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Research Article

Hölder Type Inequalities for Sugeno Integrals under Usual Multiplication Operations

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The classical Hölder inequality shows an interesting upper bound for Lebesgue integral of the product of two functions. This paper proposes Hölder type inequalities and reverse Hölder type inequalities for Sugeno integrals under usual multiplication operations for nonincreasing concave or convex functions. One of the interesting results is that the inequality, $((S)\int_0^1 f(x)^p d\mu)^{1/p}((S)\int_0^1 g(x)^q d\mu)^{1/q} \leq (p^{-q/p} - p^{-q} + 1) \vee (q^{-p/q} - q^{-p} + 1)(S)\int_0^1 f(x)g(x)d\mu$, where $1 < p < \infty, 1/p + 1/q = 1$ and μ is the Lebesgue measure on $\mathbb{R}$, holds if f and g are nonincreasing and concave functions. As a special case, we consider Cauchy-Schwarz type inequalities for Sugeno integrals involving nonincreasing concave or convex functions. Some examples are provided to illustrate the validity of the proposed inequalities.

1. Introduction and Preliminaries

A number of studies have examined the Sugeno integral since its introduction in 1974 [1]. Ralescu and Adams [2] generalized a range of fuzzy measures and provided several equivalent definitions of fuzzy integrals. Wang and Klir [3] provided an overview of fuzzy measure theory.

Caballero and Sadarangani [4, 5] proved a Hermite-Hadamard type inequality and a Fritz Carlson's inequality for fuzzy integrals. Román-Flores et al. [6–9] presented several new types of inequalities for Sugeno integrals, including a Hardy type inequality, a Jensen type inequality, and some convolution type inequalities. Flores-Franulič et al. [10, 11] presented Chebyshev's inequality and Stolarsky's inequality for fuzzy integrals. Mesiar and Ouyang [12] generalized Chebyshev type inequalities for Sugeno integrals. Ouyang and Fang [13] generalized their main results to prove some optimal upper bounds for the Sugeno integral of the monotone function in [8]. Ouyang et al. [14] generalized a Chebyshev type inequality for the fuzzy integral of monotone functions based on an arbitrary fuzzy measure. Hong [15] extended previous research by presenting a Hardy-type inequality for Sugeno integrals in [6]. Hong [16] proposed a Liapunov type inequality for Sugeno integrals and presented two interesting classes of functions for which the classical Liapunov type inequality for Sugeno integrals holds. In Hong et al. [17] we consider Steffensen's integral inequality for the Sugeno integral where f is a nonincreasing and convex function and g is a nonincreasing function defined on $[0, 1]$. Hong [18] proposed a Berwald type inequality and a Favard type inequality for Sugeno integrals. Many researchers [19, 20] have also studied the inequalities for other fuzzy integrals.

Recently, Wu et al. [21] considered Hölder type inequalities for Sugeno integrals. However, they did not examine their results under usual multiplication operations and did not make the essential assumption of $1 < p < \infty, 1/p + 1/q = 1$ for the classical Hölder inequality.

In this paper, we propose Hölder type inequalities for Sugeno integrals and find optimal constants for which these inequalities hold for nonincreasing concave or convex functions under usual multiplication operations. We also propose a reverse Hölder type inequality for Sugeno integrals. As a special case, we consider Cauchy-Schwarz type inequalities for Sugeno integrals involving nonincreasing concave or

convex functions. Some examples are provided to illustrate the validity of the proposed inequalities.

Definition 1. Let Σ be σ-algebra of subsets of $\mathbb{R}$ and let $\mu : \Sigma \longrightarrow [0,\infty]$ be nonnegative, extended real-valued set function. We say that μ is a fuzzy measure if and only if

(a) $\mu(\emptyset) = 0$.

(b) $E, F \in \Sigma$ and $E \subseteq F$ imply $\mu(E) \leq \mu(F)$ (monotonicity)

(c) $\{E_p\} \subseteq \Sigma$ and $E_1 \subseteq E_2 \subseteq \cdots$ imply $\lim_{p\longrightarrow\infty}\mu(E_p) = \mu(\bigcup_{p=1}^{\infty} E_p)$ (continuity form below)

(d) $\{E_p\} \subseteq \Sigma$, $E_1 \supseteq E_2 \supseteq \cdots$, and $\mu(E_1) < \infty$ imply $\lim_{p\longrightarrow\infty}\mu(E_p) = \mu(\bigcap_{p=1}^{\infty} E_p)$ (continuity form above)

If f is a nonnegative real-valued function defined on $\mathbb{R}$, then we denote by $F_\alpha = \{x \in \mathbb{R} \mid f(x) \geq \alpha\} = \{f \geq \alpha\}$ the α-level of f, for $\alpha > 0$, and $F_0 = \overline{\{x \in \mathbb{R} \mid f(x) > 0\}} = \operatorname{supp}(f)$ is the support of f.

We note that

$$\alpha \leq \beta \Longrightarrow \{f \geq \beta\} \subseteq \{f \geq \alpha\} \tag{1}$$

If μ is a fuzzy measure on $A \subset \mathbb{R}$, then we define the following:

$$\mathfrak{F}^{\mu}(A) = \{f : A \longrightarrow [0,\infty) \mid f \text{ is } \mu\text{-measurable}\}. \tag{2}$$

Definition 2. Let μ be a fuzzy measure on $(\mathbb{R}, \Sigma)$. If $f \in \mathfrak{F}^{\mu}(\mathbb{R})$ and $A \in \Sigma$; then the Sugeno integral (or the fuzzy integral) of f on A, with respect to the fuzzy measure μ, is defined as

$$(S)\int_A f\,d\mu = \sup_{\alpha\in[0,\infty)} [\alpha \wedge \mu(A \cap F_\alpha)]. \tag{3}$$

In particular, if $A = \mathbb{R}$, then

$$(S)\int_{\mathbb{R}} f\,d\mu = (S)\int f\,d\mu = \sup_{\alpha\in[0,\infty)} [\alpha \wedge \mu(F_\alpha)]. \tag{4}$$

The following properties of the Sugeno integral are well known and can be found in [1].

