

Review Article

Combinatorics in $\mathcal{N} = 1$ Heterotic Vacua

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We briefly review an algorithmic strategy to explore the landscape of heterotic $E_8 \times E_8$ vacua, in the context of compactifying smooth Calabi-Yau threefolds with vector bundles. The Calabi-Yau threefolds are algebraically realised as hypersurfaces in toric varieties, and a large class of vector bundles are constructed thereon as monads. In the spirit of searching for standard-like heterotic vacua, emphasis is placed on the integer combinatorics of the model-building programme.

1. Introduction

Compactifications of $E_8 \times E_8$ heterotic theory [1, 2] and heterotic M-theory [3–7] on smooth Calabi-Yau threefolds provide a simple and compelling way to reach $\mathcal{N} = 1$ supersymmetry at four dimensions. A Calabi-Yau threefold necessarily admits a Ricci-flat metric $g_{\alpha\bar{\beta}}$, where $\alpha, \bar{\beta} = 1, 2, 3$ are, respectively, the holomorphic and antiholomorphic indices. One also turns on an internal gauge field, in a subalgebra \mathcal{G} of the full E_8 , resulting in the reduction of the four-dimensional gauge group down to the commutant \mathcal{H} of $\mathcal{G} \subset E_8$. To preserve supersymmetry, the gauge field should satisfy the Hermitian Yang-Mills equations:

$$F_{\alpha\beta} = F_{\bar{\alpha}\bar{\beta}} = 0, \quad g^{\alpha\bar{\beta}} F_{\alpha\bar{\beta}} = 0, \quad (1.1)$$

where F is the associated field strength. Although these equations cannot be solved analytically, the Donaldson-Uhlenbeck-Yau theorem [8, 9] states that, on a holomorphic (poly) stable bundle, there exists a unique connection that solves (1.1).

So each of the heterotic vacua comes in two pieces: a Calabi-Yau threefold X and a holomorphic stable vector bundle V thereon. Studying the detailed geometry, however, is not an easy task. To begin with, we do not even know the Ricci-flat metrics on Calabi-Yau threefolds. Fortunately, as will be seen shortly, it turns out that the topology of a vacuum already determines many interesting features of the four-dimensional effective theory.

Table 1: A vector bundle V with structure group $\mathcal{G} = SU(4)$ or $SU(5)$, respectively, breaks the E_8 group of the heterotic string into the Grand Unifying group $\mathcal{H} = SO(10)$ or $SU(5)$. The four-dimensional low-energy representation contents of \mathcal{H} arise from the branching of the **248** adjoint of E_8 under $\mathcal{G} \times \mathcal{H} \subset E_8$. The particle spectrum is obtained as various bundle-valued cohomology groups.

\mathcal{G}	\mathcal{H}	Branching of 248 under $\mathcal{G} \times \mathcal{H} \subset E_8$	Particle spectrum
$SU(4)$	$SO(10)$	$(\mathbf{1}, \mathbf{45}) \oplus (\mathbf{4}, \mathbf{16}) \oplus (\bar{\mathbf{4}}, \bar{\mathbf{16}}) \oplus (\mathbf{6}, \mathbf{10}) \oplus (\mathbf{15}, \mathbf{1})$	$n_{16} = h^1(X, V)$
			$n_{\bar{16}} = h^1(X, V^*) = h^2(V)$
			$n_{10} = h^1(X, \wedge^2 V)$
			$n_1 = h^1(X, V \otimes V^*)$
$SU(5)$	$SU(5)$	$(\mathbf{1}, \mathbf{24}) \oplus (\mathbf{5}, \mathbf{10}) \oplus (\bar{\mathbf{5}}, \bar{\mathbf{10}}) \oplus (\mathbf{10}, \bar{\mathbf{5}}) \oplus (\bar{\mathbf{10}}, \mathbf{5}) \oplus (\mathbf{24}, \mathbf{1})$	$n_{10} = h^1(X, V)$
			$n_{\bar{10}} = h^1(X, V^*) = h^2(V)$
			$n_5 = h^1(X, \wedge^2 V^*)$
			$n_{\bar{5}} = h^1(X, \wedge^2 V)$
			$n_1 = h^1(X, V \otimes V^*)$

