Review Article

Gauge-Higgs Unification Models in Six Dimensions with S^2/Z_2 Extra Space and GUT Gauge Symmetry

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We review gauge-Higgs unification models based on gauge theories defined on six-dimensional spacetime with S^2/Z_2 topology in the extra spatial dimensions. Nontrivial boundary conditions are imposed on the extra S^2/Z_2 space. This review considers two scenarios for constructing a four-dimensional theory from the six-dimensional model. One scheme utilizes the $SO(12)$ gauge symmetry with a special symmetry condition imposed on the gauge field, whereas the other employs the E_6 gauge symmetry without requiring the additional symmetry condition. Both models lead to a standard model-like gauge theory with the $SU(3) \times SU(2)_L \times U(1)_Y (\times U(1)^2)$ symmetry and SM fermions in four dimensions. The Higgs sector of the model is also analyzed. The electroweak symmetry breaking can be realized, and the weak gauge boson and Higgs boson masses are obtained.

1. Introduction

The Higgs sector of the standard model (SM) plays an essential role in the spontaneous symmetry breaking (SSB) from the $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge group down to $SU(3)_C \times U(1)_{EM}$, thereby giving masses to the SM elementary particles. However, the SM does not address the most fundamental nature of the Higgs sector, such as the mass and self-coupling constant of the Higgs boson. Therefore, the Higgs sector is not only the last territory in the SM to be discovered, but will also provide key clues to new physics at higher energy scales.

Gauge-Higgs unification is one of many attractive approaches to physics beyond the SM in this regard [1–3] (for recent approaches, see [4–20]). In this approach, the Higgs particles originate from the extradimensional components of the gauge field defined on

spacetime with the number of dimensions greater than four (for cases where). In other words, the Higgs sector is embraced into the gauge interactions in the higher-dimensional model, and many fundamental properties of Higgs boson are dictated by the gauge interactions.

In our recent studies, we have shown interesting properties of one type of gauge-Higgs unification models based on grand unified gauge theories defined on six-dimensional (6D) spacetime, with the extradimensional space having the topological structure of two-sphere orbifold S^2/Z_2 [21, 22].

In the usual coset space dimensional reduction (CSDR) approach [1, 23–26], one imposes on the gauge fields the symmetry condition which identifies the gauge transformation as the isometry transformation of S^2 due to its coset space structure $S^2 = \text{SU}(2)/\text{U}(1)$. The dimensional reduction is explicitly carried out by applying the solution of the symmetry condition. A background gauge field is introduced as part of the solution [1]. Such a background gauge field is also necessary for obtaining chiral fermions in four-dimensional (4D) spacetime, even without the symmetry condition. After the dimensional reduction, no Kaluza-Klein (KK) mode appears because of the imposed symmetry condition. The symmetry condition also restricts the gauge symmetry and the scalar contents originated from the extra gauge field components in the 4D spacetime. Moreover, a suitable potential for the scalar sector can be obtained to induce SSB at tree level.

In this paper, we consider two scenarios for constructing the 4D theory from a 6D model: one utilizing the symmetry condition for the gauge field with $\text{SO}(12)$ symmetry [21], whereas the other without it for the gauge field with E_6 symmetry [22]. In the first scenario, however, we do not impose the condition on the fermions as used in other CSDR models. We then have massive KK modes for fermions but not the gauge and scalar fields in 4D. We can thus obtain a dark matter candidate under assumed KK parity. In the case without the symmetry condition, we find that the background gauge field is able to restrict the gauge symmetry and massless particle contents in 4D. Also, there are KK modes for each field, with the mass spectrum determined according to the model. Generically, massless modes do not appear in the KK mass spectrum because of the positive curvature of the S^2 space [27]. With the help of the background gauge field, however, we obtain massless KK modes for the gauge bosons and fermions.

In general, the gauge symmetry of a grand unified theory (GUT) tends to remain in 4D in these dimensional reduction approaches [24, 28–32]. Also, it is usually difficult to obtain an appropriate Higgs potential to break the GUT gauge symmetry to the SM-like one because of the gauge group structure. A GUT gauge symmetry can be broken to the SM-like gauge symmetry by imposing nontrivial boundary conditions (for cases with orbifold extra space, see, e.g., [4–8, 11, 12, 16–18, 33, 34]). Therefore, to solve the above-mentioned problems, we impose on the fields of the 6D model a set of nontrivial boundary conditions on the S^2/Z_2 space. Therefore, the gauge symmetry, scalar contents, and massless fermions are determined by these boundary conditions and the background gauge field. We find that in both scenarios, with or without the symmetry condition for the gauge field, the electroweak symmetry breaking (EWSB) can be realized spontaneously. The Higgs boson mass is predicted by analyzing the Higgs potential in the respective models.

This paper is organized as follows. In Section 2, we review two schemes for constructing a 4D theory from gauge models defined on 6D spacetime whose extra space has the S^2/Z_2 topology with a set of nontrivial boundary conditions. In Section 3, we show the models based on $\text{SO}(12)$ and E_6 gauge symmetries, with the former being imposed with the symmetry condition on the gauge field and the latter without. We summarize our results in Section 4.

2. The 6D Gauge-Higgs Unification Model Construction Scheme with Extra S^2/Z_2 Space

There are two schemes for constructing a 4D theory from a 6D gauge theory, where the extra space is a two-sphere orbifold S^2/Z_2 . Use of the symmetry condition is made on the first scheme but not the other. We apply nontrivial boundary condition in both schemes.

