

## Research Article

# A Deconstruction Lattice Description of the D1/D5 Brane World-Volume Gauge Theory

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I generalize the deconstruction lattice formulation of Endres and Kaplan to two-dimensional super-QCD with eight supercharges, denoted by  $(4,4)$ , and bifundamental matter. I specialize to a particularly interesting  $(4,4)$  gauge theory, with gauge group  $U(N_c) \times U(N_f)$ , and  $U(N_f)$  being weakly gauged. It describes the infrared limit of the D1/D5 brane system, which has been studied extensively as an example of the  $AdS_3/CFT_2$  correspondence. The construction here preserves two supercharges exactly and has a lattice structure quite similar to that which has previously appeared in the deconstruction approach, that is, site, link, and diagonal fields with both the Bose and Fermi statistics. I remark on possible applications of the lattice theory that would test the  $AdS_3/CFT_2$  correspondence, particularly one that would exploit the recent worldsheet instanton analysis of Chen and Tong.

## 1. Introduction

Supersymmetric large  $N_c$  gauge theory seems to afford a window on quantum gravity, through the  $AdS/CFT$  correspondence [1–4]. Recent formulations of lattice supersymmetry give some hope that we may be able to study these ideas on the lattice. In particular, to what extent does the correspondence hold at intermediate  $N_c$ , at finite temperature, and for non-BPS quantities?

Many promising lattice formulations of supersymmetric field theories occur in two dimensions (2D). (For an extensive list of references on lattice formulations of supersymmetric field theories, both old and new, see [5, 6].) In some cases, convincing perturbative arguments can be made that the correct continuum limit is obtained without fine tuning [7–16]. (Another interesting approach, involving noncommutativity on the scale of the lattice, deserves further study of its quantum continuum limit [17, 18].) In other cases, super-renormalizability implies that fine-tuning a small set of one-loop diagrams allows one

to obtain the desired continuum limit in perturbation theory, as in [19]. (A 3D analogue is described in [20].)

Broadly speaking, it is the softer ultraviolet (UV) divergences in 2D that generically make it easier to obtain the desired continuum limit in perturbation theory. Whether or not this property holds nonperturbatively is an open question, which at this point can only be answered empirically. In this regard, it is important to note that 2D field theories are more practical to study numerically; a small computer cluster can obtain reasonably accurate results. For some 2D examples, Monte Carlo simulation results have provided information on nonperturbative renormalization. For example, recent simulations of 2D supersymmetric theories that preserve a nilpotent subalgebra seem entirely consistent with continuum expectations [21–26]. In this author’s opinion, the encouraging results in 2D suggest that it is time to look for interesting applications of the lattice supersymmetry ideas that have been developed thus far.

A well-known example of the AdS/CFT correspondence occurs in the Type IIB superstring, at the intersection of D1 and D5 branes, with four of the directions of the D5 brane wrapped on, say, a torus  $T^4$ . The IR limit of the world-volume intersection theory is a 2D (4,4) supersymmetric gauge theory. It can be understood as the dimensional reduction of a 4d  $\mathcal{N} = 2$  super-QCD [27–29] where  $N_f$  flavors of matter are contained in hypermultiplets, and transform in the fundamental representation of the  $U(N_c)$  gauge group. These flavors are minimally coupled, so that there would be a  $U(N_f)$  flavor symmetry. In actuality, the  $U(N_f)$  symmetry is weakly gauged, and the flavors are bifundamentals of  $U(N_c) \times U(N_f)$ .

In this paper, the (2,2) supersymmetric formulation of Endres and Kaplan (EK) [30] will be generalized to (4,4) theories that describe the world-volume gauge theory of the D1/D5 brane system. That work is based on the orbifold approach to lattice supersymmetry, which has been reviewed in [6]. It will be seen that, in the gauge sector, a slight modification of the original (4,4) pure SYM construction of Cohen et al. (CKKU) [10] is required. EK have shown, in general terms, how to construct (2,2) theories with bifundamental matter under a quiver gauge group  $U(N)^m$  (“Example 2” at the end of their paper). The target theory for the D1/D5 system which is aimed at here is (4,4) 2D super-QCD with gauge group  $U(N_c) \times U(N_f)$ . In this context, we want the flexibility to choose  $N_c$  and  $N_f$  independently. Also, as already been mentioned, the flavor group  $U(N_f)$  should be weakly gauged; that is, the corresponding gauge couplings should satisfy  $g_{N_f} \ll g_{N_c}$  for all scales of interest. It will be seen below that it is not difficult to modify EK’s technique to obtain a quiver of two factors,  $U(N_c)$  and  $U(N_f)$ , with  $N_c \neq N_f$ . The nontrivial task is to devise a trick to make the  $U(N_f)$  weakly gauged in the EK construction with bifundamental matter. The trick that is used is quite standard and has an interesting interpretation, as will be described below. With these various generalizations of the EK formulation, the lattice theory described here is specially tailored to describe the D1/D5 world-volume theory. Having at hand a fully latticed description of this theory, one can contemplate various nonperturbative studies that would be of interest, both numerical (Monte Carlo simulations) and analytical (strong coupling expansions).

I will now summarize the remainder of this paper.

- (i) *Section 2 (Summary of the Target Theory)*. Here I describe the continuum  $U(N_c) \times U(N_f)$  (4,4) gauge theory that is aimed at, using the language of 4d  $\mathcal{N} = 1$  superfields.
- (ii) *Section 3 (Lattice Construction)*. First I explain in general terms how a (4,4) theory with gauge group  $U(N_c) \times U(N_f)^n$  and bifundamental matter is obtained.

The  $U(N_f)^n$  quiver is introduced in order to obtain a weakly coupled diagonal subgroup,  $U(N_f)_{\text{diag}}$ . It will be explained how this can be interpreted in terms of a deconstructed third dimension. The  $U(N_c)$  gauge multiplet does not propagate in this direction but is stuck to the 2D subspace. It is interesting that this mimics what occurs in the D1/D5 system, where  $U(N_c)$  gauge fields are stuck to the D1 brane, and  $g_{N_f} \ll g_{N_c}$  is due to “volume suppression”. In the 4d  $\mathcal{N} = 1$  language, the lowest components of hypermultiplets are  $SO(1,3) \times U(1)_R$  neutral and form a doublet of  $SU(2)_R$ . It follows that in the conventions of CKKU, the hypermultiplets would have fractional  $N$ -ality with respect to the  $Z_N \times Z_N$  that is used to define the lattice theory. This unacceptable situation calls for a minor modification of the choice of global charges used in the  $Z_N \times Z_N$  orbifold, relative to CKKU [10]. Finally, I describe how to break to  $U(N_f)_{\text{diag}}$ , and the conditions that must be satisfied for the Kaluza-Klein (KK) states of the corresponding third dimension to decouple from the  $U(N_c) \times U(N_f)_{\text{diag}}$  effective theory.

- (iii) *Section 4 (Application)*. Here I describe a simple study of the characteristics of instantons in the sector with unit first Chern class. The distribution of these characteristics—instanton size and orientation—has been shown recently to have  $\text{AdS}_3 \times S^3$  geometry [31]. This support for the  $\text{AdS}_3/\text{CFT}_2$  correspondence would be interesting to study on the lattice where we can access intermediate  $N_c$  and finite temperature. In this regard, the recent results of Rey and Hikida [32] provide continuum results that could be compared to.
- (iv) *Section 5 (Conclusions)*. Here I summarize this work and outline research that is in progress.
- (v) *Appendices*. Some technical details and lengthy formulae have been separated from the main text.