Proposition 3 (see [1]). *If μ is a fuzzy measure on $\mathbb{R}$ and $f, g \in \mathfrak{F}^{\mu}(\mathbb{R})$, then*

(i) $(S)\int_A f\,d\mu \leq \mu(A)$

(ii) $(S)\int_A K\,d\mu = K \wedge \mu(A)$ *for any constant* $K \in [0,\infty)$

(iii) $(S)\int_A f\,d\mu \leq (S)\int_A g\,d\mu$, *if* $f \leq g$ *on* A

(iv) $\mu(A \cap \{f \geq \alpha\}) \geq \alpha \Longrightarrow (S)\int_A f\,d\mu \geq \alpha$

(v) $\mu(A \cap \{f \geq \alpha\}) \leq \alpha \Longrightarrow (S)\int_A f\,d\mu \leq \alpha$

(vi) $(S)\int_A f\,d\mu < \alpha \Longleftrightarrow$ *there exists* $\gamma < \alpha$ *such that* $\mu(A \cap \{f \geq \gamma\}) < \alpha$

(vii) $(S)\int_A f\,d\mu > \alpha \Longleftrightarrow$ *there exists* $\gamma > \alpha$ *such that* $\mu(A \cap \{f \geq \gamma\}) > \alpha$

Theorem 4 (see [13]). *Let $f : [0,\infty) \longrightarrow [0,\infty)$ be continuous and nonincreasing or nondecreasing functions and μ be the Lebesgue measure on $\mathbb{R}$. Let $(S)\int_0^a f(x)d\mu = p$. If $0 < p < a$, then $f(p) = p$ and $f(a-p) = p$, respectively.*

2. Hölder Type Inequalities

The classical Hölder inequality in probability theory provides the following inequality [22]:

$$\begin{aligned}&\int_0^1 f(x)\,g(x)\,d\mu \\ &\quad\leq \left(\int_0^1 f(x)^p\,d\mu\right)^{1/p}\left(\int_0^1 g(x)^q\,d\mu\right)^{1/q}\end{aligned} \tag{5}$$

where $1/p+1/q = 1$ or $\infty > p > 1$ and $f, g : [0,1] \longrightarrow [0,\infty)$ are integrable functions. inequality (5) shows an interesting upper bound for the Lebesgue integral of the product of two functions. In general, inequality (5) does not hold for the Sugeno integral as demonstrated by the following example.

Example 5. Let

$$\begin{aligned}f(x) &= \alpha^{1/p}, \\ g(x) &= \frac{\beta^{1/q}}{\beta-1}(x-1)\end{aligned} \tag{6}$$

for $0 < \alpha \leq \beta \leq 1$. Then, some straightforward calculus shows that

$$\begin{aligned}(S)\int_0^1 f(x)^p\,d\mu &= \alpha, \\ (S)\int_0^1 g(x)^q\,d\mu &= \beta\end{aligned} \tag{7}$$

and because $f(x)g(x)$ is nonincreasing and continuous, by Theorem 4,

$$(S)\int_0^1 f(x)\,g(x)\,d\mu = \frac{\alpha^{1/p}\beta^{1/q}}{\alpha^{1/p}\beta^{1/q}} - \beta + 1. \tag{8}$$

Consider that

$$\begin{aligned}&\frac{\left((S)\int_0^1 f(x)^p\,d\mu\right)^{1/p}\left((S)\int_0^1 g(x)^q\,d\mu\right)^{1/q}}{(S)\int_0^1 f(x)\,g(x)\,d\mu} \\ &\quad= \alpha^{1/p}\beta^{1/q} - \beta + 1.\end{aligned} \tag{9}$$

Let $\alpha = 1/100$ and $\beta = 9/10$. Then

$$\begin{aligned}\alpha^{1/p}\beta^{1/q} - \beta + 1 &= \frac{1}{100}^{1/p+1/q}90^{1/q} + \frac{1}{10} \\ &= \frac{1}{100}90^{1/q} + \frac{1}{10} < 1.\end{aligned} \tag{10}$$

Therefore, inequality (5) does not hold for Sugeno integrals.

In this context, this section presents Hölder type inequalities derived from (5) for Sugeno integrals. For this we first consider the following lemma.

Lemma 6. *Let f and g be any nonincreasing concave functions on $[0,1]$ and $0<\alpha\le\beta\le 1$, $0<p<q$. Then we have*

$$x^* = \sup\left\{(S)\int_0^1 f(x)\,g(x)\,d\mu \mid (S)\int_0^1 f(x)^p\,d\mu = \alpha, (S)\int_0^1 g(x)^q\,d\mu = \beta\right\} = \frac{\alpha^{1/p}\beta^{1/q}}{\alpha^{1/p}\beta^{1/q}-\beta+1}. \tag{11}$$

Proof. Let

$$(S)\int_0^1 f(x)^p\,d\mu = \alpha, \qquad (S)\int_0^1 g(x)^q\,d\mu = \beta. \tag{12}$$

Then, by Theorem 4, we have

$$f(\alpha) = \alpha^{1/p}, \qquad g(\beta) = \beta^{1/q}. \tag{13}$$

We consider the case of $0<\alpha\le\beta\le 1$ and that of $0<\beta\le\alpha\le 1$ to be similar. Note that because $\alpha = \alpha^{1/p+1/q} \le \alpha^{1/p}\beta^{1/q} \le \beta^{1/p+1/q} = \beta$, we have

$$f(\alpha)\,g(\alpha) \ge f(\alpha)\,g(\beta) = \alpha^{1/p}\beta^{1/q} \ge \alpha \tag{14}$$

and

$$f(\beta)\,g(\beta) \le f(\alpha)\,g(\beta) = \alpha^{1/p}\beta^{1/q} \le \beta. \tag{15}$$