In order for the heterotic models to be “Standard-like,” they must give rise to the correct gauge group, $SU(3)_C \times SU(2)_L \times U(1)_Y$, possibly with an extra $U(1)_{B-L}$ factor, as well as a correct spectrum for light particles coming in three generations. Firstly, the choices $\mathcal{G} = SU(4)$ and $SU(5)$ for the structure group of V reduce the E_8 to the four-dimensional gauge groups $\mathcal{H} = SO(10)$ and $SU(5)$, respectively, which are desirable in the viewpoint of Grand Unification¹. The light particles then arise from the branching of the adjoint **248** of E_8 into $\mathcal{G} \times \mathcal{H}$, and the spectrum is determined by various bundle-valued cohomologies on the Calabi-Yau threefold [2], as summarised in Table 1. Of course, the gauge group \mathcal{H} should be further broken down to a standard-like one and discrete Wilson-lines are made use of, if there ever exists any, for this second breaking.

In this paper, we will make it clear how the construction of standard-like heterotic vacua turns into the integer combinatorics for a discrete system. Specifically, the Calabi-Yau threefolds will be *torically* constructed and described by the combinatorics of *reflexive* lattice polytopes [10].² Next, *monad* vector bundles [11] will be constructed thereon, equivalent of turning on internal gauge fluxes over the Calabi-Yau threefolds.

The remainder of this paper is structured as follows. In the ensuing two sections, we lay down the foundation by explaining the basic mathematical toolkit for describing $\mathcal{N} = 1$ heterotic vacua. Next, in Section 4, further constraints will be imposed on the internal geometry so that the resulting $\mathcal{N} = 1$ four-dimensional effective theory may mimic the standard model. We will conclude in Section 5 with a summary and outlook.

2. Toric Construction of Calabi-Yau Threefolds

Soon after the famous 7890 Calabi-Yau threefolds were realised as complete intersections of hypersurfaces in multiprojective spaces [12–16], Kreuzer and Skarke have classified the Calabi-Yau threefolds that arise as codimension-one hypersurfaces in toric fourfolds, comprising a much bigger dataset [17–19]. This construction, first proposed by Batyrev [10], involves an extensive usage of toric geometry. Here, we do not intend by any means to give a pedagogical introduction to toric geometry. The readers interested in the details of this subject are referred either to the maths texts [20–23] or to the excellent, introductory reviews for physicists [24, 25].

2.1. Ambient Toric Fourfolds

The Calabi-Yau threefolds X are embedded in toric fourfolds \mathcal{A} as hypersurfaces and, therefore, we will start with the description of these ambient toric varieties. A toric fourfold is described by the combinatorial data called a *fan* in \mathbb{R}^4 , which is a collection of convex *cones* in \mathbb{R}^4 with their common apex at the origin $O = (0, 0, 0, 0)$. For the sake of Calabi-Yau subvarieties, however, every fan is not appropriate. We first define a certain class of convex *polytopes* in \mathbb{R}^4 , of which fans of a special kind are made.

The polytopes considered here must contain the origin O as the unique interior lattice point and all the vertices must lie in the lattice $\mathbb{Z}^4 \subset \mathbb{R}^4$. Such polytopes are called *reflexive*. It can be shown that, for a given reflexive polytope Δ in \mathbb{R}^4 , the *dual* polytope Δ° defined by

$$\Delta^\circ = \left\{ \mathbf{v} \in \mathbb{R}^4 \mid \langle \mathbf{m}, \mathbf{v} \rangle \geq -1 \quad \forall \mathbf{m} \in \Delta \right\} \quad (2.1)$$

also has all its vertices on the lattice \mathbb{Z}^4 , like the original polytope Δ does. To this dual polytope Δ° , we can associate a collection of the convex cones over all its faces, forming the fan for our toric fourfold \mathcal{A} .