The purpose of this paper is to give a brief outline of the lattice construction and its potential applications. A more thorough discussion of details associated with the lattice system (superspace description, renormalization, etc.), as well as intensive studies of the possible applications mentioned in Section 4, is left to future work.

## 2. Summary of the Target Theory

The 2D theory is most easily obtained from a dimensional reduction of the 4d theory written in  $\mathcal{N} = 1$  superspace. (See [33] for a review of this formalism.) The  $U(N_c)$   $\mathcal{N} = 2$  vector multiplet is written in terms of an  $\mathcal{N} = 1$  vector superfield  $V$  and an adjoint  $N = 1$  chiral superfield  $\Phi$ . For  $U(N_f)$  the notation will be  $\tilde{V}, \tilde{\Phi}$ . The action is compactly described in terms of the (real) Kähler potential  $K$  and the (holomorphic) superpotential  $W$ .

For the gauge multiplet, we have

$$\begin{aligned}
 K_{\text{gse}} &= \frac{1}{g^2} \Phi^{\dagger A} \left( e^V \right)_A{}^B \Phi_B + \frac{1}{\tilde{g}^2} \tilde{\Phi}^{\dagger M} \left( e^{\tilde{V}} \right)_M{}^N \tilde{\Phi}_N, \\
 W_{\text{gse}} &= \frac{1}{4g^2} W_A^\alpha W_\alpha^A + \frac{1}{4\tilde{g}^2} \tilde{W}_M^\alpha \tilde{W}_\alpha^M,
 \end{aligned} \tag{2.1}$$

written in terms of the usual chiral field strength spinor superfields  $W^\alpha(V)$  and  $\widetilde{W}^\alpha(\widetilde{V})$  and adjoint representation matrices  $t^A_C{}^B$  and  $t^L_N{}^M$ , for  $U(N_c)$  and  $U(N_f)$ , respectively.

The hypermultiplet is written in terms of two chiral multiplets, denoted by  $Q$  and  $\widetilde{Q}$ . The  $U(N_c) \times U(N_f)$  representations for these superfields are

$$Q_a{}^m = (N_c, \overline{N}_f), \quad \widetilde{Q}_m{}^a = (\overline{N}_c, N_f). \quad (2.2)$$

The indices range according to  $a = 1, \dots, N_c$ ;  $m = 1, \dots, N_f$ . It will be convenient to regard  $Q$  as an  $N_c \times N_f$  matrix, and  $\widetilde{Q}$  and an  $N_f \times N_c$  matrix. Correspondingly,  $Q^\dagger$  will be an  $N_f \times N_c$  matrix and  $\widetilde{Q}^\dagger$  will be an  $N_c \times N_f$  matrix. We can then write the Kähler potential as

$$\begin{aligned} K_{\text{mat}} &= \text{Tr } Q^\dagger e^V Q e^{-\widetilde{V}} + \text{Tr } \widetilde{Q}^\dagger e^{\widetilde{V}} \widetilde{Q} e^{-V} \\ &= (Q^\dagger)_m{}^a (e^V)_a{}^b Q_b{}^n (e^{-\widetilde{V}})_n{}^m + (\widetilde{Q}^\dagger)_a{}^m (e^{\widetilde{V}})_m{}^n \widetilde{Q}_n{}^b (e^{-V})_b{}^a. \end{aligned} \quad (2.3)$$

Note that  $V$  is expressed in terms of  $U(N_c)$  fundamental representation generators  $t^A_a{}^b$ ; a similar statement holds for  $\widetilde{V}$ , except that the group is  $U(N_f)$ . The normalization convention that is assumed in the following is defined by  $\text{Tr } t^A t^B = (1/2)\delta^{AB}$  for the fundamental representation. In the second step of (2.3), the indices have been written explicitly in order to make the matrix notation clear. Below, such details will be left implicit.

The superpotential is the minimal one, which preserves  $U(1)_R$ :

$$W_{\text{mat}} = \sqrt{2} \text{Tr } \widetilde{Q} \Phi Q - \sqrt{2} \text{Tr } Q \widetilde{\Phi} \widetilde{Q}. \quad (2.4)$$

Here,  $\Phi$  and  $\widetilde{\Phi}$  are expressed in terms of  $U(N_c)$  and  $U(N_f)$  fundamental representation generators, respectively. It is easy to check gauge invariance, which acts holomorphically on the chiral superfields:

$$\begin{aligned} Q &\longrightarrow e^\Lambda Q e^{-\widetilde{\Lambda}}, & \widetilde{Q} &\longrightarrow e^{\widetilde{\Lambda}} \widetilde{Q} e^{-\Lambda}, & e^V &\longrightarrow e^{-\Lambda^\dagger} e^V e^{-\Lambda}, & e^{\widetilde{V}} &\longrightarrow e^{-\widetilde{\Lambda}^\dagger} e^{\widetilde{V}} e^{-\widetilde{\Lambda}}, \\ \Phi &\longrightarrow e^\Lambda \Phi e^{-\Lambda}, & \widetilde{\Phi} &\longrightarrow e^{\widetilde{\Lambda}} \widetilde{\Phi} e^{-\widetilde{\Lambda}}, \end{aligned} \quad (2.5)$$

where  $\Lambda$  and  $\widetilde{\Lambda}$  are chiral superfields valued in the Lie algebras of  $U(N_c)$  and  $U(N_f)$ , respectively.

The action is given by a Grassmann integral over superspace coordinates  $\theta^\alpha, \bar{\theta}_{\dot{\alpha}}$ :

$$S = \int d^4x \left\{ \int d^4\theta (K_{\text{gqe}} + K_{\text{mat}}) + \left[ \int d^2\theta (W_{\text{gqe}} + W_{\text{mat}}) + \text{h.c.} \right] \right\}. \quad (2.6)$$

### 3. Lattice Construction

The EK approach [30] includes matter, in a generalization of earlier work by Kaplan et al., especially CKKU [8–11]. The Kaplan et al. “deconstruction lattice” approach was

an outgrowth of *dimensional deconstruction* [34, 35]. In “Example 2” given by EK, quiver gauge theories with bifundamental matter were formulated. In this section, I generalize EK’s quiver construction to the case of (4,4) 2D super-QCD with bifundamental matter that is charged under a gauge group  $U(N_c) \times U(N_f)$ . A minor modification of the (4,4) setup of CKKU [10] will prove necessary, due to the R-charges of the hypermultiplets that are being added to the theory. The other difficulty will be that we need to have  $U(N_f)$  weakly gauged relative to  $U(N_c)$ . This will be addressed through extending to a quiver gauge theory  $U(N_c) \times U(N_f)^n$  and then breaking  $U(N_f)^n$  to its diagonal subgroup.