Because $f(x)g(x)$ is nonincreasing and continuous, by the Intermediate Value Theorem and Theorem 4, there exists $x_0 \in [\alpha,\beta]$ such that

$$(S)\int_0^1 f(x)\,g(x)\,d\mu = f(x_0)\,g(x_0) = x_0. \tag{16}$$

It is then easy to check that

$$f^*(x) = \sup\left\{f(x) : \text{non-increasing and concave} \mid (S)\cdot\int_0^1 f(x)^p\,d\mu = \alpha\right\} = \alpha^{1/p} \quad \text{on } [\alpha,\beta] \tag{17}$$

and that

$$g^*(x) = \sup\left\{g(x) : \text{non-increasing and concave} \mid (S)\cdot\int_0^1 g(x)^q\,d\mu = \beta\right\} = \frac{\beta^{1/q}}{\beta-1}(x-1) \quad \text{on } [\alpha,\beta] \tag{18}$$

Therefore, x^* should satisfy the equation

$$x^* = (S)\int_0^1 f^*(x)\,g^*(x)\,d\mu = f^*(x^*)\,g^*(x^*) = \frac{\alpha^{1/p}\beta^{1/q}}{\beta-1}(x^*-1); \tag{19}$$

that is,

$$x^* = \frac{\alpha^{1/p}\beta^{1/q}}{\alpha^{1/p}\beta^{1/q}-\beta+1}. \tag{20}$$

□

Lemma 7. *Let f and g be any nonincreasing convex functions on $[0,1]$ and $0<\alpha\le\beta\le 1$, $0<p<q$. Then we have*

$$x^* = \sup\left\{(S)\int_0^1 f(x)\,g(x)\,d\mu \mid (S)\int_0^1 f(x)^p\,d\mu = \alpha, (S)\int_0^1 g(x)^q\,d\mu = \beta\right\} = \beta. \tag{21}$$

Proof. Let

$$(S)\int_0^1 f(x)^p\,d\mu = \alpha, \qquad (S)\int_0^1 g(x)^q\,d\mu = \beta. \tag{22}$$

Then, by Theorem 4, we have

$$f(\alpha) = \alpha^{1/p}, \qquad g(\beta) = \beta^{1/q}. \tag{23}$$

We consider the case of $0<\alpha\le\beta\le 1$ and that of $0<\beta\le\alpha\le 1$ to be similar. Note that because $\alpha \le \alpha^{1/p}\beta^{1/q} \le \beta$, we have

$$f(\alpha)\,g(\alpha) \ge f(\alpha)\,g(\beta) = \alpha^{1/p}\beta^{1/q} \ge \alpha \tag{24}$$

and

$$f(\beta)\,g(\beta) \le f(\alpha)\,g(\beta) = \alpha^{1/p}\beta^{1/q} \le \beta. \tag{25}$$

Because $f(x)g(x)$ is nonincreasing and continuous, by the Intermediate Value Theorem and Theorem 4, there exists $x_0 \in [\alpha,\beta]$ such that

$$(S)\int_0^1 f(x)\,g(x)\,d\mu = f(x_0)\,g(x_0) = x_0. \tag{26}$$

It is then easy to check that

$$f^*(x) = \sup\left\{f : \text{non-increasing and convex} \mid (S)\cdot\int_0^1 f(x)^p\,d\mu = \alpha\right\} = \alpha^{1/p} \quad \text{on } [\alpha,\beta]. \tag{27}$$

Let $g_\epsilon(x) = (\beta^{1/q}/\epsilon)(\beta+\epsilon-x)\vee 0$ for any $\epsilon > 0$. Then $(S)\int_0^1 g_\epsilon(x)^q d\mu = \beta$, and we have, for any $\epsilon > 0$,

$$x^* \ge (S)\int_0^1 \alpha^{1/p} g_\epsilon(x)\,d\mu = (S)\int_0^1 \frac{\alpha^{1/p}\beta^{1/q}}{\epsilon}(\beta+\epsilon-x)\vee 0\,d\mu = \frac{\alpha^{1/p}\beta^{1/q}(\beta+\epsilon)}{\alpha^{1/p}\beta^{1/q}+\epsilon} \tag{28}$$

Because $\epsilon > 0$ is arbitrary, we have

$$x^* \geq \beta, \tag{29}$$

and $x^* \leq \beta$ is trivial, which completes the proof. □

Proposition 8. *Suppose that f and g are non-increasing concave functions on $[0,1]$ and that μ is the Lebesgue measure on $\mathbb{R}$. Then for $1 < p < \infty$, $1/p + 1/q = 1$, there is no $c > 0$ such that the Hölder type inequality*

$$\begin{aligned} &c\,(S)\int_0^1 f(x)\,g(x)\,d\mu \\ &\quad\leq \left((S)\int_0^1 f(x)^p\,d\mu\right)^{1/p}\left((S)\int_0^1 g(x)^q\,d\mu\right)^{1/q} \end{aligned} \tag{30}$$

holds.

Proof. Let

$$\begin{aligned} &H_{p,q} \\ &\quad= \inf\left\{\frac{\left((S)\int_0^1 f(x)^p\,d\mu\right)^{1/p}\left((S)\int_0^1 g(x)^q\,d\mu\right)^{1/q}}{(S)\int_0^1 f(x)\,g(x)\,d\mu} \mid \right. \\ &\qquad\left. f,g : \text{non-increasing concave on } [0,1]\right\}. \end{aligned} \tag{31}$$

Then by Lemma 6,

$$\begin{aligned} H_{p,q} &= \inf\left\{\alpha^{1/p}\beta^{1/q} - \beta + 1 \mid 0 < \alpha \leq \beta \leq 1\right\} \\ &\quad\wedge \inf\left\{\alpha^{1/p}\beta^{1/q} - \alpha + 1 \mid 0 < \beta \leq \alpha \leq 1\right\}. \end{aligned} \tag{32}$$

Letting $\alpha \downarrow 0$ and $\beta \uparrow 1$, then

$$\inf\left\{\alpha^{1/p}\beta^{1/q} - \beta + 1 \mid 0 < \alpha \leq \beta \leq 1\right\} = 0, \tag{33}$$

and, similarly,

$$\inf\left\{\alpha^{1/p}\beta^{1/q} - \alpha + 1 \mid 0 < \beta \leq \alpha \leq 1\right\} = 0. \tag{34}$$

Therefore, $H_{p,q} = 0$, which completes the proof. □

Proposition 9. *Suppose that f and g are nonincreasing convex functions on $[0,1]$ and that μ is the Lebesgue measure on $\mathbb{R}$. Then, for $1 < p < \infty$, $1/p + 1/q = 1$, there is no $c > 0$ such that the Hölder type inequality*

$$\begin{aligned} &c\,(S)\int_0^1 f(x)\,g(x)\,d\mu \\ &\quad\leq \left((S)\int_0^1 f(x)^p\,d\mu\right)^{1/p}\left((S)\int_0^1 g(x)^q\,d\mu\right)^{1/q} \end{aligned} \tag{35}$$

holds.