Now, as for the construction of toric fourfold from a given fan in \mathbb{R}^4 , several equivalent methods are known. What best suits our purpose amongst them is Cox's homogeneous-coordinate approach [26], where a complex homogeneous coordinate x_ρ is associated to each one-dimensional cone ρ in the fan. Thus, if the fan has k edges, then there are k homogeneous coordinates (x_1, \dots, x_k) for \mathbb{C}^k . The next task is to identify a certain measure zero subset Z of \mathbb{C}^k which should be removed. Let S be a set of edges that do not span any cone in the fan and let $Z(S) \subset \mathbb{C}^k$ be the linear subspace defined by setting $x_\rho = 0$, for all $\rho \in S$. Now, let $Z \subset \mathbb{C}^k$ be the union of the subspaces $Z(S)$ for all such S . Then, the toric fourfold is constructed as a quotient of $\mathbb{C}^k - Z$ by the following $(\mathbb{C}^*)^{k-4}$ action:

$$(x_1, \dots, x_k) \sim (\lambda_r^{\beta_r^1} x_1, \dots, \lambda_r^{\beta_r^k} x_k), \quad \lambda_r \in \mathbb{C}^* \text{ for } r = 1, \dots, k-4, \quad (2.2)$$

where the coefficients β_r^ρ are defined by the linear relations $\sum_{\rho=1}^k \beta_r^\rho \mathbf{v}_\rho = 0$ amongst the edges. Hence, β_r^ρ form a $(k-4) \times k$ matrix which is often referred to as a *charge matrix* [27]. The identification rule in (2.2) can be schematically written as

$$\mathcal{A} = \frac{(\mathbb{C}^k - Z)}{(\mathbb{C}^*)^{k-4}}. \quad (2.3)$$

Note that the construction of toric fourfolds in (2.3) naturally generalises that of projective space \mathbb{P}^4 , the simplest toric fourfold, in which case $Z = \{O\}$ and $k = 5$; that is,

$$\mathbb{P}^4 = \frac{(\mathbb{C}^5 - \{O\})}{\mathbb{C}^*}. \quad (2.4)$$

2.2. Calabi-Yau Threefolds

A Calabi-Yau hypersurface X to the toric fourfold \mathcal{A} is constructed in a straightforward manner without requiring any further data: as long as the polytope Δ is reflexive, it also defines $X \subset \mathcal{A}$. Note that, in this case, Δ° is also a reflexive polytope since $(\Delta^\circ)^\circ = \Delta$. To a reflexive polytope Δ in \mathbb{R}^4 , we can associate a family of Calabi-Yau threefolds X defined as

the vanishing loci of the polynomials of the form

$$P_{\{C_m\}}(x_1, \dots, x_k) = \sum_{\mathbf{m} \in \Delta} C_{\mathbf{m}} \prod_{\rho=1}^k x_{\rho}^{\langle \mathbf{m}, \mathbf{v}_{\rho} \rangle + 1}, \quad (2.5)$$

where $x_{\rho=1, \dots, k}$ are the k homogeneous coordinates of \mathcal{A} associated to the lattice vertices $\mathbf{v}_{\rho=1, \dots, k}$ of Δ° and $C_{\mathbf{m}}$ are numerical coefficients parameterising the complex structure of X .

Heterotic compactifications ask for compact Calabi-Yau threefolds that are smooth. However, a toric fourfold \mathcal{A} constructed by (2.3) usually bears singularities and they in general descend to the hypersurfaces X too. In order to make X nonsingular, we partially desingularise \mathcal{A} so that the hypersurfaces may avoid the singularities of the ambient space [10]. This process corresponds to triangulating the (dual) polytope in a special way and is called an *MPCP-triangulation*.³

As for the statistics, a total of 473,800,776 reflexive polytopes in \mathbb{R}^4 have been classified [17–19], each of which gives rise to a toric fourfold \mathcal{A} as well as a family of Calabi-Yau threefolds X . It turns out that only 124 out of them describe smooth manifolds, for which no MPCP-triangulations are required.

3. Monad Construction of Vector Bundles

In the physics literature, especially in the context of heterotic string phenomenology, construction of vector bundles has been attempted in several ways. They include *spectral cover construction* [28–34], *bundle extension* [35–37], and the mixture thereof [38]. In many of them, it was essential for the base threefolds to have a torus-fibration structure. On the other hand, *monad construction* [11] does not assume any extra structure and has proved particularly useful for algorithmically scanning a vast number of vector bundles [39–43].