### 3.1. Outline

In the present theory, we begin with a matrix model that is the zero-dimensional (0d) reduction (the 0d reduction is obtained by treating all fields as independent of space-time coordinates) of 4d  $\mathcal{N} = 2$  super-QCD with gauge group  $U((N_c + nN_f)N^2)$  and fundamental matter. This theory is described by

$$\begin{aligned} K_{\text{gge}} &= \frac{1}{g^2} \Phi^{\dagger A} (e^V)_A{}^B \Phi_B, & W_{\text{gge}} &= \frac{1}{4g^2} W_A^\alpha W_\alpha^A, \\ K_{\text{mat}} &= Q^{\dagger a} (e^V)_a{}^b Q_b + \tilde{Q}^a (e^{-V})_a{}^b \tilde{Q}_b^\dagger, & W_{\text{mat}} &= \sqrt{2} \tilde{Q}^a \Phi_a{}^b Q_b. \end{aligned} \quad (3.1)$$

Here, indices  $A, B$  correspond to the adjoint representation, whereas the indices  $a, b$  correspond to the fundamental representation. We will absorb the overall space-time volume  $V_4 = \int d^4x$  associated with the 4d  $\rightarrow$  0d reduction into a redefinition of the coupling constant  $g^2$  and the matter fields  $Q, \tilde{Q}$ . The resulting 0d theory (fixed to Wess-Zumino gauge) will be referred to as the *mother theory*, following Kaplan et al.

The next step is to perform an orbifold projection on the mother theory, in order to reduce it to the *daughter theory*. This “orbifolding” proceeds in two steps. First we orbifold by a  $Z_{n+1}$  symmetry of the mother theory, to break the gauge group according to

$$U((N_c + nN_f)N^2) \longrightarrow U(N_c N^2) \times U(N_f N^2)^n. \quad (3.2)$$

Then we orbifold by a  $Z_N \times Z_N$  symmetry of the mother theory to break the gauge group further, according to

$$U(N_c N^2) \times U(N_f N^2)^n \longrightarrow U(N_c)^{N^2} \times U(N_f)^{nN^2}. \quad (3.3)$$

It is at this point that the trick to get a weakly gauged  $U(N_f)$  comes in. At the final stage of the orbifolding—the RHS of (3.3)—the gauge coupling is universal, with its strength determined by the single coupling  $g^2$  that appears in the original 0d matrix model (3.1) and the overall lattice spacing that is determined by the choice of vacuum (dynamical lattice spacing) for the deconstruction—what was called the *a-configuration* in [5]. However, we

now ‘‘Higgs’’ the subgroup  $U(N_f)^{nN^2}$  on the RHS of (3.3) with universal vacuum expectation values in bifundamental matter of this group, to break to the diagonal subgroup:

$$U(N_f)^{nN^2} \longrightarrow U(N_f)_{\text{diag}}^{N^2}. \quad (3.4)$$

Then the coupling for the diagonal group is

$$\tilde{g}_{\text{diag}}^2 = \frac{g^2}{n}. \quad (3.5)$$

For large  $n$  we obtain the desired result—a weakly gauged flavor group.

An alternative picture of this trick is the following. We may regard the factor  $n$  as counting sites in a third dimension that has been deconstructed. Only the fields with  $U(N_f)$  charge propagate in this third dimension. The  $U(N_c)$  vector multiplet is stuck to the 2D subspace. It is very interesting that this mimics what happens in the D1/D5 brane system. There, the flavored fields propagate throughout the torus  $T^4$ , since they correspond to strings that have one end on the D5 brane that wraps  $T^4$ . The D1 branes are stuck at a point in  $T^4$ , and so the purely colored fields do not propagate in the  $T^4$  direction. The difference here is that, to simplify the lattice construction, we have only a line interval in the extra dimension. It would be interesting to generalize the present construction to a  $U((N_c + n^4 N_f)N^2)$  mother theory and to make a deconstructed  $T^4$  appear in the theory. (The more exotic case of a K3 manifold in the extra four dimensions could also be attempted.)

From this perspective we see that it is necessary to keep the third dimension small so that we never see the effects of the KK states. That is, we want only the  $U(N_f)_{\text{diag}}^{N^2}$  states to be light enough to play a role at the scales that we study. In fact, this is exactly what happens in the D1/D5 system. Dimensional reduction of the D5 theory to the 2D intersection gives a volume suppression:

$$\frac{g_{\text{D5 reduc.}}^2}{g_{\text{D1}}^2} \approx \frac{\ell_s^4}{V_4}, \quad (3.6)$$

where  $V_4$  is the volume of the torus  $T^4$  and  $\ell_s$  is the string length. For  $V_4 \gg \ell_s^4$ , the 2D  $U(N_f)$  is weakly gauged, and the KK states are supermassive on the scale (recall that in 2D,  $[g_{\text{D1}}] = 1$  and that this is the scale of non-KK modes)  $g_{\text{D1}}$ .

In the discussion of Section 3.5 below, details associated with decoupling the KK states along the third dimension will be addressed.

### 3.2. Mother Theory

In  $\mathcal{N} = 1$  superfield notation, the mother theory is the 0d reduction of (3.1). It is straightforward to work out the 0d reduction of the component field action in the mother theory. I denote component fields (in Wess-Zumino gauge):

$$V = (v_\mu, \lambda, \bar{\lambda}, D), \quad \Phi = (\phi, \psi, G), \quad Q = (Q, \chi, F), \quad \tilde{Q} = (\tilde{Q}, \tilde{\chi}, \tilde{F}). \quad (3.7)$$

The result, after euclideanization, is:

$$S_{\text{gge}} = \frac{1}{2g^2} \text{Tr}[v_\mu, v_\nu][v_\mu, v_\nu] + \frac{2}{g^2} \text{Tr}[v_\mu, \phi][v_\mu, \phi^\dagger] + \frac{1}{g^2} \text{Tr}(D^2 + 2D[\phi, \phi^\dagger]) \\ + \frac{2}{g^2} \text{Tr}G^\dagger G + \frac{2}{g^2} \text{Tr}\bar{\Psi}[v_\mu \gamma_\mu, \Psi] + \frac{2\sqrt{2}i}{g^2} \text{Tr}(\lambda[\psi, \phi^\dagger] - [\phi, \bar{\psi}]\bar{\lambda}), \quad (3.8)$$

$$S_{\text{mat}} = -Q^\dagger v_\mu v_\mu Q + F^\dagger F + Q^\dagger DQ - \tilde{Q} v_\mu v_\mu \tilde{Q}^\dagger + \tilde{F}\tilde{F}^\dagger \\ - \tilde{Q}D\tilde{Q}^\dagger + \sqrt{2}(\tilde{F}\phi Q + \tilde{Q}GQ + \tilde{Q}\phi F + \text{h.c.}) \\ + \bar{\Lambda}v_\mu \gamma_\mu \Lambda - \sqrt{2}(\tilde{\chi}\phi\chi + \bar{\chi}\phi^\dagger\bar{\chi} + \tilde{\chi}\psi Q + \tilde{Q}\psi\chi + Q^\dagger\bar{\psi}\bar{\chi} + \bar{\chi}\bar{\psi}\tilde{Q}^\dagger) \\ + i\sqrt{2}(Q^\dagger\lambda\chi - \bar{\chi}\bar{\lambda}Q - \tilde{\chi}\lambda\tilde{Q}^\dagger + \tilde{Q}\bar{\lambda}\bar{\chi}). \quad (3.9)$$