Proof. Let

$$\begin{aligned} &H_{p,q} \\ &\quad= \inf\left\{\frac{\left((S)\int_0^1 f(x)^p\,d\mu\right)^{1/p}\left((S)\int_0^1 g(x)^q\,d\mu\right)^{1/q}}{(S)\int_0^1 f(x)\,g(x)\,d\mu} \mid \right. \\ &\qquad\left. f,g : \text{non-increasing convex on } [0,1]\right\}. \end{aligned} \tag{36}$$

Then, by Lemma 7,

$$\begin{aligned} H_{p,q} &= \inf\left\{\alpha^{1/p}\beta^{1/q-1} \mid 0 < \alpha \leq \beta \leq 1\right\} \\ &\quad\wedge \inf\left\{\alpha^{1/p-1}\beta^{1/q} \mid 0 < \beta \leq \alpha \leq 1\right\}. \end{aligned} \tag{37}$$

Let $\alpha \downarrow 0$ and $\beta \uparrow 1$. Then,

$$\inf\left\{\alpha^{1/p}\beta^{1/q-1} \mid 0 < \alpha \leq \beta \leq 1\right\} = 0, \tag{38}$$

and, similarly,

$$\inf\left\{\alpha^{1/p-1}\beta^{1/q} \mid 0 < \beta \leq \alpha \leq 1\right\} = 0. \tag{39}$$

Therefore $H_{p,q} = 0$, which completes the proof. □

Theorem 10 (Hölder type inequality for concave functions). *Suppose that f and g are nonincreasing concave functions on $[0,1]$ such that $f(0), g(0) \geq c > 0$ and that μ is the Lebesgue measure on $\mathbb{R}$. Then, for $0 < c \leq 1$, $1 < p < \infty$, $1/p+1/q = 1$, the inequality*

$$\begin{aligned} &c\,(S)\int_0^1 f(x)\,g(x)\,d\mu \\ &\quad\leq \left((S)\int_0^1 f(x)^p\,d\mu\right)^{1/p}\left((S)\int_0^1 g(x)^q\,d\mu\right)^{1/q} \end{aligned} \tag{40}$$

holds.

Proof. Let

$$\begin{aligned} &H_{p,q} \\ &\quad= \inf\left\{\frac{\left((S)\int_0^1 f(x)^p\,d\mu\right)^{1/p}\left((S)\int_0^1 g(x)^q\,d\mu\right)^{1/q}}{(S)\int_0^1 f(x)\,g(x)\,d\mu} \mid \right. \\ &\qquad\left. f,g : \text{non-increasing concave on } [0,1]\right\}. \end{aligned} \tag{41}$$

Then, by Lemma 6,

$$\begin{aligned} H_{p,q} &= \inf\left\{\alpha^{1/p}\beta^{1/q} - \beta + 1 \mid 0 < \alpha \leq \beta \leq 1\right\} \\ &\quad\wedge \inf\left\{\alpha^{1/p}\beta^{1/q} - \alpha + 1 \mid 0 < \beta \leq \alpha \leq 1\right\}. \end{aligned} \tag{42}$$

Now, consider that, for $0 < \alpha \le \beta \le 1$,

$$\begin{aligned}\frac{\partial\left(\alpha^{1/p}\beta^{1/q}-\beta+1\right)}{\partial\alpha} &= \frac{1}{p}\alpha^{1/p-1}\beta^{1/q} > 0,\\ \frac{\partial\left(\alpha^{1/p}\beta^{1/q}-\beta+1\right)}{\partial\beta} &= \frac{1}{q}\alpha^{1/p}\beta^{1/q-1}-1\\ &= \frac{1}{q}\alpha^{1/p}\beta^{1/q}\frac{1}{\beta}-1 \le \frac{1}{q}-1 < 0;\end{aligned} \tag{43}$$

that is, $\alpha^{1/p}\beta^{1/q}-\beta+1$ is increasing with respect to α and decreasing with respect to β. Suppose that $f(0), g(0) \ge c > 0$; then $c \le f^*(x) = \alpha^{1/p}$, because f^* is concave and constant on $[\alpha, \beta]$, and, hence, we have

$$c^p \le (S)\int_0^1 f^*(x)^p\, d\mu = \alpha. \tag{44}$$

Then, by letting $c^p = \alpha,\ \beta = 1$, we have

$$\inf\left\{\alpha^{1/p}\beta^{1/q}-\beta+1 \mid c^p \le \alpha \le \beta \le 1\right\} = c, \tag{45}$$

and, similarly, we have, for $0 < \beta \le \alpha \le 1$,

$$\inf\left\{\alpha^{1/p}\beta^{1/q}-\alpha+1 \mid c^q \le \beta \le \alpha \le 1\right\} = c. \tag{46}$$

Therefore, we have

$$H_{p,q} = c. \tag{47}$$

□

Corollary 11 (Cauchy-Schwarz type inequality for concave functions). *Suppose that f and g are nonincreasing concave functions on $[0,1]$ such that $f(0), g(0) \ge c > 0$ and that μ is the Lebesgue measure on $\mathbb{R}$. Then, for $0 < c \le 1$, the inequality*

$$\begin{aligned}&c\,(S)\int_0^1 f(x)\,g(x)\,d\mu\\ &\le \left((S)\int_0^1 f(x)^2\,d\mu\right)^{1/2}\left((S)\int_0^1 g(x)^2\,d\mu\right)^{1/2}\end{aligned} \tag{48}$$

holds.