2.1. A Gauge Theory on 6D Spacetime with S^2/Z_2 Extraspace

2.1.1. The 6D Spacetime with S^2/Z_2 Extraspace

We begin by considering a 6D spacetime M^6 that is assumed to be a direct product of the 4D Minkowski spacetime M^4 and two-sphere orbifold S^2/Z_2 , that is, $M^6 = M^4 \times S^2/Z_2$. The two-sphere S^2 is a unique two-dimensional coset space and can be written as $S^2 = \text{SU}(2)_I/\text{U}(1)_I$, where $\text{U}(1)_I$ is a subgroup of $\text{SU}(2)_I$. This coset space structure of S^2 requires that S^2 have the isometry group $\text{SU}(2)_I$ and that the $\text{U}(1)_I$ group be embedded in the group $\text{SO}(2)$ which is in turn a subgroup of the full Lorentz group $\text{SO}(1,5)$. We denote the coordinates of M^6 by $X^M = (x^\mu, y^\theta = \theta, y^\phi = \phi)$, where x^μ and $\{\theta, \phi\}$ are M^4 coordinates and S^2 spherical coordinates, respectively. The spacetime index M runs over $\mu \in \{0, 1, 2, 3\}$ and $\alpha \in \{\theta, \phi\}$. The orbifold S^2/Z_2 is defined by the identification of (θ, ϕ) and $(\pi - \theta, -\phi)$ [35], leaving two fixed points: $(\pi/2, 0)$ and $(\pi/2, \pi)$. The metric g_{MN} of M^6 is written as

$$g_{MN} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & -g_{\alpha\beta} \end{pmatrix}, \quad (2.1)$$

where $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and $g_{\alpha\beta} = R^2 \text{diag}(1, \sin^2\theta)$ are the metrics for M^4 and S^2 , respectively, and R is the radius of S^2 . We define the vielbeins e_A^M that connect the metric of M^6 and that of the tangent space of M^6 , denoted by h_{AB} , through the relation $g_{MN} = e_M^A e_N^B h_{AB}$. Here $A = (\mu, a)$, where $a \in \{4, 5\}$, is the index for the coordinates of tangent space of M^6 . The explicit forms of the vielbeins are

$$e_\theta^4 = R, \quad e_\phi^5 = R \sin \theta, \quad e_\phi^4 = e_\theta^5 = 0. \quad (2.2)$$

Also the nonzero components of the spin connection are

$$R_\phi^{45} = -R_\phi^{54} = -\cos \theta. \quad (2.3)$$

2.1.2. Lagrangian on 6D Spacetime with S^2/Z_2 Extra Space

We now discuss the general structure of a gauge theory on M^6 . We first introduce a gauge field $A_M(x, y) = (A_\mu(x, y), A_a(x, y))$, which belongs to the adjoint representation of a gauge group G , and fermions $\Psi(x, y)$, which lies in a representation F of G . The action of this theory is then given by

$$S = \int dx^4 R^2 \sin \theta d\theta d\phi \left(\bar{\Psi} i \Gamma^\mu D_\mu \Psi + \bar{\Psi} i \Gamma^a e_a^\alpha D_\alpha \Psi - \frac{1}{4g^2} g^{MN} g^{KL} \text{Tr}[F_{MK} F_{NL}] \right), \quad (2.4)$$

where $F_{MN} = \partial_M A_N(X) - \partial_N A_M(X) - [A_M(X), A_N(X)]$ is the field strength, D_M is the covariant derivative including the spin connection, and Γ_A represents the Dirac matrices satisfying the 6D Clifford algebra. Here D_M and Γ_A can be written explicitly as

$$\begin{aligned} D_\mu &= \partial_\mu - A_\mu, & D_\theta &= \partial_\theta - A_\theta, & D_\phi &= \partial_\phi - i \frac{\Sigma_3}{2} \cos \theta - A_\phi, \\ \Gamma_\mu &= \gamma_\mu \otimes \mathbf{I}_2, & \Gamma_4 &= i\gamma_5 \otimes \sigma_1, & \Gamma_5 &= i\gamma_5 \otimes \sigma_2, \end{aligned} \quad (2.5)$$

where $\{\gamma_\mu, \gamma_5\}$ are the 4D Dirac matrices, σ_i ($i = 1, 2, 3$) are the Pauli matrices, $\mathbf{I}_d$ is the $d \times d$ identity matrix, and $\Sigma_3 = \mathbf{I}_4 \otimes \sigma_3$. The covariant derivative D_ϕ has the spin connection term $i(\Sigma_3/2) \cos \theta$ which is needed for space with a nonzero curvature- like S^2 and applied only to fermions. In 6D spacetime, one can define the chirality of fermions and the corresponding projection operators are

$$\Gamma_\pm = \frac{1 \pm \Gamma_7}{2}, \quad (2.6)$$

where $\Gamma_7 \equiv \gamma_5 \otimes \sigma_3$ is the chiral operator. The chiral fermions on 6D spacetime are thus

$$\Psi_\pm = \Gamma_\pm \Psi, \quad \Gamma_7 \Psi_\pm = \pm \Psi_\pm. \quad (2.7)$$

The 6D chiral fermions can be also written in terms of 4D chiral fermions $\psi_{L(R)}$ as

$$\Psi_+ = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \quad (2.8)$$

$$\Psi_- = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}. \quad (2.9)$$

Here we note in passing that the mass dimensions of A_μ , A_α , Ψ and g in the 6D model are 1, 0, 5/2 and -1, respectively.