Here, the following notations are used ( $\alpha = 1, 2$ ):

$$\Psi = \begin{pmatrix} \lambda_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}, \quad \bar{\Psi} = (\psi^\alpha, \bar{\lambda}_{\dot{\alpha}}), \quad \Lambda = \begin{pmatrix} \chi_\alpha \\ \bar{\tilde{\chi}}^{\dot{\alpha}} \end{pmatrix}, \quad \bar{\Lambda} = (\tilde{\chi}^\alpha, \bar{\tilde{\chi}}_{\dot{\alpha}}), \quad (3.10) \\ \gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix}, \quad \sigma_\mu = (\vec{\sigma}, i), \quad \bar{\sigma}_\mu = (-\vec{\sigma}, i), \quad \text{Tr}T^A T^B = \frac{1}{2}\delta^{AB}.$$

It is not difficult to relate (3.8) to the mother theory action of CKKU. The translation is

$$z_1 = \frac{1}{\sqrt{2}}(v_1 + iv_2), \quad z_2 = \frac{1}{\sqrt{2}}(v_3 + iv_4), \quad z_3 = i\phi^\dagger, \quad (3.11) \\ \Psi = (\xi_2, \xi_1, \xi_3, \lambda), \quad \bar{\Psi} = (\psi_1, -\psi_2, \chi, -\psi_3).$$

Note that  $\lambda, \chi$  here are not the two-component fermions  $\lambda_\alpha, \chi_\alpha$  of the 4d notation (3.10). The  $U(1)^4$  subgroup of  $SO(6) \times SU(2)_R$  that CKKU chooses for their orbifold procedure is then

$$q_1 = \Sigma_{1,2} = \frac{1}{4i}[\gamma_1, \gamma_2], \quad q_2 = \Sigma_{3,4} = \frac{1}{4i}[\gamma_3, \gamma_4], \quad (3.12) \\ q_3 = -\frac{1}{2}Q_R, \quad q_4 = -T_R^3.$$

I have expressed the last two charges in terms of the conventional  $SU(2)_R \times U(1)_R$   $R$ -symmetry of the 4d  $\mathcal{N} = 2$  theory. The  $U(1)_R$  generator  $Q_R$  is normalized such that gluinos (denoted  $\lambda_\alpha, \bar{\psi}^{\dot{\alpha}}$  in (3.8)–(3.10)) have  $Q_R = 1$ .  $T_R^3 = (1/2)\sigma^3$  is the diagonal generator of  $SU(2)_R$ . The charges of all gauge multiplet fields are summarized in Table 1. The charges of

**Table 1:**  $U(1)^4$  charges and  $Z_N \times Z_N$  orbifold action  $N$ -alities for the gauge multiplet, after modification of (3.24) to accommodate hypermultiplets. The second line connects to the CKKU notation for the fields  $z_3, \xi_2, \xi_1$ , and so forth.

	$z_1$	$z_2$	$i\phi^\dagger$	$\Psi_1$	$\Psi_2$	$\Psi_3$	$\Psi_4$	$\bar{\Psi}_1$	$\bar{\Psi}_2$	$\bar{\Psi}_3$	$\bar{\Psi}_4$	$G$	$D$
			$z_3$	$\xi_2$	$\xi_1$	$\xi_3$	$\lambda$	$\varphi_1$	$-\varphi_2$	$\chi$	$-\varphi_3$		
$q_1$	1	0	0	-1/2	+1/2	-1/2	+1/2	+1/2	-1/2	+1/2	-1/2	0	0
$q_2$	0	1	0	+1/2	-1/2	-1/2	+1/2	-1/2	+1/2	+1/2	-1/2	0	0
$q_3$	0	0	1	-1/2	-1/2	+1/2	+1/2	-1/2	-1/2	+1/2	+1/2	0	0
$q_4$	0	0	0	-1/2	-1/2	-1/2	-1/2	+1/2	+1/2	+1/2	+1/2	-1	0
$r_1$	1	0	-1	0	1	-1	0	1	0	0	-1	0	0
$r_2$	0	1	-1	1	0	-1	0	0	1	0	-1	0	0

**Table 2:**  $U(1)^4$  charges and  $Z_N \times Z_N$  orbifold action  $N$ -alities for the matter hypermultiplets.

	$Q$	$\tilde{Q}$	$\Lambda_1$	$\Lambda_2$	$\Lambda_3$	$\Lambda_4$	$\bar{\Lambda}_1$	$\bar{\Lambda}_2$	$\bar{\Lambda}_3$	$\bar{\Lambda}_4$	$F$	$\tilde{F}$
$q_1$	0	0	-1/2	+1/2	-1/2	+1/2	+1/2	-1/2	+1/2	-1/2	0	0
$q_2$	0	0	+1/2	-1/2	-1/2	+1/2	-1/2	+1/2	+1/2	-1/2	0	0
$q_3$	0	0	+1/2	+1/2	-1/2	-1/2	+1/2	+1/2	-1/2	-1/2	1	1
$q_4$	+1/2	+1/2	0	0	0	0	0	0	0	0	-1/2	-1/2
$r_1$	0	0	-1	0	0	1	0	-1	1	0	-1	-1
$r_2$	0	0	0	-1	0	1	-1	0	1	0	-1	-1

all hypermultiplet fields are summarized in Table 2. For the fermions, the notation is related to (3.8)–(3.10) by

$$(\Psi_1, \Psi_2, \Psi_3, \Psi_4) = (\lambda_1, \lambda_2, \bar{\varphi}^1 \bar{\varphi}^2), \quad (\bar{\Psi}_1, \bar{\Psi}_2, \bar{\Psi}_3, \bar{\Psi}_4) = (\varphi^1, \varphi^2, \bar{\lambda}_1 \bar{\lambda}_2) \quad (3.13)$$

and a similar translation for  $\Lambda, \bar{\Lambda}$ . The upper placement and lower placement of indices are significant because of the implicit spinor sums that are in (3.8)–(3.9). For example, in the last line of (3.9), one has the term

$$Q^\dagger \lambda_\chi = Q^\dagger \lambda^\alpha \chi_\alpha = Q^\dagger (\epsilon^{12} \lambda_2 \chi_1 + \epsilon^{21} \lambda_1 \chi_2) = Q^\dagger (\Psi_2 \Lambda_1 - \Psi_1 \Lambda_2). \quad (3.14)$$

Here, the conventions of [33] have been used:  $e_{21} = e^{12} = 1$ ,  $e_{12} = e^{21} = -1$ . These details were important in writing down the explicit daughter theory action that is given in Appendix B.