A *monad* vector bundle is essentially the quotient of two Whitney sums of line bundles. More precisely, a monad bundle V over a Calabi-Yau threefold X is defined by the short exact sequence of the form

$$0 \longrightarrow V \longrightarrow \bigoplus_{i=1}^{r_b} \mathcal{O}_X(\mathbf{b}_i) \xrightarrow{f} \bigoplus_{j=1}^{r_c} \mathcal{O}_X(\mathbf{c}_j) \longrightarrow 0, \quad (3.1)$$

where \mathbf{b}_i and \mathbf{c}_j are integer vectors of length $h^{1,1}(X)$, representing the first Chern classes of the summand line bundles $\mathcal{O}_X(\mathbf{b}_i)$ and $\mathcal{O}_X(\mathbf{c}_j)$. The bundle V is a holomorphic $U(n)$ -bundle, where

$$n = r_b - r_c \quad (3.2)$$

is the rank of V .

From (3.1), one can readily read off the Chern class of V :

$$\begin{aligned} c_1(V) &= \left(\sum_{i=1}^{r_b} b_i^r - \sum_{j=1}^{r_c} c_j^r \right) J_r, \\ c_2(V) &= \frac{1}{2} d_{rst} \left(\sum_{j=1}^{r_c} c_j^s c_j^t - \sum_{i=1}^{r_b} b_i^s b_i^t \right) \nu^r, \\ c_3(V) &= \frac{1}{3} d_{rst} \left(\sum_{i=1}^{r_b} b_i^r b_i^s b_i^t - \sum_{j=1}^{r_c} c_j^r c_j^s c_j^t \right), \end{aligned} \quad (3.3)$$

where $J_r \in H^{1,1}(X, \mathbb{R})$ represent the harmonic $(1, 1)$ -forms $c_1(\mathcal{O}_X(\mathbf{e}_r))$, the d_{rst} are the triple intersection numbers defined by

$$d_{rst} = \int_X J_r \wedge J_s \wedge J_t, \quad (3.4)$$

and the ν^r are the 4-forms furnishing the dual basis to the Kähler generators J_r , subject to the duality relation

$$\int_X J_r \wedge \nu^s = \delta_r^s. \quad (3.5)$$

As can be seen from (3.3), the Chern class of V only depends on the integer parameters \mathbf{b}_i and \mathbf{c}_j , as well as the topology of the base manifold X . Choosing an appropriate morphism f in the defining sequence (3.1) corresponds to the tuning of more refined invariants of V .

4. Towards the Standard Model

Sections 2 and 3 have shown that the vacuum topology is essentially described by lattice vertices and integer parameters, both of which are discrete and combinatorial in nature. One can therefore attempt to construct $\mathcal{N} = 1$ heterotic vacua in an algorithmic way. Torically constructed Calabi-Yau threefolds form a dataset of reflexive polytopes represented by the lattice vertices, and monad bundles are explored on each of the base manifolds by varying the integer parameters.

4.1. Phenomenological Constraints on the Vacua

With the geometric constraints so far explained, one would only be able to guarantee the right number of supersymmetry at low energy, that is, $\mathcal{N} = 1$ at $D = 4$. Since the goal of string phenomenology is to obtain (supersymmetric versions of) the standard model, more criteria should further be imposed on the $\mathcal{N} = 1$ vacua. To make things clear, let us emphasize that in this paper the terminology “standard-like” model will imply the following:

- (i) Gauge invariance under $SU(3)_C \times SU(2)_L \times U(1)_Y$, possibly with an extra $U(1)_{B-L}$ factor;
- (ii) three generations of quarks and leptons, and no exotics;
- (iii) cancellation of heterotic anomaly.

Here, we translate the above three phenomenological constraints into the conditions on the vacuum topology.

4.1.1. Gauge Group

As explained in Section 1, the structure group \mathcal{G} of the visible sector bundle sits in E_8 and its commutant \mathcal{H} becomes the low-energy gauge group. In order to obtain $\mathcal{H} = SO(10)$ and $SU(5)$, one must choose $\mathcal{G} = SU(4)$ and $SU(5)$, respectively. In particular, the rank of the bundle should be either 4 or 5 and, hence, by (3.2),

$$r_b - r_c = 4 \text{ or } 5, \quad (4.1)$$

where r_b and r_c are the ranks of the two vector bundles in the defining sequence (3.1) of V . What is more, since the structure group should be “special” unitary, the first Chern class $c_1(V)$ of V is to vanish. By (3.4), this corresponds to

$$\sum_{i=1}^{r_b} \mathbf{b}_i = \sum_{j=1}^{r_c} \mathbf{c}_j, \quad (4.2)$$

where $\mathbf{b}_i = (b_i^1, \dots, b_i^{h^{1,1}})$ and $\mathbf{c}_j = (c_j^1, \dots, c_j^{h^{1,1}})$ are the $h^{1,1}$ -tuples of integers labelling the summand line bundles and, hence, parameterising the monad V .

