((X, \tau_1, \tau_2, \tau_3)\) is a FTTS and \(G \in P(X)\), then
\[(i) \Downarrow \operatorname{ab}_{(1,2,3)}(G) = \alpha cl_{(1,2,3)}(G) \cap \alpha cl_{(1,2,3)}(X \sim G);\]
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Definition 29. If \((X, \tau, \tau_1, \tau_2, \tau_3)\) is a FTTS, then the class of all nets in \(X\) is defined as \(N(X) = \{S \mid S : D \rightarrow X, (D, \geq)\text{ is a directed set}\}\).

Definition 30. If \((X, \tau_1, \tau_2, \tau_3)\) is a FTTS, then the binary fuzzy predicates \(\triangleright_{(1,2,3)}\), \(\preceq_{(1,2,3)}\) are defined as

\[
S^a_{(1,2,3)} x = \forall G (G \in \alpha N_{x}^{(1,2,3)} \rightarrow S \subseteq G),
\]

\[
S^a_{(1,2,3)} x = \forall G (G \in \alpha N_{x}^{(1,2,3)} \rightarrow S \subseteq G), \quad S \in N(X),
\]

where \(S^a_{(1,2,3)} x\) stand for \("S is (1, 2, 3)-α-convergence to x"\) and \(S^a_{(1,2,3)} x\) stand for \("x is (1, 2, 3)-α-accumulation point of S\). Also, the binary crisp predicate \(\preceq\) is \("almost in\) and \(\preceq\) is \("often in\)\).

Definition 31. Let \(T \in N(X)\). One has the following fuzzy sets:

\[
\lim_{(1,2,3)}^a T(x) = [T^a_{(1,2,3)} x] = (1, 2, 3)-\lim(T)\]

\[
\text{adh}^a_{(1,2,3)} T(x) = [T^a_{(1,2,3)} x] = (1, 2, 3)-\text{adh}(T).
\]

Theorem 32. If \((X, \tau_1, \tau_2, \tau_3)\) is a FTTS \(x \in X, G \in P(X),\) and \(S \in N(X)\), then

\[(i) \exists S ((S \subseteq G \rightarrow \{x\}) \land (S^a_{(1,2,3)} x) \rightarrow x \in \text{adh}_{(1,2,3)} G); \]

\[(ii) \exists S ((S \subseteq G) \land (S^a_{(1,2,3)} x) \rightarrow x \in \alpha l_{(1,2,3)} G); \]

\[(iii) G \in \alpha l_{(1,2,3)} \quad \forall S ((S \subseteq G) \land \lim_{(1,2,3)}^a S \subseteq G); \]

\[(iv) \exists T ((T < S) \land (T^a_{(1,2,3)} x) \rightarrow S^a_{(1,2,3)} x, \text{where } T < S \text{ standing for } T \text{ is a subset of } S).\]

Proof.

(i) \[\exists S ((S \subseteq G \rightarrow \{x\}) \land (S^a_{(1,2,3)} x) \rightarrow x \in \text{adh}_{(1,2,3)} G);\]

(ii) \[\exists S ((S \subseteq G) \land (S^a_{(1,2,3)} x) \rightarrow x \in \alpha l_{(1,2,3)} G);\]

(iii) \[G \in \alpha l_{(1,2,3)} \quad \forall S ((S \subseteq G) \land \lim_{(1,2,3)}^a S \subseteq G);\]

(iv) \[\exists T ((T < S) \land (T^a_{(1,2,3)} x) \rightarrow S^a_{(1,2,3)} x, \text{where } T < S \text{ standing for } T \text{ is a subset of } S).\]
\[ [G \in \alpha \mathcal{F}_{1,2,3}] = [G = \alpha c_{1,2,3}(G)] = [G \subseteq \alpha c_{1,2,3}(G)] \land [\alpha c_{1,2,3}(G) \subseteq G] \leq [\alpha c_{1,2,3}(G) \subseteq G] = [X \sim G \subseteq X \sim \alpha c_{1,2,3}(G)] = \inf_{x \in X} \inf_{x \in X} (1 - \alpha N_{x}^{(1,2,3)}(G(x))) \leq \inf_{x \in X} \sup_{x \in X} (1 - \alpha N_{x}^{(1,2,3)}(H)) = \inf_{x \in X} \inf_{x \in X} (1 - \alpha N_{x}^{(1,2,3)}(H)) = [\forall S (S \subseteq G \rightarrow \lim \alpha_{(1,2,3)}^a S \subseteq G)]. \]

(iv) We have if \( S \not\subseteq G \), then \( S \not\subseteq G \), for any \( S \in \mathcal{N}(X) \) and any \( G \subseteq X \). Therefore

\[ \exists T ((T < S) \land (T \triangleright^a_{1,2,3} x)) = \sup_{T \subseteq S} \min_{T \subseteq S} (1 - \alpha N_{x}^{(1,2,3)}(G)) \leq \inf_{T \subseteq S} \min_{T \subseteq S} (1 - \alpha N_{x}^{(1,2,3)}(G)) \]

\( \triangleright^a_{1,2,3} T \).