### 3.3. Orbifolding Details

#### 3.3.1. Projections, Generally

Denote  $U((N_c + nN_f)N^2)$  indices collectively by

$$S \equiv Im_1 m_2, \quad I \in \{1, \dots, N_c + nN_f\}, \quad m_1, m_2 \in \{0, \dots, N\}. \quad (3.15)$$

The domain of the index  $I$  should be thought of as follows:

$$I = 1, \dots, N_c; N_c + 1, \dots, N_c + N_f; (N_c + N_f) + 1, \dots, (N_c + N_f) + N_f; \dots; (N_c + (n-1)N_f) + 1, \dots, (N_c + (n-1)N_f) + N_f. \quad (3.16)$$

The interpretation is in terms of a block diagonal matrix, with an  $N_c \times N_c$  block, followed by  $n$  blocks of size  $N_f \times N_f$ . The index  $S$  then indicates, say, that the entries of the  $N_c \times N_c$  matrix are themselves  $N^2 \times N^2$  matrices, and so on. In what follows, “diag” will indicate a block diagonal matrix, with only block entries given explicitly. For example the unit matrix in the mother theory is given by

$$\mathbf{1}_{(N_c+nN_f)N^2} = \text{diag}\left(\mathbf{1}_{N_c N^2}, \mathbf{1}_{N_f N^2}, \dots, \mathbf{1}_{N_f N^2}\right), \quad (3.17)$$

with  $n$  entries of  $\mathbf{1}_{N_f N^2}$ . Other matrices of this form follow.

Introduce “clock operators” that involve roots of unity  $\omega_k \equiv \exp(2\pi i/k)$ :

$$\begin{aligned} P &= \text{diag}\left(\mathbf{1}_{N_c N^2}, \omega_{n+1} \mathbf{1}_{N_f N^2}, \dots, \omega_{n+1}^n \mathbf{1}_{N_f N^2}\right), \\ \Omega_N &= \text{diag}\left(1, \omega_N, \dots, \omega_N^{N-1}\right), \\ C_N^{1,k} &= \mathbf{1}_k \otimes \Omega_N \otimes \mathbf{1}_N, \quad C_N^{2,k} = \mathbf{1}_k \otimes \mathbf{1}_N \otimes \Omega_N, \\ C_N^1 &= \text{diag}\left(C_N^{1,N_c}, C_N^{1,N_f}, \dots, C_N^{1,N_f}\right), \\ C_N^2 &= \text{diag}\left(C_N^{2,N_c}, C_N^{2,N_f}, \dots, C_N^{2,N_f}\right). \end{aligned} \quad (3.18)$$

Orbifold projections for any field  $\mathcal{A}$  are defined by

$$\mathcal{A} \equiv \omega_{n+1}^s P \mathcal{A} P^\dagger, \quad \mathcal{A} \equiv \omega_N^{r_1} C_N^1 \mathcal{A} C_N^{1\dagger}, \quad \mathcal{A} \equiv \omega_N^{r_2} C_N^2 \mathcal{A} C_N^{2\dagger}. \quad (3.19)$$

The charges  $s, r_1, r_2$  will correspond to, respectively,  $(n+1)$ -ality,  $N$ -ality,  $N$ -ality. The origin of the  $Z_{n+1}$  symmetry in the mother theory—corresponding to the  $(n+1)$ -ality—will be discussed shortly. The two  $N$ -alities correspond to a  $Z_N \times Z_N$  subgroup of the  $SO(6) \times SU(2)_R$  symmetry of the mother theory.

To understand the effect of (3.19), it is best to look at it in stages. For the  $Z_{n+1}$  projection,

$$\begin{aligned} \mathcal{A}_{s=0} &\longrightarrow \left(\text{adj } U(N_c N^2), 1, \dots, 1\right) \oplus \left(1, \text{adj } U(N_f N^2), 1, \dots, 1\right) \\ &\quad \oplus \dots \oplus \left(1, 1, \dots, \text{adj } U(N_f N^2)\right), \\ \mathcal{A}_{s=1} &\longrightarrow \left(N_c N^2, \overline{N_f N^2}, 1, \dots, 1\right) \oplus \left(1, N_f N^2, \overline{N_f N^2}, 1, \dots, 1\right) \\ &\quad \oplus \dots \oplus \left(1, 1, \dots, 1, N_f N^2, \overline{N_f N^2}\right) \oplus \left(\overline{N_c N^2}, 1, \dots, 1, N_f N^2\right), \end{aligned} \quad (3.20)$$

and  $s = -1$  is conjugate to the latter. This yields “sites” and “links” of the quiver gauge theory (3.2). This structure will persist in the lattice theory and its continuum limit.

The minimal coupling superpotential of the mother theory has the  $U(1)$  global symmetry

$$Q \longrightarrow e^{i\alpha} Q, \quad \tilde{Q} \longrightarrow e^{-i\alpha} \tilde{Q}, \quad (3.21)$$

with all other fields neutral and all components in  $Q, \tilde{Q}$  transforming identically (it is not an R-symmetry). This is the symmetry that we use for the  $Z_{n+1}$  orbifold. That is, we assign  $s = 1$  to all components of  $Q$ ,  $s = -1$  to all components of  $\tilde{Q}$ , and  $s = 0$  to all components of  $V, \Phi$ . In this way,  $Q, \tilde{Q}$  will be bifundamentals (“links”) of  $U(N_c) \times U(N_f)$  quiver gauge theory, whereas  $V, \Phi$  will be adjoints (“sites”). The notion of “links” and “sites” used here is distinct from that associated with the 2D lattice that is described next. Note that because (3.21) is not an R-symmetry (i.e., it commutes with the supercharges of the mother theory), the  $Z_{n+1}$  orbifold projection leaves all eight supercharges intact.

The  $Z_N \times Z_N$  projections involve clock matrices  $C_N^{1,2}$ , which only act on the “site” indices  $m_1, m_2$  of (3.15). They have the usual effect of the deconstruction lattice formulation. Label any of the fields in the decomposition (3.20) by  $\mathcal{A}_{m_1, m_2, n_1, n_2}$ , ignoring indices of the surviving gauge group. Then the surviving components of  $\mathcal{A}_{m_1, m_2, n_1, n_2}$  after the  $Z_N \times Z_N$  orbifold are those that satisfy

$$m_1 - n_1 + r_1 = 0 \pmod{N}, \quad m_2 - n_2 + r_2 = 0 \pmod{N}. \quad (3.22)$$

This yields site, horizontal link, vertical link, and diagonal link interpretations, depending on  $r_1, r_2$ . The fields are then labeled by site indices  $m \equiv (m_1, m_2)$ . Next I discuss particulars with respect to the various fields of the mother theory.

### 3.3.2. Daughter Theory Gauge Action

A problem arises for the construction of CKKU [10] when we include hypermultiplets. The scalar components are neutral with respect to the  $SO(6)$  global symmetry of the mother theory, which decomposes to  $SO(4) \times U(1)_R$  in the 4d theory. In the notation of CKKU,  $q_1 = q_2 = q_3 = 0$ . On the other hand these scalars transform as doublets  $(\tilde{Q}^\dagger, Q)$  under  $SU(2)_R$ , and as a consequence  $q_4 = 1/2$  for  $Q, \tilde{Q}$ . The  $N$ -alities defined by CKKU are

$$r_1 = q_1 + q_4, \quad r_2 = q_2 + q_4, \quad (3.23)$$

which would lead to half-integral  $r_1, r_2$  for the scalars  $Q, \tilde{Q}$ . Nonintegral  $N$ -alities do not make sense in the lattice interpretation of the orbifolded theory. Therefore we must modify the  $N$ -ality assignments of CKKU in order to include hypermultiplets. It will be seen that this is easily accomplished. The lattice that is obtained is quite similar to the one of CKKU. It is of particular importance that two supercharges are preserved exactly.