2.1.3. Nontrivial Boundary Conditions on the Two-Sphere Orbifold

On the two-sphere orbifold, one can consider parity operations $P : (\theta, \phi) \rightarrow (\pi - \theta, -\phi)$ and azimuthal translation $T_\phi : (\theta, \phi) \rightarrow (\theta, \phi + 2\pi)$. Notice that here the periodicity $\phi \rightarrow \phi + 2\pi$ is not associated with the orbifolding. We can impose the following two types of boundary conditions on both gauge and fermion fields under the two operations:

$$A_\mu(x, \pi - \theta, -\phi) = P_1 A_\mu(x, \theta, \phi) P_1, \quad A_\mu(x, \pi - \theta, 2\pi - \phi) = P_2 A_\mu(x, \theta, \phi) P_2, \quad (2.10)$$

$$A_\alpha(x, \pi - \theta, -\phi) = -P_1 A_\alpha(x, \theta, \phi) P_1, \quad A_\alpha(x, \pi - \theta, 2\pi - \phi) = -P_2 A_\alpha(x, \theta, \phi) P_2, \quad (2.11)$$

$$\Psi(x, \pi - \theta, -\phi) = \pm \gamma_5 P_1 \Psi(x, \theta, \phi), \quad \Psi(x, \pi - \theta, 2\pi - \phi) = \pm \gamma_5 P_2 \Psi(x, \theta, \phi), \quad (2.12)$$

or

$$A_\mu(x, \pi - \theta, -\phi) = P_1 A_\mu(x, \theta, \phi) P_1, \quad A_\mu(x, \theta, \phi + 2\pi) = \hat{P}_2 A_\mu(x, \theta, \phi) \hat{P}_2, \quad (2.13)$$

$$A_\alpha(x, \pi - \theta, -\phi) = -P_1 A_\alpha(x, \theta, \phi) P_1, \quad A_\alpha(x, \theta, \phi + 2\pi) = \hat{P}_2 A_\alpha(x, \theta, \phi) \hat{P}_2, \quad (2.14)$$

$$\Psi(x, \pi - \theta, -\phi) = \pm \gamma_5 P_1 \Psi(x, \theta, \phi), \quad \Psi(x, \theta, \phi + 2\pi) = \pm \hat{P}_2 \Psi(x, \theta, \phi), \quad (2.15)$$

where the former conditions are associated with P operation and combination of P and T_ϕ operations, while the latter conditions are associated with the P or T_ϕ operation individually. More explicitly, P_1 , P_2 , and $\hat{P}_2$ correspond to operations P , PT_ϕ , and T_ϕ , respectively. These boundary conditions are determined by requiring invariance of the 6D action under the transformation $(\theta, \phi) \rightarrow (\pi - \theta, -\phi)$ and $\phi \rightarrow \phi + 2\pi$. Note that at the poles ($\sin \theta = 0$), the coordinate ϕ is not well-defined and the translation T_ϕ is irrelevant. Thus, only the components which are even under $\phi \rightarrow \phi + 2\pi$ can exist without contradiction.

The projection matrices $P_{1,2}$ act on the gauge group representation space and have eigenvalues ± 1 . They assign different parities for different representation components. For fermion boundary conditions, the sign in front of γ_5 can be either $+$ or $-$ since the fermions always appear in bilinear forms in the action. The 4D action is then restricted by these parity assignments.

2.2. Dimensional Reduction Scheme with Symmetry Condition

Here we review the dimensional reduction scheme in which a symmetry condition is applied to the gauge field [21].

2.2.1. The Symmetry Condition

We impose on the gauge field $A_M(X)$ the symmetry which connects $SU(2)_I$ isometry transformation on S^2 and the gauge transformation of the field in order to carry out dimensional reduction. Moreover, the nontrivial boundary conditions of S^2/Z_2 are also utilized to restrict the 4D theory. The symmetry demands that the $SU(2)_I$ coordinate transformation should be compensated by a gauge transformation [1, 23]. It further leads to the following set of the symmetry condition on the gauge field:

$$\begin{aligned} \xi_i^\beta \partial_\beta A_\mu &= \partial_\mu W_i + [W_i, A_\mu], \\ \xi_i^\beta \partial_\beta A_\alpha + \partial_\alpha \xi_i^\beta A_\beta &= \partial_\alpha W_i + [W_i, A_\alpha], \end{aligned} \quad (2.16)$$

where ξ_i^α are the killing vectors that generate the $SU(2)_I$ symmetry, and W_i are some fields that generate an infinitesimal gauge transformation of G . Here the index $i = 1, 2, 3$ corresponds to that of the $SU(2)$ generators. The explicit forms of ξ_i^α s for S^2 are

$$\begin{aligned} \xi_1^\theta &= \sin \phi, & \xi_1^\phi &= \cot \theta \cos \phi, \\ \xi_2^\theta &= -\cos \phi, & \xi_2^\phi &= \cot \theta \sin \phi, \\ \xi_3^\theta &= 0, & \xi_3^\phi &= -1. \end{aligned} \quad (2.17)$$

The LHS's and RHS's of (2.16) are infinitesimal isometry transformations and the corresponding infinitesimal gauge transformations, respectively.

2.2.2. Dimensional Reduction and Lagrangian in 4D Spacetime

The dimensional reduction of the gauge sector is explicitly carried out by applying the solutions of the symmetry condition equations (2.16). These solutions are given by Manton [1]

$$A_\mu = A_\mu(x), \quad A_\theta = -\Phi_1(x), \quad A_\phi = \Phi_2(x) \sin \theta - \Phi_3 \cos \theta, \quad (2.18)$$

$$W_1 = -\Phi_3 \frac{\cos \phi}{\sin \theta}, \quad W_2 = -\Phi_3 \frac{\sin \phi}{\sin \theta}, \quad W_3 = 0, \quad (2.19)$$

where $\Phi_1(x)$ and $\Phi_2(x)$ are scalar fields and the Φ_3 term for A_ϕ corresponds to the background gauge field [36]. They satisfy the following constraints:

$$[\Phi_3, A_\mu] = 0, \quad (2.20)$$

$$[-i\Phi_3, \Phi_i(x)] = i\epsilon_{3ij}\Phi_j(x), \quad (2.21)$$

where the LHS shows the gauge transformation associated with Φ_3 and the RHS shows the $U(1)_I$ transformation embedded in Lorentz group $SO(2)$. These constraints can be satisfied when $U(1)_I$ is embedded in the gauge group G and $-i\Phi_3$ should be chosen as the corresponding generator.