Example 12. Let μ be the usual Lebesgue measure on $[0,1]$. If we take the functions f, g defined by $f(x) = 1-x$, $g(x) = 1-x/2$, then $f(0) = g(0) = 1 = c$. Then some straightforward calculus shows that

$$\begin{aligned}(S)\int_0^1 f(x)^2\,d\mu &= \frac{3-\sqrt{5}}{2},\\ (S)\int_0^1 g(x)^2\,d\mu &= 4-\sqrt{12},\end{aligned} \tag{49}$$

and

$$(S)\int_0^1 f(x)\,g(x)\,d\mu = \frac{5-\sqrt{17}}{2}. \tag{50}$$

Therefore,

$$H_{2,2}\,(S)\int_0^1 f(x)\,g(x)\,d\mu \approx 0.4384, \tag{51}$$

and

$$\begin{aligned}&\left((S)\int_0^1 f(x)^2\,d\mu\right)^{1/2}\left((S)\int_0^1 g(x)^2\,d\mu\right)^{1/2}\\ &= (0.618)\,(0.7321) = 0.4524\end{aligned} \tag{52}$$

Therefore, inequality (48) holds.

The next example shows that the constant $H_{p,q}$ is optimal.

Example 13. Let μ be the usual Lebesgue measure on $[0,1]$. If we take the functions f, g defined by $f(x) = 1$, $g(x) = n(1-x)$, $n > 1$, then $f(0), g(0) \ge 1 = c$, and, hence, $H_{2,2} = 1$. Then some straightforward calculus shows that

$$\begin{aligned}(S)\int_0^1 f(x)^2\,d\mu &= 1,\\ (S)\int_0^1 g(x)^2\,d\mu &= \frac{2n^2+1-\sqrt{4n^2+1}}{2n^2},\end{aligned} \tag{53}$$

and

$$(S)\int_0^1 f(x)\,g(x)\,d\mu = (S)\int_0^1 g(x)\,d\mu = \frac{n}{n+1}. \tag{54}$$

Therefore,

$$H_{2,2}\,(S)\int_0^1 f(x)\,g(x)\,d\mu = \frac{n}{n+1} \tag{55}$$

and

$$\begin{aligned}&\left((S)\int_0^1 f(x)^2\,d\mu\right)^{1/2}\left((S)\int_0^1 g(x)^2\,d\mu\right)^{1/2}\\ &= \left(\frac{2n^2+1-\sqrt{4n^2+1}}{2n^2}\right)^{1/2}.\end{aligned} \tag{56}$$

As $n \longrightarrow \infty$, we obtain that inequality (48) holds.

Theorem 14 (Hölder type inequality for convex functions). *Suppose that f and g are nonincreasing convex functions on $[0,1]$ such that $f(1), g(1) \ge c > 0$ and that μ is the Lebesgue measure on $\mathbb{R}$. Then, for $0 < c \le 1$, $1 < p < \infty$, $1/p+1/q = 1$, the inequality*

$$\begin{aligned}&c\,(S)\int_0^1 f(x)\,g(x)\,d\mu\\ &\le \left((S)\int_0^1 f(x)^p\,d\mu\right)^{1/p}\left((S)\int_0^1 g(x)^q\,d\mu\right)^{1/q}\end{aligned} \tag{57}$$

holds.

Proof. Let

$$H_{p,q} = \inf\left\{\frac{\left((S)\int_0^1 f(x)^p\,d\mu\right)^{1/p}\left((S)\int_0^1 g(x)^q\,d\mu\right)^{1/q}}{(S)\int_0^1 f(x)\,g(x)\,d\mu} \mid f, g : \text{non-increasing convex on } [0,1]\right\}. \tag{58}$$

Then, by Lemma 7,

$$H_{p,q} = \inf\left\{\alpha^{1/p}\beta^{1/q-1} \mid 0 < \alpha \le \beta \le 1\right\} \wedge \inf\left\{\alpha^{1/p-1}\beta^{1/q} \mid 0 < \beta \le \alpha \le 1\right\}. \tag{59}$$

Now, consider that, for $0 < \alpha \le \beta \le 1$,

$$\begin{aligned}\frac{\partial\left(\alpha^{1/p}\beta^{1/q-1}\right)}{\partial\alpha} &= \frac{1}{p}\alpha^{1/p-1} > 0,\\ \frac{\partial\left(\alpha^{1/p}\beta^{1/q-1}\right)}{\partial\beta} &= \left(\frac{1}{q}-1\right)\alpha^{1/p}\beta^{1/q-2}\\ &= \left(\frac{1}{q}-1\right)\alpha^{1/p}\beta^{1/q}\beta^{-2}\\ &\le \left(\frac{1}{q}-1\right)\beta^{-1} < 0.\end{aligned} \tag{60}$$

Suppose that $f(1) \ge c > 0$. Then $c \le f(x)$, and, hence, we have

$$c^p \le (S)\int_0^1 f(x)^p\,d\mu = \alpha. \tag{61}$$

Then we have

$$\inf\left\{\alpha^{1/p}\beta^{1/q-1} \mid c^p \le \alpha \le \beta \le 1\right\} = c^{p^{1/p}}1^{1/q-1} = c. \tag{62}$$

Similarly, we have, for $0 < \beta \le \alpha \le 1$,

$$\inf\left\{\alpha^{1/p-1}\beta^{1/q} \mid c^q \le \beta \le \alpha \le 1\right\} = c. \tag{63}$$

Therefore, we have

$$H_{p,q} = c. \tag{64}$$

□

Corollary 15 (Cauchy-Schwarz type inequality for convex functions). *Suppose that f and g are nonincreasing convex functions on $[0,1]$ such that $f(1), g(1) \ge c > 0$ and that μ is the Lebesgue measure on $\mathbb{R}$. Then, for $0 < c \le 1$, the inequality*

$$c\,(S)\int_0^1 f(x)\,g(x)\,d\mu \le \left((S)\int_0^1 f(x)^2\,d\mu\right)^{1/2}\left((S)\int_0^1 g(x)^2\,d\mu\right)^{1/2} \tag{65}$$

holds.