In the modification, we want to leave the  $N$ -alities of link bosons  $z_1, z_2$  unchanged, since these must ultimately get a vacuum expectation value that links neighboring sites (Actually, it is an interesting question whether or not a dynamical lattice spacing can be

associated with, say, diagonal link bosons. I will not pursue this here.) We must choose  $r_1, r_2$  such that all fields have integer  $N$ -ality. Also we would like to preserve two supercharges, as in the pure gauge construction of CKKU. According to the CKKU rubric, we therefore must choose  $r_1, r_2$  such that two components of the fermions are neutral. Here we choose to keep  $\Psi_4 = \lambda$  neutral, as in the CKKU construction (other choices are of course possible, but lead to similar results, due to the symmetries of the mother theory). Then it is easy to show (cf. Appendix A) that the unique choice that satisfies all of our requirements is

$$r_1 = q_1 - q_3, \quad r_2 = q_2 - q_3. \quad (3.24)$$

In addition to  $\Psi_4$ , the fermion component  $\bar{\Psi}_3 = \chi$  is  $r_1, r_2$  neutral. The charges for the vector multiplet are summarized in Table 1.

Relative to the formulation of CKKU, only the following five fields of the gauge multiplet change their nature:

$$\begin{aligned} z_3: \text{site} &\longrightarrow (-) \text{ diagonal link,} \\ \xi_2: (-\hat{e}_1) \text{ link} &\longrightarrow \hat{e}_2 \text{ link,} \\ \xi_1: (-\hat{e}_2) \text{ link} &\longrightarrow \hat{e}_1 \text{ link,} \\ \chi: \text{diagonal link} &\longrightarrow \text{site,} \\ \psi_3: \text{site} &\longrightarrow (-) \text{ diagonal link.} \end{aligned} \quad (3.25)$$

This merely leads to modest changes in the site labels for the daughter theory action of CKKU, their (1.2) and (1.4). These changes are all obvious from the  $r_1, r_2$  assignments of Table 1. For instance, in their bosonic action one replaces (CKKU uses the notation  $\bar{z}$  while  $z^\dagger$  is used here)

$$\left[ z_{3,m'}^\dagger, z_{3,m} \right] \longrightarrow z_{3,m}^\dagger z_{3,m+\hat{e}_1+\hat{e}_2} - z_{3,m} z_{3,m-\hat{e}_1-\hat{e}_2}^\dagger, \quad (3.26)$$

to take into account that  $z_3, z_3^\dagger$  are now  $-/+$  diagonal link fields. The fermion action is still of the form

$$S_{F,g} = \frac{2\sqrt{2}}{g^2} \sum_m \text{Tr} \left\{ \Delta_m(\lambda, z_a^\dagger, \psi_a) - \Delta_m(\chi, z_a^\dagger, \xi_a) + \epsilon_{abc} \Delta_m(\psi_a, z_b, \xi_c) \right\}, \quad (3.27)$$

with  $\Delta(A, B, C) = ABC - ACB$  and site labels assigned according to the nature of the fields that appear.

Referring to Table 1, we note that there is a twofold degeneracy for the  $r_1, r_2$  charges among the fermions. The reason for this is that the orbifold charges (3.24) do not involve the  $SU(2)_R$  diagonal generator  $q_4$ . Thus the  $SU(2)_R$  symmetry of the mother theory is preserved, unlike that which occurs in the CKKU construction. Since the (4,4) gluinos fall into doublets of  $SU(2)_{R'}$ , we are guaranteed to have the twofold degeneracy with respect to  $r_1, r_2$ .

Note also that the  $r_1, r_2$  neutral fermions are those that have  $q_1 = q_2 = q_3$ . It follows that the two supercharges that are preserved in the daughter theory are those that have  $q_1 = q_2 = q_3$ .

### 3.3.3. Daughter Theory Matter Action

Having explained how the gauge action is modified, we next turn to the matter action. The daughter theory is obtained in a simple application of the orbifold procedure to the mother theory (3.9), as determined by the  $r_1, r_2$  assignments that appear in Table 2. Due to the CKKU calculus, we are assured to obtain the correct classical continuum limit, just as in the EK examples.

We have already seen from the discussion of the daughter theory gauge action that there are two supercharges that are neutral with respect to the  $Z_N \times Z_N$  charges  $r_1, r_2$ . This symmetry of the matter mother theory action will be an exact supersymmetry of the matter daughter theory as well. Upon inspection of Table 2, one sees that the  $r_1, r_2$  neutral fermions are once again those that have  $q_1 = q_2 = q_3$ . It follows that the two supercharges that are preserved in the daughter theory matter action are those that have  $q_1 = q_2 = q_3$ . It is no accident that this is identical to what occurs in the daughter theory gauge action: the supercharges are inherited from the mother theory. This illustrates the usefulness of the orbifold technique of CKKU.

Straightforward manipulations yield the daughter theory matter action. One merely writes out the fermion components in (3.9) explicitly, reexpresses  $v_\mu$  in terms of  $z_i, z_i^\dagger$ , and adds site labels as determined by the  $r_1, r_2$  charges given in Table 2. Because the result is somewhat lengthy, it has been relegated to Appendix B.

## 3.4. Higgsing Details

To ‘‘Higgs’’ the theory, such that only the  $U(N_f)_{\text{diag}}$  subgroup of  $U(N_f)^n$  survives at the scale  $g_c = ga$  of the  $U(N_c)$  gauge theory, we only require the application of the deconstruction idea to the  $U(N_f)^n$  quiver. This 1d quiver is similar to that considered in [34], in that it is an extra dimensional interval (in this case a third dimension),  $U(N_f)_1 \times \cdots \times U(N_f)_n$ , and it is not necessary to rework all the details.

In terms of  $\mathcal{N} = 1$  superfields, the quiver is described by the 0d reduction of the theory with

$$K_{\text{mat}} = \sum_{i=1}^{n-1} \text{Tr} \left\{ Q_i^\dagger e^{V_i} Q_i e^{-V_{i+1}} + \tilde{Q}_i^\dagger e^{V_{i+1}} \tilde{Q}_i e^{-V_i} \right\},$$

$$W_{\text{mat}} = \sqrt{2} \sum_{i=1}^{n-1} \text{Tr} \left\{ \tilde{Q}_i \Phi_i Q_i - Q_i \Phi_{i+1} \tilde{Q}_i \right\}.$$
(3.28)

Formally, this is quite similar to the quiver theory studied in [36]. I do not write  $K_{\text{gge}}, W_{\text{gge}}$  since it is just an  $n$ -fold replication of terms of the form (3.1). Holomorphic gauge invariance is given by

$$Q_i \longrightarrow e^{\Lambda_i} Q_i e^{-\Lambda_{i+1}}, \quad \tilde{Q}_i \longrightarrow e^{\Lambda_{i+1}} \tilde{Q}_i e^{-\Lambda_i},$$

$$\Phi_i \longrightarrow e^{\Lambda_i} \Phi_i e^{-\Lambda_i}, \quad e^{V_i} \longrightarrow e^{-\Lambda_i^\dagger} e^{V_i} e^{-\Lambda_i}.$$
(3.29)

One then gives an expectation value to the  $(N_f^{(i)}, \overline{N}_f^{(i+1)})$ ,  $i = 1, \dots, n-1$ , bifundamentals and their conjugates:

$$\langle Q_i \rangle = \langle \tilde{Q}_i \rangle = \frac{1}{\sqrt{2}a_3}. \quad (3.30)$$