We still have to break the GUT group further down to a standard-like one, and this second breaking will require $\pi_1(X)$ -Wilson-lines. However, given the observation that most of the torically constructed Calabi-Yau threefolds have a trivial first fundamental group [44], they must be quotiented out by freely-acting discrete symmetries so that we may turn on appropriate Wilson lines. Therefore, we will eventually have to look for a discrete symmetry group G that acts freely on X and make a quotient space $\hat{X} = X/G$, which will then have a nontrivial first fundamental group $\pi_1(\hat{X}) \simeq G$.

4.1.2. Cancellation of Heterotic Anomaly

Heterotic models need to satisfy a well-known anomaly condition. So far, we have only mentioned one holomorphic vector bundle V for the visible sector but the theory has another bundle \tilde{V} for the hidden sector. Heterotic vacua can also have five-branes whose strong-coupling origin is M5-branes. In order to keep the four-dimensional Lorentz symmetry, their world volumes must stretch along the external Minkowski M_4 . The remaining two dimensions should then wrap holomorphic two cycles in X for unbroken supersymmetry. Thus, the homology classes associated with these two cycles must be *effective* and, hence, belong to the Mori cone in $H_2(X, \mathbb{Z})$. In other words, the corresponding four-forms must belong to the corresponding cone in $H^4(X, \mathbb{Z})$.

In this most general setup, heterotic anomaly cancellation imposes a topological constraint relating the Calabi-Yau threefold, the two vector bundles, and the five-brane classes. When $c_1(TX) = c_1(V) = c_1(\tilde{V}) = 0$, the anomaly condition can be expressed, at the level of cohomology, as

$$c_2(TX) - c_2(V) - c_2(\tilde{V}) = W, \quad (4.3)$$

where $W = \sum_i W_i$ is the sum of the five-brane classes. Note that W itself should also belong to the Mori cone of X as all the summands W_i do. In our discussion, however, without mentioning the second bundle \tilde{V} , we presume a trivial bundle for the hidden sector. Thus, the anomaly constraint in (4.3) says that $c_2(TX) - c_2(V)$ is effective.

4.1.3. Particle Spectra

Table 1 shows how the low-energy particle spectra are determined from various bundle-valued cohomology groups. Assuming that V is a stable bundle⁴ implies that $h^0(X, V) = 0 = h^3(X, V)$, and, hence, to obtain three net generations of quarks and leptons, we must have

$$-\frac{1}{2} \int_X c_3(V) = h^1(X, V) - h^2(X, V) = 3|G|, \quad (4.4)$$

Table 2: The list of standard-like constraints on the $\mathcal{N} = 1$ heterotic vacua, each described by a Calabi-Yau threefolds X and a \mathcal{G} -bundle V thereon; the integers b_i^r and c_j^r parameterise the bundle V as in (3.1), where $i = 1, \dots, r_b$, $j = 1, \dots, r_c$, and $r = 1, \dots, h^{1,1}(X)$. The second column states the constraints on the background geometry, and the third column expresses the corresponding algebraic equations that the parameters b_i^r and c_j^r must obey, where $d_{rst} = \int_X J_r \wedge J_s \wedge J_t$ are the intersection numbers for a given basis $\{J_r\}_{1 \leq r \leq h^{1,1}(X)}$ for $H^{1,1}(X, \mathbb{R})$.

