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Research Article

Møller's Energy in the Kantowski-Sachs Space-Time

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We have shown that the fourth component of Einstein's complex for the Kantowski-Sachs space-time is not identically zero. We have calculated the total energy of this space-time by using the energy-momentum definitions of Møller in the theory of general relativity and the tetrad theory of gravity.

1. Introduction

Since the birth of the theory of general relativity, this theory has been accepted as a superb theory of space-time and gravitation, as many physical aspects of nature have been experimentally verified in this theory. However, this theory is still incomplete theory; namely, it lacks definition of energy and momentum. In this theory many physicists have introduced different types of energy-momentum complexes [1–5], each of them being a pseudotensor, to solve this problem. The nontensorial property of these complexes is inherent in the way they have been defined and so much so it is quite difficult to conceive of a proper definition of energy and momentum of a given system. The recent attempt to solve this problem is to replace the theory of general relativity by another theory, concentrated on the gauge theories for the translation group, the so called teleparallel equivalent of general relativity. We were hoping that the theory of teleparallel gravity would solve this problem. Unfortunately, the localization of energy and momentum in this theory is still an open, unsolved, and disputed problem as in the theory of general relativity.

Møller modified the theory of general relativity by constructing a gravitational theory based on Weitzenböck space-time. This modification was to overcome the problem of the energy-momentum complex that appears in Riemannian space. In a series of paper [6–8], he was able to obtain a general expression for a satisfactory energy-momentum complex in

the absolute parallelism space. In this theory the field variable are 16 tetrad components $h_a{}^\mu$, from which the Riemannian metric arises as

$$g_{\mu\nu} = \eta_{ab} h^a{}_\mu h^b{}_v. \tag{1.1}$$

The basic purpose of this paper is to obtain the total energy of the Kantowski-Sachs space-time by using the energy-momentum definitions of Møller in the theory of general relativity and the tetrad theory of gravity.

The standard representation of Kantowski and Sachs space-times is given by [9]

$$ds^2 = dt^2 - A^2(t)dr^2 - B^2(t)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{1.2}$$

where the functions $A(t)$ and $B(t)$ are function in t and determined from the field equations.

For more detailed descriptions of the geometry and physics of this space-time (see [9–11]).

2. Fourth Component of Einstein's Complex

Prasanna has shown that space-times with purely time-dependent metric potentials have their components of total energy and momentum for any finite volume $(T_i^4 + t_i^4)$ identically zero. He had used the Einstein complex for the general Riemannian metric

$$ds^2 = g_{ij}\left(x^0\right)dx^i dx^j \tag{2.1}$$

and concluded the following: for space-times with metric potentials g_{ij} being functions of time variable alone and independent of space variable, the components $(T_i^4 + t_i^4)$ vanish identically as a consequence of conservation law.

Unfortunately the conclusion above is not the solution to the problem considered, in the sense that it does not give the same result, for all metrics have form of (2.1), using Einstein complex. If (2.1) is given in spherical coordinates, then Prasanna's conclusion is correct by using Møller's complex but not correct for all metrics by using Einstein's complex. Because Møller's complex could be utilized to any coordinate system, Einstein's complex gives meaningful result if it is evaluated in Cartesian coordinates. In the present paper we have found that the total energy for the Kantowski-Sachs space-time is identically zero by using Møller's complex, but not zero by using Einstein's complex.

In a recent paper [12], Gad and Fouad have found the energy and momentum distribution of Kantowski-Scahs space-time, using Einstein, Bergmann-Thomson, Landau-Lifshitz, and Papapetrou energy momentum complexes. In this section we restrict our attention to the Einstein's complex which is defined by [13]

$$\theta_i^k = T_i^k + t_i^k = u_{i,j}^{[kj]}, \tag{2.2}$$

with

$$u_i^{[kj]} = \frac{1}{16\pi}\frac{g_{in}}{\sqrt{-g}}\left[-g\left(g^{kn}g^{jm} - g^{jn}g^{km}\right)\right]_{,m}. \tag{2.3}$$