Then, for instance, the quadratic terms in the 2D Lagrangian for the gauge bosons are (here I am hiding all the details of the 2D lattice theory and just emphasizing the quiver in the third dimension. The modes of the lattice theory that are getting mass here are just the  $z_{1,m}, z_{2,m}$  that transform as adjoints of the  $U(N_f)^n$  group, excepting the combination corresponding to  $U(N_f)_{\text{diag}}$ )

$$\sum_{i=1}^{n-1} \frac{g^2}{4a_3^2} (A_{i+1}^{\mu,m} - A_i^{\mu,m})^2, \quad (3.31)$$

where a contraction over the 4d index  $\mu$  and  $U(N_f)$  index  $m = 1, \dots, N_f^2$  is implied. The scaling  $A \rightarrow \hat{g}A$  has been performed to make the kinetic terms for gauge bosons canonical. Here,  $\hat{g} = ga^2$  is the dimensionless coupling, that is, the coupling of the matrix model expressed in units of the 2D lattice. It follows immediately from the considerations of [34] that only  $U(N_f)_{\text{diag}}$  has a massless gauge boson. All other modes are quanta with configuration energies of order  $1/(na_3)$ , corresponding to discrete momenta in the third dimension. To be precise, the spectrum is

$$M_n^2 = \frac{\hat{g}^2}{a_3^2} \sin^2 \frac{j\pi}{n}, \quad j = 0, \dots, n-1. \quad (3.32)$$

The radius  $R$  of this third deconstructed dimension and the KK mass scale  $M$  are therefore

$$R \approx \frac{na_3}{\hat{g}}, \quad M = \frac{\pi}{R}. \quad (3.33)$$

The effective gauge coupling of the  $U(N_f)_{\text{diag}}$  theory is given by (3.5).

### 3.5. Decoupling KK States

The condition that the KK states decouple from the  $U(N_c) \times U(N_f)_{\text{diag}}$  gauge theory is just  $M \gg g_c = ga$ . Various realizations of this could be imagined. A strong one is that we set the KK scale at the UV cutoff of the  $U(N_c) \times U(N_f)_{\text{diag}}$  gauge theory:  $R \equiv a$ . This translates into

$$n = \frac{a\hat{g}}{a_3} = g_c \frac{a^2}{a_3}. \quad (3.34)$$

Thus as we take the continuum limit  $a \rightarrow 0$  in the 2D  $U(N_c) \times U(N_f)_{\text{diag}}$  gauge theory, with  $n, g_c$  held fixed, we have the scaling  $a_3 \sim a^2$ . This would decouple the effects of the  $U(N_f)^n$

quiver at the UV scale of the  $U(N_c) \times U(N_f)_{\text{diag}}$  gauge theory, and just represents a slightly different UV completion that should not have physical consequences—based on universality arguments.

A less aggressive prescription is to take  $g_c R$  fixed and small. This should also decouple the KK states before important  $U(N_c) \times U(N_f)_{\text{diag}}$  physics sets in. This translates into

$$\frac{a_3}{a} = \frac{f}{n}, \quad f \ll 1. \quad (3.35)$$

Holding the factors  $f, n$  fixed, we see that a scaling  $a_3 \sim a$  is prescribed as the continuum limit is taken.

## 4. Application

Here I mention one possible application of the lattice theory. Recently, Chen and Tong have studied the D1/D5 effective worldsheet instanton partition function on the Higgs branch. In the gauge theory one looks at the distribution of instanton size  $\rho$  and orientational modes  $\widehat{\Omega}$ , where the latter are points on  $S^3$ . Indeed, it is found that the distribution has the  $\text{AdS}_3 \times S^3$  geometry in the sector with first Chern class  $k = 1$ , that is, a unit of winding in the  $U(1)_{\text{diag}}$  of the color group.

In a numerical study of this phenomenon, one would build up a histogram in the  $k = 1$  topological sector. Twisted boundary conditions could be imposed to force nontrivial topology for the gauge fields. The histogram would count configurations with a given instanton size  $\rho$  and orientation  $\widehat{\Omega}$ . If the weight is identical to the  $\text{AdS}_3 \times S^3$  density, it would provide evidence of the correspondence. In particular, it is interesting to explore the correspondence for intermediate values of  $N_c$ , given the current fashion for applying AdS/QCD ideas to real-world QCD, where  $N_c = 3$ .

It would also be interesting to explore the correspondence at finite temperature, since continuum methods start to break down if the temperature is too far from zero. The recent results of Rey and Hikida for small 't Hooft coupling and finite temperature [32] provide continuum results that could be compared to. Finally, one would like to study correlation functions that are not BPS saturated. Again, continuum methods are generally unreliable in that case.

## 5. Conclusions

In this paper I have generalized the EK construction to 2D (4,4) gauge theories. I have specialized to a  $U(N_c) \times U(N_f)^n$  quiver theory. Next, I showed how to treat the  $U(N_f)^n$  quiver as a deconstructed third dimension and how to obtain a weakly coupled 2D remnant  $U(N_f)_{\text{diag}}$ , mimicking what really happens in the D1/D5 brane system. I described a simple test of  $\text{AdS}_3/\text{CFT}_2$  that could be conducted numerically. It is worth noting that it should be straightforward to include the Fayet-Iliopoulos (FI) terms in the mother theory, and thus in the lattice theory, indeed, this has already been illustrated by EK in their “Example 2.”

Work in progress includes a careful study of renormalization in the lattice theory, the number of counterterms that need to be fine-tuned, their exact calculation in perturbation theory (the lattice theory is super-renormalizable since the coupling has positive mass

dimension), and a numerical study of the correspondence. Renormalization of the theory, such as has been studied in [37], is certainly a pressing question in the presence of matter. It remains to be seen the extent to which complex phase problems of the pure gauge lattice theory [38, 39] persist once matter is introduced. If FI terms are introduced and the theory is studied on the Higgs branch, the complex phase may be less of a problem.

Finally, it is of some interest to work out a superfield description of the daughter theory in this model. This would be useful in a super-Feynman diagram perturbative analysis, as well as for understanding the renormalizations to the tree-level action that are possible.

## Appendices

### A. Uniqueness of $r_1, r_2$ with Conditions Imposed

The conditions that we will impose are the following.

- (i) The link bosons  $z_1, z_2$  should have  $(r_1, r_2) = (1, 0)$  and  $(0, 1)$ , respectively.
- (ii) The fermion component  $\Psi_4$  should have  $(r_1, r_2) = (0, 0)$ .
- (iii) At least one other fermion component in  $\Psi, \bar{\Psi}$  should have  $(r_1, r_2) = (0, 0)$ .
- (iv) All fields should have integral values of  $r_1, r_2$ .