Substituting the solutions, (2.18), into $A_M(X)$ in the action, (2.4), one can easily obtain the 4D action by integrating out coordinates θ and ϕ in the gauge sector.

$$\begin{aligned} S_{4D}^{(\text{gauge})} = \int d^4x & \left(-\frac{R^2}{4g^2} \text{Tr}[F_{\mu\nu}F^{\mu\nu}(x)] \right. \\ & - \frac{1}{2g^2} \text{Tr}[D'_\mu \Phi_1(x) D'^\mu \Phi_1(x) + D'_\mu \Phi_2(x) D'^\mu \Phi_2(x)] \\ & \left. - \frac{1}{2g^2 R^2} \text{Tr}[(\Phi_3 + [\Phi_1(x), \Phi_2(x)])(\Phi_3 + [\Phi_1(x), \Phi_2(x)])] \right), \end{aligned} \quad (2.22)$$

where $D'_\mu \Phi = \partial_\mu \Phi - [A_\mu, \Phi]$.

For fermions, we do not impose the symmetry condition. Then the gauge interaction term is not invariant under the coordinate transformation on S^2/Z_2 . The fermion sector of the 4D action is thus obtained by expanding fermions in terms of the normal modes of S^2/Z_2 and then integrating out the S^2/Z_2 coordinates in the 6D action. As a result, the fermions have massive KK modes which can provide a dark matter candidate. Generally, the KK modes do

not contain massless modes because of the positive curvature of S^2 [27]. Nevertheless, we can show that the fermion components satisfying the condition

$$-i\Phi_3\Psi = \frac{\Sigma_3}{2}\Psi \quad (2.23)$$

do have massless modes. The squared masses of the KK modes are eigenvalues of the square of the extradimensional Dirac-operator $-i\hat{D}$. In the S^2 case,

$$\begin{aligned} -i\hat{D} &= -ie^{aa}\Gamma_a D_a \\ &= -\frac{i}{R}\left[\Sigma_1\left(\partial_\theta + \frac{\cot\theta}{2}\right) + \Sigma_2\left(\frac{1}{\sin\theta}\partial_\phi + \Phi_3\cot\theta\right)\right], \end{aligned} \quad (2.24)$$

where $\Sigma_i = \mathbf{I}_4 \times \sigma_i$. Hence,

$$\begin{aligned} (-i\hat{D})^2 &= -\frac{1}{R^2}\left[\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{\sin^2\theta}\partial_\phi^2 + i(2(-i\Phi_3) - \Sigma_3)\frac{\cos\theta}{\sin^2\theta}\partial_\phi\right. \\ &\quad \left.-\frac{1}{4} - \frac{1}{4\sin^2\theta} + \Sigma_3(-i\Phi_3)\frac{1}{\sin^2\theta} - (-i\Phi_3)^2\cot^2\theta\right]. \end{aligned} \quad (2.25)$$

By acting the above operator on a fermion $\Psi(X)$ that satisfies (2.23), we obtain the relation

$$(-i\hat{D})^2\Psi = -\frac{1}{R^2}\left[\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{\sin^2\theta}\partial_\phi^2\right]\Psi. \quad (2.26)$$

The eigenvalues of the operator on the RHS are less than or equal to zero. Hence, the fermion components satisfying (2.23) have massless modes, while other components have only massive KK modes. Note that the massless mode ψ_0 should be independent of S^2 coordinates θ and ϕ , that is,

$$\psi_0 = \psi(x). \quad (2.27)$$

The existence of massless fermions signifies the meaning and importance of the symmetry condition. Although the energy density of the gauge sector in the presence of the background field is higher than that with no background field, the massless fermions may help render a true ground state as a whole. In other words, the existence of the background field will give a positive contribution to the energy density of the gauge sector, indicating that the gauge sector with the background field alone is not at the ground state. Nevertheless, it gives rise to a negative contribution to the energy density of the fermion sector to induce massless fermions. We therefore expect that once both the gauge and fermion sectors are considered together, the existence of the background field renders the system at the ground state. We also note that one could impose symmetry condition on fermions [24, 37]. In that case, we obtain the massless condition equation (2.23) from the symmetry condition of fermion, and the solution of symmetry condition is independent of the S^2 coordinates: $\psi = \psi(x)$ with no massive KK mode. Therefore, the same discussion as before can be applied for this case if one only focuses on the massless mode in our scheme.

2.2.3. Gauge Symmetry and Particle Contents in 4D Spacetime

The symmetry condition and the nontrivial boundary conditions substantially constrain the 4D gauge group and the representations of the particle contents.

First, we show the prescriptions to identify gauge symmetry and field components which satisfy the constraint equations (2.20), (2.21), and (2.23). The gauge group H that satisfy the constraint equation (2.20) is identified as

$$H = C_G(U(1)_I), \quad (2.28)$$

where $C_G(U(1)_I)$ denotes the centralizer of $U(1)_I$ in G [23]. Note that this implies $G \supset H = H' \times U(1)_I$, where H' is some subgroup of G . In this way, the gauge group G is reduced to its subgroup $H = H' \times U(1)_I$ by the symmetry condition.