The energy and momentum in the Einstein's prescription are given by

$$P_i = \iiint \theta_i^0 dx^1\, dx^2\, dx^3. \tag{2.4}$$

The Einstein energy-momentum complex satisfies the local conservation law

$$\frac{\partial \theta_i^k}{\partial x^k} = 0. \tag{2.5}$$

The energy density for the space-time under consideration, in the Cartesian coordinates, obtained in [12] is

$$\theta_0^0 = \frac{1}{8\pi A r^4}\left(A^2 r^2 - B^2\right), \tag{2.6}$$

and the total energy is

$$E_{Ein} = P_0 = \frac{1}{2Ar}\left(A^2 r^2 + B^2\right). \tag{2.7}$$

Following the approach in [12], we obtain the following components of θ_0^k :

$$\begin{aligned}
\theta_0^1 &= -\frac{x}{8\pi A^2 r^4}\left[\dot{A}A^2 r^2 + B\left(2A\dot{B} - B\dot{A}\right)\right],\\
\theta_0^2 &= -\frac{y}{8\pi A^2 r^4}\left[\dot{A}A^2 r^2 + B\left(2A\dot{B} - B\dot{A}\right)\right],\\
\theta_0^3 &= -\frac{z}{8\pi A^2 r^4}\left[\dot{A}A^2 r^2 + B\left(2A\dot{B} - B\dot{A}\right)\right].
\end{aligned} \tag{2.8}$$

The components (2.6) and (2.8) satisfy the conservation law (2.5).

Hence from (2.6) and (2.8), we have $\theta_0^i = T_0^i + t_0^i \neq 0$; consequently $\theta_0^0 = T_0^0 + t_0^0$ is not identically zero.

3. Energy in the Theory of General Relativity

In the general theory of relativity, the energy-momentum complex of Møller in a four-dimensional background is given as [6]

$$\mathfrak{I}_i^k = \frac{1}{8\pi}\chi_{i,l}^{kl}, \tag{3.1}$$

where the antisymmetric superpotential χ_i^{kl} is

$$\chi_i^{kl} = -\chi_i^{lk} = \sqrt{-g}\left(\frac{\partial g_{in}}{\partial x^m} - \frac{\partial g_{im}}{\partial x^n}\right) g^{km} g^{nl}, \tag{3.2}$$

$\mathfrak{I}_0^0$ is the energy density and $\mathfrak{I}_\alpha^0$ are the momentum density components. Also, the energy-momentum complex $\mathfrak{I}_i^k$ satisfies the local conservation laws:

$$\frac{\partial \mathfrak{I}_i^k}{\partial x^k} = 0. \tag{3.3}$$

The energy and momentum components are given by

$$P_i = \iiint \mathfrak{I}_i^0 dx^1\, dx^2\, dx^3 = \frac{1}{8\pi} \int\int\int \frac{\partial \chi_i^{0l}}{\partial x^l} dx^1\, dx^2\, dx^3. \tag{3.4}$$

For the line element (1.2), the only nonvanishing components of χ_i^{kl} are

$$\begin{aligned} \chi_1^{01} &= -\frac{B^2(t)}{A(t)} \sin\theta, \\ \chi_2^{02} &= -A(t) \sin\theta, \\ \chi_3^{03} &= -\frac{A(t)}{\sin\theta}. \end{aligned} \tag{3.5}$$

Using these components in (3.1), we get the energy and momentum densities as follows

$$\begin{gathered} \mathfrak{I}_0^0 = 0, \\ \mathfrak{I}_1^0 = \mathfrak{I}_3^0 = 0, \qquad \mathfrak{I}_2^0 = -A(t) \cos\theta. \end{gathered} \tag{3.6}$$