Example 16. Let μ be the usual Lebesgue measure on $[0,1]$. If we take the functions f, g defined by $f(x) = 1/2$, $g(x) = 1 - x/2$ then $f(1) = g(1) = 1/2 = c$. Then some straightforward calculus shows that

$$\begin{aligned}(S)\int_0^1 f(x)^2\,d\mu &= \frac{1}{4},\\ (S)\int_0^1 g(x)^2\,d\mu &= 4 - \sqrt{12} \approx 0.5359,\end{aligned} \tag{66}$$

and

$$(S)\int_0^1 f(x)\,g(x)\,d\mu = \frac{2}{5}. \tag{67}$$

Therefore,

$$c\,(S)\int_0^1 f(x)\,g(x)\,d\mu = 0.2, \tag{68}$$

and

$$\left((S)\int_0^1 f(x)^2\,d\mu\right)^{1/2}\left((S)\int_0^1 g(x)^2\,d\mu\right)^{1/2} = (0.5)(0.7321) = 0.3661. \tag{69}$$

Thus, inequality (65) holds.

3. Reverse Hölder Type Inequalities

In this section, we consider a reverse Hölder type inequality derived from (5) for Sugeno integrals. For this, we first consider the following lemma.

Lemma 17. *Let f and g be any nonincreasing concave functions on $[0,1]$ and $0 < \alpha \le \beta \le 1$, $0 < p < q$. Then we have*

$$x_* = \inf\left\{(S)\int_0^1 f(x)\,g(x)\,d\mu \mid (S)\int_0^1 f(x)^p\,d\mu = \alpha, (S)\int_0^1 g(x)^q\,d\mu = \beta\right\} = \frac{\alpha^{1/p}\beta^{1/q}}{\alpha^{1/p}\beta^{1/q} - \alpha + 1}. \tag{70}$$

Proof. Let

$$\begin{aligned}(S)\int_0^1 f(x)^p\,d\mu &= \alpha,\\ (S)\int_0^1 g(x)^q\,d\mu &= \beta.\end{aligned} \tag{71}$$

Then, by Theorem 4, we have

$$\begin{aligned}f(\alpha) &= \alpha^{1/p},\\ g(\beta) &= \beta^{1/q}.\end{aligned} \tag{72}$$

We consider the case of $0 < \alpha \le \beta \le 1$ and that of $0 < \beta \le \alpha \le 1$ to be similar. Note that because $\alpha = \alpha^{1/p+1/q} \le \alpha^{1/p}\beta^{1/q} \le \beta^{1/p+1/q} = \beta$, we have

$$f(\alpha)\,g(\alpha) \ge f(\alpha)\,g(\beta) = \alpha^{1/p}\beta^{1/q} \ge \alpha \tag{73}$$

and

$$f(\beta)\, g(\beta) \le f(\alpha)\, g(\beta) = \alpha^{1/p}\beta^{1/q} \le \beta. \tag{74}$$

Because $f(x)g(x)$ is nonincreasing and continuous, by the Intermediate Value Theorem and Theorem 4, there exists $x_0 \in [\alpha, \beta]$ such that

$$(S)\int_0^1 f(x)\, g(x)\, d\mu = f(x_0)\, g(x_0) = x_0. \tag{75}$$

It is then easy to check that

$$\begin{aligned} f_*(x) = \inf\Big\{ & f(x) : \text{non-increasing and concave} \mid (S) \\ & \cdot \int_0^1 f(x)^p\, d\mu = \alpha \Big\} = \frac{\alpha^{1/p}}{\alpha - 1}(x-1) \quad \text{on } [\alpha, \beta] \end{aligned} \tag{76}$$

and

$$\begin{aligned} g_*(x) = \inf\Big\{ & g(x) : \text{non-increasing and concave} \mid (S) \\ & \cdot \int_0^1 g(x)^q\, d\mu = \beta \Big\} = \beta^{1/q} \quad \text{on } [\alpha, \beta]. \end{aligned} \tag{77}$$

Therefore, x^* should satisfy the equation

$$\begin{aligned} x_* &= (S)\int_0^1 f_*(x)\, g_*(x)\, d\mu = f_*(x_*)\, g_*(x_*) \\ &= \frac{\alpha^{1/p}\beta^{1/q}}{\alpha - 1}(x_* - 1); \end{aligned} \tag{78}$$

that is,

$$x_* = \frac{\alpha^{1/p}\beta^{1/q}}{\alpha^{1/p}\beta^{1/q} - \alpha + 1}. \tag{79}$$

□

Lemma 18. *Let f and g be any nonincreasing convex functions on $[0,1]$ and $0 < \alpha \le \beta \le 1$, $0 < p < q$. Then we have*

$$\begin{aligned} x_* = \inf\Big\{ & (S)\int_0^1 f(x)\, g(x)\, d\mu \mid (S)\int_0^1 f(x)^p\, d\mu \\ & = \alpha, (S)\int_0^1 g(x)^q\, d\mu = \beta \Big\} = \alpha. \end{aligned} \tag{80}$$

Proof. Let

$$\begin{aligned} (S)\int_0^1 f(x)^p\, d\mu &= \alpha, \\ (S)\int_0^1 g(x)^q\, d\mu &= \beta. \end{aligned} \tag{81}$$

Then, by Theorem 4, we have

$$\begin{aligned} f(\alpha) &= \alpha^{1/p}, \\ g(\beta) &= \beta^{1/q}. \end{aligned} \tag{82}$$

We consider the case of $0 < \alpha \le \beta \le 1$ and that of $0 < \beta \le \alpha \le 1$ to be similar. Note that since $\alpha \le \alpha^{1/p}\beta^{1/q} \le \beta$, we have

$$f(\alpha)\, g(\alpha) \ge f(\alpha)\, g(\beta) = \alpha^{1/p}\beta^{1/q} \ge \alpha \tag{83}$$

and

$$f(\beta)\, g(\beta) \le f(\alpha)\, g(\beta) = \alpha^{1/p}\beta^{1/q} \le \beta. \tag{84}$$

Because $f(x)g(x)$ is nonincreasing and continuous, by the Intermediate Value Theorem and Theorem 4, there exists $x_0 \in [\alpha, \beta]$ such that

$$(S)\int_0^1 f(x)\, g(x)\, d\mu = f(x_0)\, g(x_0) = x_0. \tag{85}$$

It is now easy to check that

$$\begin{aligned} g_*(x) = \inf\Big\{ & g : \text{non-increasing and convex} \mid (S) \\ & \cdot \int_0^1 g(x)^q\, d\mu = \beta \Big\} = \beta^{1/q} \quad \text{on } [\alpha, \beta]. \end{aligned} \tag{86}$$