Physics origin	Background geometry	Algebraic constraints on \mathbf{b}_i and \mathbf{c}_j
Gauge group	(a) $\mathcal{G} = SU(n)$, for $n = 4, 5$ (b) X has a discrete free action G	(a1) $r_b - r_c = 4$ or 5 (a2) $\sum_{i=1}^{r_b} b_i^r = \sum_{j=1}^{r_c} c_j^r$, for all r
Anomaly	$c_2(TX) - c_2(V)$ is effective	$c_2(TX)_r - (1/2)d_{rst}(\sum_{j=1}^{r_c} c_j^s c_j^t - \sum_{i=1}^{r_b} b_i^s b_i^t) > 0$, for all r
Particle spectra	$(1/2) \int_X c_3(V) = -3 G $	$d_{rst}(\sum_{i=1}^{r_b} b_i^r b_i^s b_i^t - \sum_{j=1}^{r_c} c_j^r c_j^s c_j^t) = -18 G $

where the Aiyah-Singer index theorem [45] has been applied to the differential operator \not{D}_V on X and $|G|$ is the order of the discrete symmetry group G , with which we will have to quotient the ‘‘upstairs’’ threefolds X .

4.2. Discrete System for Standard-Like Vacua

In this subsection, we briefly summarise the model-building requirements that have so far been discussed.

- (i) Calabi-Yau threefold: a reflexive polytope $\Delta \subset \mathbb{R}^4$ describes a Calabi-Yau threefold X . In the computer package PALP [46], by inserting the list of lattice vertices of Δ or, equivalently, the corresponding ‘‘weight system’’ [18], one can obtain all the topological invariants of X relevant to the heterotic compactification.
- (ii) Monad vector bundle: the $r_b + r_c$ integer vectors \mathbf{b}_i and \mathbf{c}_j of length $h^{1,1}(X)$, each labelling a line bundle summand, parameterise our monad bundle V .
- (iii) Standard-like constraints: the internal backgrounds are also constrained by the standard-like phenomenology. It turns out that given a Calabi-Yau threefold X , that is, for a fixed topology of X , the monad parameters \mathbf{b}_i and \mathbf{c}_j must obey the algebraic equations of degree 0, 1, 2, and 3 shown in Table 2.

Note that the integer combinatorics under the algebraic constraints of small degrees has formed a simple discrete system for standard-like heterotic vacua. However, this system of vacua is far from being finite yet. To begin with, there are no upper bounds on r_b and r_c that count the number of monad parameters. Before initiating an exploration of the landscape, one first needs to add more constraints to make the system finite and those extra constraints had better be related to preferred phenomenology. Now, if the line bundle summands in (3.1) are ample or, equivalently, if all the monad parameters b_i^r and c_j^r are positive,⁵ then, by Kodaira’s vanishing theorem [45, 47], the cohomology group $H^2(X, V)$ vanishes and, hence, the low-energy effective theory acquires no antigerations. Of course this is a phenomenologically preferred feature, although not necessary. We call a monad *positive* if it is only parameterised by positive integers and *semipositive* if all its parameters are either positive or zero. Secondly, one can also constrain the relative size of these monad parameters so that each entry of the vector $\mathbf{b}_i - \mathbf{c}_j$ may be nonnegative, for all i and j . In this case, we will call the monad *generic*

since the monad map f in (3.1), thought of as an $r_c \times r_b$ matrix of polynomials, may generically have all the entries nonzero.

4.3. Exploring a Region of the Landscape

As for the first step, one can think of exploring generic, positive monads over a “small” class of Calabi-Yau threefolds. As was mentioned in Section 2.2, the total dataset of torically constructed Calabi-Yau threefolds are way too large to grasp altogether. Therefore, at the initial stage, those in “smooth” ambient spaces have first been considered amongst the total of 500 million [40]. It turns out that over these $O(100)$ manifolds, the generic, positive monads are finite in number under the constraints in Table 2 and the standard-like vacua have indeed been classified, resulting in 61 candidate models.

Based on this experience, one can become more ambitious and extend the vacuum search, both bundle-wise and Calabi-Yau-wise. Firstly, with the positivity condition a bit relaxed, the generic, semipositive monads have been explored over the same $O(100)$ Calabi-Yau threefolds [40]. The standard-like vacua with the monads of this type turn out to form an infinite class, and, hence, they have been explored under an artificial upper bound on the monad parameters, resulting in 85 models. Secondly, the programme has also been extended to include singular ambient manifolds with small $h^{1,1}$ [41]. A total of $O(300)$ torically constructed Calabi-Yau threefolds have the Hodge number $h^{1,1} \leq 3$, and the generic, positive monads have been classified thereon, giving rise to new candidate models.