Lemma 34. If \((x, r_1, r_2, r_3) \) is a FTTS, then \( (T \triangleright^a_{1,2,3} x) \)

\( \triangleright^a_{1,2,3} T \).

Proof. \( \lim \alpha_{(1,2,3)}^a T(x) = \inf_{T \subseteq T} (1 - \alpha N_{x}^{(1,2,3)}(G)) = \inf_{T \subseteq T} (1 - \alpha N_{x}^{(1,2,3)}(G)) = \alpha \triangleright^a_{1,2,3} T(x) \).

Theorem 33. If \((x, r_1, r_2, r_3) \) is a FTTS and \( T \) is a universal net, then \( \lim \alpha_{(1,2,3)}^a T = \alpha \triangleright^a_{1,2,3} T \).

Proof. \( \lim \alpha_{(1,2,3)}^a T(x) = \inf_{T \subseteq T} (1 - \alpha N_{x}^{(1,2,3)}(G)) = \inf_{T \subseteq T} (1 - \alpha N_{x}^{(1,2,3)}(G)) = \alpha \triangleright^a_{1,2,3} T(x) \).

Lemma 34. If \((x, r_1, r_2, r_3) \) is a FTTS, then \( (T \triangleright^a_{1,2,3} x) \)

\( \triangleright^a_{1,2,3} T \).

Proof. If \( H \subseteq G \) and \( T \not\subseteq G \), then \( T \not\subseteq G \).

\[ [T \triangleright^a_{1,2,3} x] = \inf_{T \subseteq T} (1 - \alpha N_{x}^{(1,2,3)}(G)) = 1 - \sup_{T \subseteq T} (1 - \alpha N_{x}^{(1,2,3)}(G)) \geq 1 - \sup_{T \subseteq T} (1 - \alpha N_{x}^{(1,2,3)}(G)) \geq \inf_{T \subseteq T} (1 - \alpha N_{x}^{(1,2,3)}(G)) \geq \inf_{T \subseteq T} (1 - \alpha N_{x}^{(1,2,3)}(G)) \geq [T \triangleright^a_{1,2,3} x]. \]

Theorem 37. If \((x, r_1, r_2, r_3) \) is a FTTS, then we have the following.

(1) \( T \in \mathcal{N}(X) \) and \( K^T \) is the filter corresponding to \( T \), i.e., \( K^T = [G : T \subseteq G] \), then

\( i \triangleright \alpha^a_{(1,2,3)} K^T = \alpha^a_{(1,2,3)} T \);

(2) \if \( K \in \mathcal{N}(X) \) and \( K^T \) is the net corresponding to \( K \), i.e., \( T^K : D \rightarrow X, (x, G) \rightarrow x, (x, G) \in D \), where \( D = \{x, G : x \in X, G \subseteq K\} \), \( (x, G) \geq (y, H) \) iff \( G \subseteq H \), then

\( i \triangleright \alpha^a_{(1,2,3)} T^K = \alpha^a_{(1,2,3)} K \);

\( k \triangleright \alpha^a_{(1,2,3)} T^K = \alpha^a_{(1,2,3)} K \).

8. Conclusion

The main contribution of the present paper is to give characterization of tri-\( \alpha \)-open sets in fuzzifying topological space. We also define the concepts of tri-\( \alpha \)-closed sets, tri-\( \alpha \)-neighborhood system, tri-\( \alpha \)-interior, tri-\( \alpha \)-closure, tri-\( \alpha \)-derived, tri-\( \alpha \)-boundary, tri-\( \alpha \)-exterior, and tri-\( \alpha \)-convergence in fuzzifying topological spaces and some basics of such spaces. We present some problems for future study.

(1) Study the results of the present paper by considering the quad-\( \alpha \)-open sets in fuzzifying quad-topological spaces.
(2) Investigate relations between fuzzifying quad-topology, tritopology, bitopology and fuzzifying topology.
(3) Study of quad-\(\alpha\)-separation axioms in fuzzifying quad-topological spaces.
(4) Generalize the results in the present work to soft fuzzifying topology.

Data Availability
No data were used to support this study.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

References