It is completely general to write

$$r_1 = \sum_{i=1}^4 c_{1i} q_i, \quad r_2 = \sum_{i=1}^4 c_{2i} q_i. \quad (\text{A.1})$$

Condition (i) yields immediately  $c_{11} = c_{22} = 1, c_{21} = c_{12} = 0$ . Condition (ii) gives  $c_{14} = c_{13} + 1, c_{24} = c_{23} + 1$ . Thus the charges reduce to

$$r_1 = (q_1 + q_4) + c_{13}(q_3 + q_4), \quad r_2 = (q_2 + q_4) + c_{23}(q_3 + q_4). \quad (\text{A.2})$$

It is easy to see from Table 2 that the components of the matter fermions  $\Lambda, \bar{\Lambda}$  have  $(q_1 + q_4) = \pm 1/2, (q_2 + q_4) = \pm 1/2, (q_3 + q_4) = \pm 1/2$ . It follows that we must take  $c_{13}$  and  $c_{23}$  to be odd integers, in order to satisfy condition (iv). The remaining condition (iii) then has a unique solution, as can be checked from Table 1. It is  $c_{13} = c_{23} = -1$  which gives (3.24).

### B. Daughter Theory Matter Action

The action can be expressed as three terms,

$$S_{\text{mat}} = S_1 + S_2 + S_3, \quad (\text{B.1})$$

where

$$\begin{aligned}
S_1 = & -Q_m^\dagger \left( z_{i,m} z_{i,m+\hat{e}_i}^\dagger + z_{i,m}^\dagger z_{i,m-\hat{e}_i} \right) Q_m \\
& - \tilde{Q}_m \left( z_{i,m} z_{i,m+\hat{e}_i}^\dagger + z_{i,m}^\dagger z_{i,m-\hat{e}_i} \right) \tilde{Q}_m^\dagger \\
& + F_m^\dagger F_{m+\hat{e}_1+\hat{e}_2} + \tilde{F}_m \tilde{F}_{m-\hat{e}_1-\hat{e}_2}^\dagger + Q_m^\dagger D_m Q_m - \tilde{Q}_m D_m \tilde{Q}_m^\dagger \\
& + \sqrt{2} \left( \tilde{F}_m \phi_{m-\hat{e}_1-\hat{e}_2} Q_m + \tilde{Q}_m G_m Q_m + \tilde{Q}_m \phi_m F_{m+\hat{e}_1+\hat{e}_2} \right. \\
& \quad \left. + Q_m^\dagger \phi_m^\dagger \tilde{F}_{m-\hat{e}_1-\hat{e}_2}^\dagger + Q_m^\dagger G_m^\dagger \tilde{Q}_m^\dagger + \tilde{F}_m^\dagger \phi_{m+\hat{e}_1+\hat{e}_2}^\dagger \tilde{Q}_m^\dagger \right), \\
S_2 = & \sqrt{2} \left[ \bar{\Lambda}_{1,m} \left( z_{1,m-\hat{e}_2}^\dagger \Lambda_{4,m-\hat{e}_1-\hat{e}_2} + z_{2,m-\hat{e}_2} \Lambda_{3,m} \right) \right. \\
& + \bar{\Lambda}_{2,m} \left( z_{1,m-\hat{e}_1} \Lambda_{3,m} - z_{2,m-\hat{e}_1}^\dagger \Lambda_{4,m-\hat{e}_1-\hat{e}_2} \right) \\
& - \bar{\Lambda}_{3,m} \left( z_{1,m+\hat{e}_1+\hat{e}_2}^\dagger \Lambda_{2,m+\hat{e}_2} - z_{2,m+\hat{e}_1+\hat{e}_2}^\dagger \Lambda_{1,m+\hat{e}_1} \right) \\
& \left. - \bar{\Lambda}_{4,m} \left( z_{1,m} \Lambda_{1,m+\hat{e}_1} + z_{2,m} \Lambda_{2,m+\hat{e}_2} \right) \right], \\
S_3 = & -\sqrt{2} \left[ \bar{\Lambda}_{1,m} \phi_{m-\hat{e}_2} \Lambda_{1,m+\hat{e}_1} + \bar{\Lambda}_{2,m} \phi_{m-\hat{e}_1} \Lambda_{2,m+\hat{e}_2} \right. \\
& + \bar{\Lambda}_{3,m} \phi_{m+\hat{e}_1+\hat{e}_2}^\dagger \Lambda_{3,m} + \bar{\Lambda}_{4,m} \phi_m^\dagger \Lambda_{4,m-\hat{e}_1-\hat{e}_2} - \left( \bar{\Lambda}_{1,m} \bar{\Psi}_{2,m-\hat{e}_2} - \bar{\Lambda}_{2,m} \bar{\Psi}_{1,m-\hat{e}_1} \right) Q_m \\
& + \tilde{Q}_m \left( \bar{\Psi}_{1,m} \Lambda_{1,m+\hat{e}_1} + \bar{\Psi}_{2,m} \Lambda_{2,m+\hat{e}_2} \right) - Q_m^\dagger \left( \Psi_{4,m} \Lambda_{3,m} - \Psi_{3,m} \Lambda_{4,m-\hat{e}_1-\hat{e}_2} \right) \\
& \left. + \left( \bar{\Lambda}_{3,m} \Psi_{3,m+\hat{e}_1+\hat{e}_2} + \bar{\Lambda}_{4,m} \Psi_{4,m} \right) \tilde{Q}_m^\dagger \right] \\
& + i\sqrt{2} \left[ Q_m^\dagger \left( \Psi_{2,m} \Lambda_{1,m+\hat{e}_1} - \Psi_{1,m} \Lambda_{2,m+\hat{e}_2} \right) - \left( \bar{\Lambda}_{3,m} \bar{\Psi}_{4,m+\hat{e}_1+\hat{e}_2} - \bar{\Lambda}_{4,m} \bar{\Psi}_{3,m} \right) Q_m \right. \\
& \left. - \left( \bar{\Lambda}_{1,m} \Psi_{1,m-\hat{e}_2} + \bar{\Lambda}_{2,m} \Psi_{2,m-\hat{e}_1} \right) \tilde{Q}_m^\dagger + \tilde{Q}_m \left( \bar{\Psi}_{3,m} \Lambda_{3,m} + \bar{\Psi}_{4,m} \Lambda_{4,m-\hat{e}_1-\hat{e}_2} \right) \right].
\end{aligned} \tag{B.2}$$

Here, the site indices  $m$  are implicitly summed.

## Acknowledgment

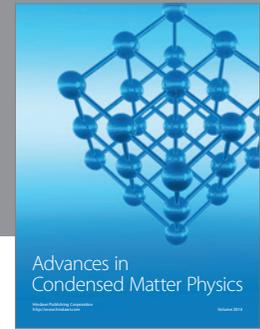
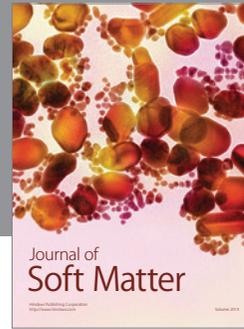
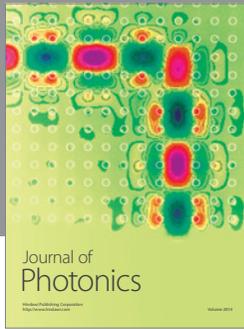
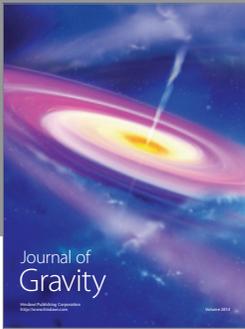
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