5. Summary and Outlook

In this paper, we have discussed a systematic approach towards standard-like heterotic vacua. The proposed algorithms have indeed been implemented in a computer package [48]. Simplicity of the integer combinatorics for the $\mathcal{N} = 1$ heterotic vacua was the essential feature that made this approach a tractable programme. It was motivated by the general observation that any carefully chosen single model is likely to fail the detailed structure of the standard model. Thus, the spirit of the programme is to obtain a large number of standard-like models, on which further constraints should be imposed later on to refine the set of candidates, eventually reaching the “genuine” standard model(s).

The combinatorics of toric geometry has been invaluable for constructing toric ambient fourfolds, to which Calabi-Yau threefolds have been embedded as hypersurfaces, and for computing their topological invariants relevant to the four-dimensional phenomenology. Smooth ambient fourfolds have been considered as a starter [40], and general ambient fourfolds have also been dealt with [41] by partially resolving the singularities, if they bear any, so that the smoothness of the hypersurface Calabi-Yau threefolds is guaranteed. In both cases, the generic, positive monads (and some semipositive ones, too, in the former case) have been probed under the standard-like criteria. We have thus obtained a set of candidate models, that are anomaly free and that have a chance to produce three generations of quarks and leptons without any antigerations.

To guarantee the three-generation property of these candidates, further study of discrete symmetries of the manifolds is essential. Braun has recently classified the free group actions on complete intersection Calabi-Yau threefolds in multiprojective spaces [49], and his algorithm can in principle be generalised to toric cases. The line-bundle cohomologies on the torically constructed Calabi-Yau threefolds are also an essential part of the model building. The starting point would be to work out the cohomologies on the ambient toric

varieties, which have already been investigated in the mathematics and physics literatures [23, 50–52]. Practical conversion of this information into the line-bundle cohomologies on the hypersurfaces is a rewarding work along the line of monad bundles and heterotic strings. As for the completion of the detailed particle spectra, the cohomologies of the monads in different representations are also to be revealed.

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Endnotes

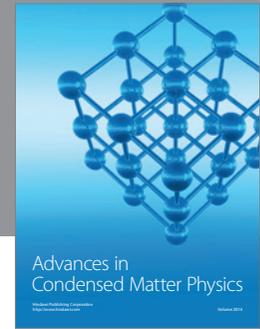
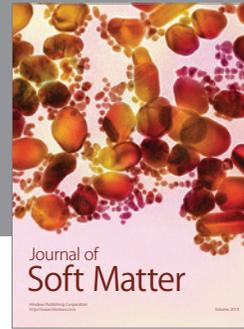
1. The choice $G = SU(3)$ also gives rise to E_6 GUT models. However, they have an inherent trouble in doublet-triplet splitting of Higgs multiplet (see, for a recent example, [39]) and, hence, we will not address the models of this type here.
2. In algebraic geometry, Calabi-Yau threefolds are, in general, realised as complete intersections of hypersurfaces in toric varieties of dimension greater than or equal to four but this paper will only be dealing with single-hypersurface cases.
3. MPCP is a short for maximal, projective, crepant and partial. A triangulation is said to be *maximal* if all lattice points of the polytope are involved, *projective* if the Kähler cone of the resolved manifold has a nonempty interior, and *crepant* if no points outside the polytope are taken. In practice, all possible MPCP-triangulations of a given reflexive polytope are searched by the computer package PALP [46].
4. Testing the bundle stability is indeed one of the crucial steps for our model construction. However, it is not at all an easy task to check if a given bundle is stable. So, our strategy is first to make use of some consequences of stability and then to check the validity at the very end of the story. In this paper, focusing on the discrete combinatorics for the vacua, we will not say more about the issue of stability.
5. To be precise, the Kodaira vanishing assumes that the vectors \mathbf{b}_i and \mathbf{c}_j lie in the Kähler cone of X . In case the Kähler cone does not coincide with the positive region, one may redefine the standard basis vectors of $H^{1,1}(X, \mathbb{R})$ to be the Kähler cone generators. For this to work, however, the cone generators should form a linearly independent basis and hence, we implicitly restrict ourselves to the Calabi-Yau threefolds of this type.

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