Let $f_\epsilon(x) = (\alpha^{1/p}/\epsilon)(\alpha + \epsilon - x) \vee 0$ for any $\epsilon > 0$. Then $(S)\int_0^1 f_\epsilon(x)^p d\mu = \alpha$, and we have, for any $\epsilon > 0$,

$$\begin{aligned} x_* &\le (S)\int_0^1 \beta^{1/q} f_\epsilon(x)\, d\mu \\ &= (S)\int_0^1 \frac{\beta^{1/q}\alpha^{1/p}}{\epsilon}(\alpha + \epsilon - x) \vee 0\, d\mu \\ &= \frac{\alpha^{1/p}\beta^{1/q}(\alpha + \epsilon)}{\alpha^{1/p}\beta^{1/q} + \epsilon}. \end{aligned} \tag{87}$$

Because $\epsilon > 0$ is arbitrary, we have

$$x_* \le \alpha \tag{88}$$

and $x_* \ge \alpha$ is trivial, which completes the proof. □

Theorem 19 (reverse Hölder type inequality for concave function). *Suppose that f and g are nonincreasing concave functions on $[0,1]$ and that μ is the Lebesgue measure on $\mathbb{R}$. Then for $1 < p < \infty$, $1/p + 1/q = 1$ the inequality*

$$\begin{aligned} & \left((S)\int_0^1 f(x)^p\, d\mu\right)^{1/p} \left((S)\int_0^1 g(x)^q\, d\mu\right)^{1/q} \\ & \quad \le H_{p,q}(S)\int_0^1 f(x)\, g(x)\, d\mu \end{aligned} \tag{89}$$

where $H_{p,q} = (p^{-q/p} - p^{-q} + 1) \vee (q^{-p/q} - q^{-p} + 1)$ holds.

Proof. Let

$$H_{p,q} = \sup\left\{\frac{\left((S)\int_0^1 f(x)^p\,d\mu\right)^{1/p}\left((S)\int_0^1 g(x)^q\,d\mu\right)^{1/q}}{(S)\int_0^1 f(x)\,g(x)\,d\mu} \;\middle|\; f,g : \text{non-increasing concave on } [0,1]\right\}. \tag{90}$$

Then, by Lemma 17,

$$\begin{aligned} H_{p,q} &= \sup\left\{\alpha^{1/p}\beta^{1/q} - \alpha + 1 \mid 0 < \alpha \le \beta \le 1\right\} \\ &\quad \vee \sup\left\{\alpha^{1/p}\beta^{1/q} - \beta + 1 \mid 0 < \beta \le \alpha \le 1\right\}. \end{aligned} \tag{91}$$

We first consider that, for $0 < \alpha \le \beta \le 1$,

$$\begin{aligned} &\sup\left\{\alpha^{1/p}\beta^{1/q} - \alpha + 1 \mid 0 < \alpha \le \beta \le 1\right\} \\ &\quad = \sup\left\{\alpha^{1/p} - \alpha + 1 \mid 0 < \alpha \le 1\right\}. \end{aligned} \tag{92}$$

Because

$$\frac{d\left(\alpha^{1/p} - \alpha + 1\right)}{d\alpha} = \frac{1}{p}\alpha^{1/p-1} - 1 = \frac{1}{p}\alpha^{-1/q} - 1 = 0, \tag{93}$$

then $\alpha = p^{-q}$. We also note that

$$\frac{d^2\left(\alpha^{1/p} - \alpha + 1\right)}{d\alpha^2} = -\frac{1}{pq}\alpha^{-1/q-1} < 0. \tag{94}$$

Thus, we have

$$\sup\left\{\alpha^{1/p} - \alpha + 1 \mid 0 < \alpha \le 1\right\} = p^{-q/p} - p^{-q} + 1. \tag{95}$$

Similarly, we have, for $0 < \beta \le \alpha \le 1$,

$$\begin{aligned} &\sup\left\{\alpha^{1/p}\beta^{1/q} - \beta + 1 \mid 0 < \beta \le \alpha \le 1\right\} \\ &\quad = q^{-p/q} - q^{-p} + 1. \end{aligned} \tag{96}$$

Therefore, we have

$$H_{p,q} = \left(p^{-q/p} - p^{-q} + 1\right) \vee \left(q^{-p/q} - q^{-p} + 1\right). \tag{97}$$

which completes the proof. □

Corollary 20 (reverse Cauchy-Schwarz type inequality for concave functions.). *Suppose that f and g are nonincreasing concave functions on $[0,1]$ and that μ is the Lebesgue measure on $\mathbb{R}$. Then the inequality*

$$\begin{aligned} &\left((S)\int_0^1 f(x)^2\,d\mu\right)^{1/2}\left((S)\int_0^1 g(x)^2\,d\mu\right)^{1/2} \\ &\quad \le \frac{5}{4}(S)\int_0^1 f(x)\,g(x)\,d\mu \end{aligned} \tag{98}$$

holds.

Example 21. Let μ be the usual Lebesgue measure on $[0,1]$. If we take the functions f, g defined by $f(x) = 1 - x$, $g(x) = 1 - x/2$. Then some straightforward calculus shows that

$$\begin{aligned} (S)\int_0^1 f(x)^2\,d\mu &= \frac{3-\sqrt{5}}{2}, \\ (S)\int_0^1 g(x)^2\,d\mu &= 4 - \sqrt{12}, \end{aligned} \tag{99}$$

and

$$(S)\int_0^1 f(x)\,g(x)\,d\mu = \frac{5-\sqrt{17}}{2} \approx 0.4384. \tag{100}$$

Then

$$\begin{aligned} &\left((S)\int_0^1 f(x)^2\,d\mu\right)^{1/2}\left((S)\int_0^1 g(x)^2\,d\mu\right)^{1/2} \\ &\quad = (0.618)(0.7321) = 0.4524 \end{aligned} \tag{101}$$

and

$$\frac{5}{4}(S)\int_0^1 f(x)\,g(x)\,d\mu = \frac{5-\sqrt{17}}{2} \approx 0.548. \tag{102}$$

Therefore, inequality (98) holds.