Secondly, the scalar field components which satisfy the constraint equations (2.21) are specified by the following prescription. Suppose that the adjoint representations of $SU(2)_I$ and G are decomposed according to the embeddings $SU(2)_I \supset U(1)_I$ and $G \supset H' \times U(1)_I$ as

$$3(\text{adj } SU(2)) = (0(\text{adj } U(1))_I) + (2) + (-2), \quad (2.29)$$

$$\text{adj } G = (\text{adj } H)(0) + 1(0(\text{adj } U(1))_I) + \sum_g h_g(r_g), \quad (2.30)$$

where h_g 's denote representations of H' , and r_g 's denote the $U(1)_I$ charges. Then the scalar components satisfying the constraints belong to h_g 's whose corresponding r_g 's in (2.30) are ± 2 .

Thirdly, the fermion components which satisfy the constraint equations (2.23) are determined as follows [37]. Let the group $U(1)_I$ be embedded in the Lorentz group $SO(2)$ in such a way that the vector representation 2 of $SO(2)$ is decomposed according to $SO(2) \supset U(1)_I$ as

$$2 = (2) + (-2). \quad (2.31)$$

This embedding specifies a decomposition of the Weyl spinor representation $\sigma_6 = 4$ of $SO(1,5)$ according to $SO(1,5) \supset SU(2) \times SU(2) \times U(1)_I$ as

$$\sigma_6 = (2,1)(1) + (1,2)(-1), \quad (2.32)$$

where the $SU(2) \times SU(2)$ representations $(2,1)$ and $(1,2)$ correspond to left-handed and right-handed spinors, respectively. We note that this decomposition corresponds to (2.8) [or (2.9)]. We then decompose F according to $G \supset H' \times U(1)_I$ as

$$F = \sum_f h_f(r_f). \quad (2.33)$$

Now the fermion components satisfying the constraints are identified as those h_f 's whose corresponding r_f 's in (2.33) are +1 for left-handed fermions and -1 for right-handed fermions.

Finally, we show which gauge symmetry and field components remain in 4D spacetime by surveying the consistency between the boundary conditions (2.13)–(2.15), the solutions in (2.18), and the massless fermion modes equation (2.27). By applying (2.18) and (2.27) to (2.13)–(2.15), we obtain the parity conditions

$$\begin{aligned}
A_\mu(x) &= P_1(\hat{P}_2)A_\mu(x)P_1(\hat{P}_2), \\
-\Phi_1(x) &= -P_1(-\Phi_1(x))P_1, \\
-\Phi_1(x) &= \hat{P}_2(-\Phi_1(x))\hat{P}_2, \\
\Phi_2(x) + \Phi_3 \cos \theta &= -P_1\Phi_2(x)P_1 + P_1\Phi_3P_1 \cos \theta, \\
\Phi_2(x) - \Phi_3 \cos \theta &= \hat{P}_2\Phi_2(x)\hat{P}_2 - \hat{P}_2\Phi_3\hat{P}_2 \cos \theta, \\
\Psi(x) &= \gamma_5 P_1 \Psi(x), \\
\Psi(x) &= \hat{P}_2 \Psi(x).
\end{aligned} \tag{2.34}$$

We find that the gauge fields, scalar fields, and massless fermions in 4D spacetime should be even for $P_1 A_\mu P_1$ and $\hat{P}_2 A_\mu \hat{P}_2$; $-P_1 \Phi_{1,2} P_1$ and $\hat{P}_2 \Phi_{1,2} \hat{P}_2$; $\gamma_5 P_1 \Psi$ and $\hat{P}_2 \Psi$, respectively. Φ_3 always remains in the spectrum because it is proportional to the $U(1)_I$ generator and commutes with $P(P')$. Therefore, the particle spectrum contains those satisfying both the constraint equations (2.20), (2.21), and (2.23) and the parity conditions (2.34). The remaining 4D gauge symmetry can be readily identified by observing which components of the gauge field remain in the spectrum.

2.3. Dimensional Reduction Scheme without the Symmetry Condition

Here we review the dimensional reduction scheme which does not require the imposition of the symmetry condition on the gauge field [22].

2.3.1. Background Gauge Field and Gauge Group Reduction

Instead of utilizing the symmetry condition, we consider the background gauge field $A_\phi^B \equiv \tilde{A}_\phi^B \sin \theta$ that corresponds to a Dirac monopole [36]

$$\tilde{A}_\phi^B = -Q \frac{\cos \theta \mp 1}{\sin \theta}, \quad \left(- : 0 \leq \theta < \frac{\pi}{2}, + : \frac{\pi}{2} \leq \theta \leq \pi \right), \tag{2.35}$$

where Q is proportional to the generator of a $U(1)$ subgroup of the original gauge group. The background gauge field A_ϕ^B corresponds to $\Phi_3 \cos \theta \subset A_\phi$ in (2.18).

Here we choose the background gauge field to belong to the $U(1)_I$ group, which is a subgroup of original gauge group G :

$$G \supset G_{\text{sub}} \otimes U(1)_I. \quad (2.36)$$

We find that there is no massless mode for gauge field components with a nonzero $U(1)_I$ charge. In fact, these components acquire masses due to the background field from the term proportional to $F_{\mu\phi}F_\phi^\mu$:

$$\begin{aligned} & \text{Tr} \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2R^2 \sin^2 \theta} F_{\mu\phi} F_\phi^\mu \right] \\ & \longrightarrow \text{Tr} \left[-\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu) (\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{2R^2 \sin^2 \theta} [A_\mu, A_\phi^B] [A^\mu, A_\phi^B] \right]. \end{aligned} \quad (2.37)$$