From (3.4) and (3.5) and applying the Gauss theorem, we obtain the total energy and momentum components in the following form:

$$\begin{aligned} P_0 &= E = 0, \\ P_\alpha &= 0. \end{aligned} \tag{3.7}$$

4. Energy in the Tetrad Theory of Gravity

The superpotential of Møller in the tetrad theory of gravity is given by (see [7, 8, 14])

$$U_\mu^{\nu\beta} = \frac{\sqrt{-g}}{2\kappa} P_{\chi\rho\sigma}^{\tau\nu\beta} \left[\Phi^\rho g^{\sigma\chi} g_{\mu\tau} - \lambda g_{\tau\mu} \gamma^{\chi\rho\sigma} - (1-2\lambda) g_{\tau\mu} \gamma^{\sigma\rho\chi}\right], \tag{4.1}$$

where

$$P^{\tau\nu\beta}_{\chi\rho\sigma} = \delta^{\tau}_{\chi} g^{\nu\beta}_{\rho\sigma} + \delta^{\tau}_{\rho} g^{\nu\beta}_{\sigma\chi} - \delta^{\tau}_{\sigma} g^{\nu\beta}_{\chi\sigma}, \tag{4.2}$$

with $g^{\nu\beta}_{\rho\sigma}$ being a tensor defined by

$$g^{\nu\beta}_{\rho\sigma} = \delta^{\nu}_{\rho}\delta^{\beta}_{\sigma} - \delta^{\nu}_{\sigma}\delta^{\beta}_{\rho}, \tag{4.3}$$

γ_{abc} is the con-torsion tensor given by

$$\gamma_{\mu\nu\beta} = h_{i\mu} h^{i}{}_{\nu;\beta} \tag{4.4}$$

and Φ_{μ} is the basic vector defined by

$$\Phi_{\mu} = \gamma^{\rho}{}_{\mu\rho}, \tag{4.5}$$

The energy in this theory is expressed by the following surface integral:

$$E = \lim_{r\to\infty} \int_{r=\text{const.}} U^{0\alpha}_{0} n_{\alpha} dS, \tag{4.6}$$

where n_{α} is the unit three vector normal to the surface element dS.

The tetrad components of the space-time (1.2), using (1.1), are as follows

$$\begin{gathered} h^{a}{}_{\mu} = [1, A(t), B(t), B(t)\sin\theta], \\ h_{a}{}^{\mu} = \left[1, A^{-1}(t), B^{-1}(t), \frac{B^{-1}(t)}{\sin\theta}\right]. \end{gathered} \tag{4.7}$$

Using these components in (4.4), we get the nonvanishing components of $\gamma_{\mu\nu\beta}$ as follows

$$\begin{gathered} \gamma_{011} = -\gamma_{101} = -A(t)\dot{A}(t), \\ \gamma_{022} = -\gamma_{202} = -B(t)\dot{B}(t), \\ \gamma_{033} = -\gamma_{303} = -B(t)\dot{B}(t)\sin^2\theta, \\ \gamma_{233} = -\gamma_{323} = -B^2(t)\sin\theta\cos\theta. \end{gathered} \tag{4.8}$$

Consequently, the only nonvanishing components of basic vector field are

$$\begin{gathered} \Phi^{0} = -2\left\{\frac{\dot{A}(t)}{A(t)} + \frac{\dot{B}(t)}{B(t)}\right\}, \\ \Phi^{2} = \frac{\cot\theta}{B^2(t)}. \end{gathered} \tag{4.9}$$

Using (4.8) and (4.9) in (4.1) and (4.6), we get

$$E = 0. \tag{4.10}$$

5. Summary and Discussion

In this paper we have shown that the fourth component of Einstein's complex for the Kantowski-Sachs space-time is not identically zero. This gives a counterexample to the result obtained by Prasanna [15]. We calculated the total energy of Kantowski-Sachs space-time using Møller's tetrad theory of gravity. We found that the total energy is zero in this space-time. This result does not agree with the previous results obtained in both theories of general relativity [12] and teleparallel gravity [16], using Einstein, Bergmann-Thomson, and Landau-Lifshitz energy-momentum complexes. In both theories the energy and momentum densities for this space-time are finite and reasonable. We notice that the results obtained by using Einstein, Bergmann-Thomson, and Papapetrou are in conflict with that given by Møller's values for the energy and momentum densities if r tends to infinity, while Landau-Lifshitz's values are not in conflict.

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