Proposition 22. *Suppose that f and g are nonincreasing convex functions on $[0,1]$ and that μ is the Lebesgue measure on $\mathbb{R}$. Then, for $1 < p < \infty$, $1/p + 1/q = 1$, there is no $c > 0$ such that the Hölder type inequality*

$$\begin{aligned} &\left((S)\int_0^1 f(x)^p\,d\mu\right)^{1/p}\left((S)\int_0^1 g(x)^q\,d\mu\right)^{1/q} \\ &\quad \le c\,(S)\int_0^1 f(x)\,g(x)\,d\mu \end{aligned} \tag{103}$$

holds.

Proof. Let

$$H_{p,q} = \sup\left\{\frac{\left((S)\int_0^1 f(x)^p\,d\mu\right)^{1/p}\left((S)\int_0^1 g(x)^q\,d\mu\right)^{1/q}}{(S)\int_0^1 f(x)\,g(x)\,d\mu} \mid f,g : \text{non-increasing convex on } [0,1]\right\}. \tag{104}$$

Then, by Lemma 18,

$$H_{p,q} = \sup\left\{\alpha^{1/p-1}\beta^{1/q} \mid 0<\alpha\le\beta\le 1\right\} \wedge \sup\left\{\alpha^{1/p}\beta^{1/q-1} \mid 0<\beta\le\alpha\le 1\right\}. \tag{105}$$

Let $\alpha\downarrow 0$ and $\beta\uparrow 1$. Then

$$\sup\left\{\alpha^{1/p}\beta^{1/q-1} \mid 0<\alpha\le\beta\le 1\right\} = \infty, \tag{106}$$

and, similarly,

$$\sup\left\{\alpha^{1/p-1}\beta^{1/q} \mid 0<\beta\le\alpha\le 1\right\} = \infty. \tag{107}$$

Therefore, $H_{p,q}=\infty$, which completes the proof. □

Theorem 23 (reverse Hölder type inequality for convex functions). *Suppose that f and g are nonincreasing convex functions on $[0,1]$ such that $f(1), g(1)\ge c>0$ and that μ is the Lebesgue measure on $\mathbb{R}$. Then for $0<c\le 1$, $1<p<\infty$, $1/p+1/q=1$ the inequality*

$$\left((S)\int_0^1 f(x)^p\,d\mu\right)^{1/p}\left((S)\int_0^1 g(x)^q\,d\mu\right)^{1/q} \le c^{1-p\vee q}(S)\int_0^1 f(x)\,g(x)\,d\mu \tag{108}$$

holds.

Proof. Let

$$H_{p,q} = \sup\left\{\frac{\left((S)\int_0^1 f(x)^p\,d\mu\right)^{1/p}\left((S)\int_0^1 g(x)^q\,d\mu\right)^{1/q}}{(S)\int_0^1 f(x)\,g(x)\,d\mu} \mid f,g : \text{non-increasing convex on } [0,1]\right\}. \tag{109}$$

Then, by Lemma 18,

$$H_{p,q} = \sup\left\{\alpha^{1/p-1}\beta^{1/q} \mid 0<\alpha\le\beta\le 1\right\} \vee \sup\left\{\alpha^{1/p}\beta^{1/q-1} \mid 0<\beta\le\alpha\le 1\right\}. \tag{110}$$

We first consider that, for $0<\alpha\le\beta\le 1$, the function $\alpha^{1/p-1}\beta^{1/q}$ is decreasing with respect to α and increasing with respect to β. Suppose that $f(1)\ge c>0$. Then $c\le f(x)$, and, hence, we have

$$c^p \le (S)\int_0^1 f(x)^p\,d\mu = \alpha. \tag{111}$$

Then we have

$$\sup\left\{\alpha^{1/p-1}\beta^{1/q} \mid c^p\le\alpha\le\beta\le 1\right\} = c^{p(1/p-1)}1^{1/q} = c^{1-p}. \tag{112}$$

Similarly, we have, for $0<\beta\le\alpha\le 1$,

$$\sup\left\{\alpha^{1/p}\beta^{1/q-1} \mid c^p\le\beta\le\alpha\le 1\right\} = c^{1-q}. \tag{113}$$

Therefore, we have

$$H_{p,q} = c^{1-p}\vee c^{1-q} = c^{1-p\vee q}. \tag{114}$$

□

Corollary 24 (reverse Cauchy-Schwarz type inequality for convex functions). *Suppose that f and g are nonincreasing convex functions on $[0,1]$ such that $f(1), g(1)\ge c>0$ and that μ is the Lebesgue measure on $\mathbb{R}$. Then for $0<c\le 1$, the inequality*

$$\left((S)\int_0^1 f(x)^2\,d\mu\right)^{1/2}\left((S)\int_0^1 g(x)^2\,d\mu\right)^{1/2} \le c^{-1}(S)\int_0^1 f(x)\,g(x)\,d\mu \tag{115}$$

holds.

Example 25. Let μ be the usual Lebesgue measure on $[0,1]$. If we take the functions f, g defined by $f(x)=1/2$, $g(x)=1-x/2$, then $f(1)=g(1)=1/2=c$. Then some straightforward calculus shows that

$$(S)\int_0^1 f(x)^2\,d\mu = \frac{1}{4}, \qquad (S)\int_0^1 g(x)^2\,d\mu = 4-\sqrt{12}\approx 0.5359, \tag{116}$$

and

$$(S)\int_0^1 f(x)\,g(x)\,d\mu = \frac{2}{5}. \tag{117}$$

Therefore,

$$2(S)\int_0^1 f(x)\,g(x)\,d\mu = 0.8, \tag{118}$$

and

$$\left((S)\int_0^1 f(x)^2\,d\mu\right)^{1/2}\left((S)\int_0^1 g(x)^2\,d\mu\right)^{1/2} = (0.5)(0.7321) = 0.3661. \tag{119}$$

Therefore, inequality (115) holds.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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