For the components of A_μ with nonzero $U(1)_I$ charge, we have

$$A_\mu^i Q_i + A_{i\mu} Q^i \in A_\mu, \quad (2.38)$$

where Q_i ($Q^i = Q_i^\dagger$) are generators corresponding to distinct components in (3.30) that have nonzero $U(1)_I$ charges, and $A_{i\mu}$ ($A_\mu^i = A_{i\mu}^\dagger$) are the corresponding components of A_μ . We find the term

$$\begin{aligned} \frac{1}{\sin^2 \theta} \text{Tr} \left[[A_\mu, A_\phi^B] [A^\mu, A_\phi^B] \right] &= \frac{(\cos \theta \mp 1)^2}{\sin^2 \theta} \text{Tr} \left[[A_\mu^i Q_i + A_{i\mu} Q^i, Q] [A^{i\mu} Q_i + A_i^\mu Q^i, Q] \right] \\ &= -2|q|^2 \frac{(\cos \theta \mp 1)^2}{\sin^2 \theta} A^{i\mu} A_{i\mu}, \end{aligned} \quad (2.39)$$

where q is the Q charge of the relevant component. Use of the facts that A_ϕ^B belongs to $U(1)_I$ and that $\text{Tr}[Q_i Q^i] = 2$ has been made in the above equation. A mass is thus associated with the lowest modes of those components of A_μ with nonzero $U(1)_I$ charges:

$$\begin{aligned} & \int d\Omega \text{Tr} \left[-\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu) (\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{2R^2 \sin^2 \theta} [A_\mu, A_B] [A^\mu, A_B] \right] \Big|_{\text{lowest}} \\ & \longrightarrow -\frac{1}{2} [\partial_\mu A_{i\nu}(x) - \partial_\nu A_{i\mu}(x)] [\partial^\mu A^{i\nu}(x) - \partial^\nu A^{i\mu}(x)] + m_B^2 A_\mu^i(x) A^{i\mu}(x), \end{aligned} \quad (2.40)$$

where the subscript “lowest” means that only the lowest KK modes are kept. Here the lowest KK modes of A_μ correspond to the term $A_\mu(x)/\sqrt{4\pi}$ in the KK expansion. In short, any representation of A_μ carrying a nonzero $U(1)_I$ charge acquires a mass m_B from

the background field contribution after one integrates over the extra spatial coordinates. More explicitly,

$$m_B^2 = \frac{|q|^2}{4\pi R^2} \int d\Omega \frac{(\cos \theta \mp 1)^2}{\sin^2 \theta} \simeq 0.39 \frac{|q|^2}{R^2} \quad (2.41)$$

for the zero mode. Therefore the gauge group G is reduced to $G_{\text{sub}} \otimes U(1)_I$ by the presence of the background gauge field. This condition is the same as the case with the symmetry condition.

2.3.2. Scalar Field Contents in 4D Spacetime

The scalar contents in 4D spacetime are obtained from the extradimensional components of the gauge field $\{A_\theta, A_\phi\}$ after integrating out the extra spatial coordinates. The kinetic term and potential term of $\{A_\theta, A_\phi\}$ are obtained from the gauge sector containing these components

$$\begin{aligned} S_{\text{scalar}} &= \int dx^4 d\Omega \left(\frac{1}{2g^2} \text{Tr} [F_{\mu\theta} F_{\theta}^\mu] + \frac{1}{2g^2 \sin^2 \theta} \text{Tr} [F_{\mu\phi} F_{\phi}^\mu] - \frac{1}{2g^2 R^2 \sin^2 \theta} \text{Tr} [F_{\theta\phi} F_{\theta\phi}] \right) \\ &\longrightarrow \int dx^4 d\Omega \left(\frac{1}{2g^2} \text{Tr} [(\partial_\mu A_\theta - i[A_\mu, A_\theta])^2] + \frac{1}{2g^2} \text{Tr} [(\partial_\mu \tilde{A}_\phi - i[A_\mu, \tilde{A}_\phi])^2] \right) \\ &\quad - \frac{1}{2g^2 R^2} \text{Tr} \left[\left(\frac{1}{\sin \theta} \partial_\theta (\sin \theta \tilde{A}_\phi + \sin \theta \tilde{A}_\phi^B) - \frac{1}{\sin \theta} \partial_\phi A_\theta - i[A_\theta, \tilde{A}_\phi + \tilde{A}_\phi^B] \right)^2 \right], \end{aligned} \quad (2.42)$$

where we have taken $A_\phi = \tilde{A}_\phi \sin \theta + \tilde{A}_\phi^B \sin \theta$. In the second step indicated by the arrow in (2.42), we have omitted terms which do not involve A_θ and $\tilde{A}_\phi$ from the right-hand side of the first equality. It is known that one generally cannot obtain massless modes for physical scalar components in 4D spacetime [14, 38]. One can see this by noting that the eigenfunction of the operator $(1/\sin \theta) \partial_\theta \sin \theta$ with zero eigenvalue is not normalizable [14]. In other words, these fields have only KK modes. However, an interesting feature is that it is possible to obtain a negative squared mass when taking into account the interactions between the background gauge field $\tilde{A}_\phi^B$ and $\{A_\theta, \tilde{A}_\phi\}$. This happens when the component carries a nonzero $U(1)_I$ charge, as the background gauge field belongs to $U(1)_I$. In this case, the $(l=1, m=1)$ modes of these real scalar components are found to have a negative squared mass in 4D spacetime. They can be identified as the Higgs fields once they are shown to belong to the correct representation under the SM gauge group. Here the numbers (ℓ, m) are the angular momentum quantum number on S^2/Z_2 , and each KK mode is characterized by these numbers. One can show that the $(l=1, m=0)$ mode has a positive squared mass and is not considered as the Higgs field. A discussion of the KK masses with general (ℓ, m) will be given in Section 3.2.5.

2.3.3. Chiral Fermions in 4D Spacetime

We introduce fermions as the Weyl spinor fields of the 6D Lorentz group $SO(1,5)$. They can be written in terms of the $SO(1,3)$ Weyl spinors as (2.8) and (2.9). In general, fermions on the two spheres do not have massless KK modes because of the positive curvature of the two spheres. The massless modes can be obtained by incorporating the background gauge field (2.35) though, for it can cancel the contribution from the positive curvature. In this case, the condition for obtaining a massless fermion mode is

$$Q\Psi = \pm \frac{1}{2}\Psi, \quad (2.43)$$

where Q comes from the background gauge field and is proportional to the $U(1)_I$ generator [35, 36, 38]. We observe that the upper [lower] component on the RHS of (2.8) [(2.9)] has a massless mode for the + [-] sign on the RHS of (2.43).

2.3.4. The Higgs Potential

The Lagrangian for the Higgs sector is derived from the gauge sector that contains extradimensional components of the gauge field $\{A_\theta, \tilde{A}_\phi\}$, as given in (2.42), by considering the lowest KK modes of them. The kinetic term and potential term are, respectively,

$$L_K = \frac{1}{2g^2} \int d\Omega \left(\text{Tr} \left[(\partial_\mu A_\theta - i[A_\mu, A_\theta])^2 \right] + \text{Tr} \left[(\partial_\mu \tilde{A}_\phi - i[A_\mu, \tilde{A}_\phi])^2 \right] \right) \Big|_{\text{lowest}},$$

$$V = \frac{1}{2g^2 R^2} \int d\Omega \text{Tr} \left[\left(\frac{1}{\sin \theta} \partial_\theta (\sin \theta \tilde{A}_\phi + \sin \theta \tilde{A}_\phi^B) - \frac{1}{\sin \theta} \partial_\phi A_\theta - i[A_\theta, \tilde{A}_\phi + \tilde{A}_\phi^B] \right)^2 \right] \Big|_{\text{lowest}}. \quad (2.44)$$

In our model, scalar components other than the Higgs field have vanishing VEV because only the Higgs field has a negative mass-squared term, coming from the interaction with the background gauge field at tree level. Therefore, only the Higgs field contributes to the spontaneous symmetry breaking. Consider the $(1,1)$ mode of the $\{(1,2)(3,-3,3) + \text{h.c.}\}$ representation in (3.31) as argued in the previous section. The gauge fields are given by the following KK expansions:

$$A_\theta = -\frac{1}{\sqrt{2}} \left[\Phi_1(x) \partial_\theta Y_{11}^-(\theta, \phi) + \Phi_2(x) \frac{1}{\sin \theta} \partial_\phi Y_{11}^-(\theta, \phi) \right] + \dots, \quad (2.45)$$

$$\tilde{A}_\phi = \frac{1}{\sqrt{2}} \left[\Phi_2(x) \partial_\theta Y_{11}^-(\theta, \phi) - \Phi_1(x) \frac{1}{\sin \theta} \partial_\phi Y_{11}^-(\theta, \phi) \right] + \dots, \quad (2.46)$$

where $\dots$ represents higher KK mode terms [35]. The function $Y_{11}^- = -1/\sqrt{2}[Y_{11} + Y_{1-1}]$ is odd under $(\theta, \phi) \rightarrow (\pi/2 - \theta, -\phi)$. We will discuss their higher KK modes and masses in

the existence of the background gauge field in Section 3.2.5. With (2.45) and (2.46), the kinetic term becomes

$$L_K(x) = \frac{1}{2g^2} (\text{Tr}[D_\mu \Phi_1(x) D^\mu \Phi_1(x)] + \text{Tr}[D_\mu \Phi_2(x) D^\mu \Phi_2(x)]), \quad (2.47)$$

where $D_\mu \Phi_{1,2} = \partial_\mu \Phi_{1,2} - i[A_\mu, \Phi_{1,2}]$ is the covariant derivative acting on $\Phi_{1,2}$. The potential term, on the other hand, is

$$\begin{aligned} V = \frac{1}{2g^2 R^2} \int d\Omega \text{Tr} \left[\left(-\sqrt{2} Y_{11}^- \Phi_2(x) + Q \right. \right. \\ \left. \left. + \frac{i}{2} [\Phi_1(x), \Phi_2(x)] \left\{ \partial_\theta Y_{11}^- \partial_\theta Y_{11}^- + \frac{1}{\sin^2 \theta} \partial_\phi Y_{11}^- \partial_\phi Y_{11}^- \right\} \right. \right. \\ \left. \left. + \frac{i}{\sqrt{2}} [\Phi_1(x), \tilde{A}_\phi^B] \partial_\theta Y_{11}^- + \frac{i}{\sqrt{2}} [\Phi_2(x), \tilde{A}_\phi^B] \frac{1}{\sin \theta} \partial_\phi Y_{11}^- \right)^2 \right], \end{aligned} \quad (2.48)$$

where $\partial_\theta (\sin \theta \tilde{A}_\phi^B) = Q \sin \theta$ from (2.35) is used. Expanding the square in the trace, we get

$$\begin{aligned} V = \frac{1}{2g^2 R^2} \int d\Omega \text{Tr} \left[2(Y_{11}^+)^2 \Phi_2^2(x) + Q^2 \right. \\ \left. - \frac{1}{4} [\Phi_1(x), \Phi_2(x)]^2 \left(\partial_\theta Y_{11}^- \partial_\theta Y_{11}^- + \frac{1}{\sin^2 \theta} \partial_\phi Y_{11}^- \partial_\phi Y_{11}^- \right)^2 \right. \\ \left. - \frac{1}{2} [\Phi_1(x), \tilde{A}_\phi^B]^2 (\partial_\theta Y_{11}^-)^2 - \frac{1}{2} [\Phi_2(x), \tilde{A}_\phi^B]^2 \left(\frac{1}{\sin \theta} \partial_\phi Y_{11}^- \right)^2 \right. \\ \left. - 2i\Phi_2(x) [\Phi_1(x), \tilde{A}_\phi^B] Y_{11}^- \partial_\theta Y_{11}^- - [\Phi_1(x), \tilde{A}_\phi^B] [\Phi_2(x), \tilde{A}_\phi^B] \partial_\theta Y_{11}^- \frac{1}{\sin \theta} \partial_\phi Y_{11}^- \right. \\ \left. + iQ [\Phi_1(x), \Phi_2(x)] \left(\partial_\theta Y_{11}^- \partial_\theta Y_{11}^- + \frac{1}{\sin^2 \theta} \partial_\phi Y_{11}^- \partial_\phi Y_{11}^- \right) \right], \end{aligned} \quad (2.49)$$

where terms that vanish after the $d\Omega$ integration are directly omitted. In the end, the potential is simplified to

$$\begin{aligned} V = \frac{1}{2g^2 R^2} \text{Tr} \left[2\Phi_2^2(x) + 4\pi Q^2 - \frac{3}{10\pi} [\Phi_1(x), \Phi_2(x)]^2 + \frac{5i}{2} Q [\Phi_1(x), \Phi_2(x)] \right. \\ \left. + \mu_1 [Q, \Phi_1(x)]^2 + \mu_2 [Q, \Phi_2(x)]^2 \right], \end{aligned} \quad (2.50)$$

where use of $\tilde{A}_\phi^B = -Q(\cos \theta \mp 1)/\sin \theta$ has been made and $\mu_1 = 1 - (3/2) \ln 2$ and $\mu_2 = (3/4)(1 - 2 \ln 2)$.

We now take the following linear combination of Φ_1 and Φ_2 to form a complex Higgs doublet,

$$\Phi(x) = \frac{1}{\sqrt{2}}(\Phi_1(x) + i\Phi_2(x)), \quad (2.51)$$

$$\Phi(x)^\dagger = \frac{1}{\sqrt{2}}(\Phi_1(x) - i\Phi_2(x)). \quad (2.52)$$

It is straightforward to see that

$$[\Phi_1(x), \Phi_2(x)] = i[\Phi(x), \Phi^\dagger(x)]. \quad (2.53)$$

The kinetic term and the Higgs potential now become

$$L_K = \frac{1}{g^2} \text{Tr} [D_\mu \Phi^\dagger(x) D^\mu \Phi(x)], \quad (2.54)$$

$$V = \frac{1}{2g^2 R^2} \text{Tr} \left[2\Phi_2^2(x) + 4\pi Q^2 + \frac{3}{10\pi} [\Phi(x), \Phi^\dagger(x)]^2 - \frac{5}{2} Q [\Phi(x), \Phi^\dagger(x)] \right. \\ \left. + \mu_1 [Q, \Phi_1(x)]^2 + \mu_2 [Q, \Phi_2(x)]^2 \right]. \quad (2.55)$$

The last three terms in the potential are contributions to the squared mass term of the Higgs boson from the background gauge field and can lead to a negative value. This means that the existence of the background gauge field makes the minimum of Higgs potential lower.

3. The Models Based on Our Schemes

In this section, we show concrete models based on the scheme introduced in previous section. We review the model based on $\text{SO}(12)$ gauge symmetry for the scheme with symmetry condition given in [21], and review the model based on E_6 gauge symmetry for the scheme without symmetry condition given in [22].

3.1. The $\text{SO}(12)$ Model with Symmetry Condition

Here we show a model based on a gauge group $G = \text{SO}(12)$ and a representation $F = 32$ of $\text{SO}(12)$ for fermions, under the scheme with symmetry condition [21]. The choice of $G = \text{SO}(12)$ and $F = 32$ is motivated by the study based on CSDR which leads to an $\text{SO}(10) \times \text{U}(1)$ gauge theory with one generation of fermion in 4D spacetime [28] (for $\text{SO}(12)$ GUT see also [39]).

3.1.1. A Gauge Symmetry and Particle Contents

First, we show the particle contents in 4D spacetime without parities equations (2.13)–(2.15). We assume that $U(1)_I$ is embedded into $SO(12)$ such as

$$SO(12) \supset SO(10) \times U(1)_I. \quad (3.1)$$

Thus we identify $SO(10) \times U(1)_I$ as the gauge group which satisfy the constraint equations (2.20), using (2.28). The $SO(12)$ gauge group is reduced to $SO(10) \times U(1)$ by the symmetry condition. We identify the scalar components which satisfy (2.21) by decomposing adjoint representation of $SO(12)$:

$$SO(12) \supset SO(10) \times U(1)_I : 66 = 45(0) + 1(0) + 10(2) + 10(-2). \quad (3.2)$$

According to the prescription below (2.28) in Section 2, the scalar components $10(2) + 10(-2)$ remains in 4D spacetime. We also identify the fermion components which satisfy (2.23) by decomposing 32 representations of $SO(12)$ as

$$SO(12) \supset SO(10) \times U(1)_I : 32 = 16(1) + \overline{16}(-1). \quad (3.3)$$

According to the prescription below (2.30) in Section 2, we have the fermion components as $16(1)$ for a left-handed fermion and $\overline{16}(-1)$ for a right-handed fermion, respectively, in 4D spacetime.

Next, we specify the parity assignment of $P_1(\hat{P}_2)$ in order to identify the gauge symmetry and the particle contents that actually remain in 4D spacetime. We choose a parity assignment so as to break gauge symmetry as $SO(12) \supset SO(10) \times U(1)_I \supset SU(5) \times U(1)_X \times U(1)_I \supset SU(3) \times SU(2)_L \times U(1)_Y \times U(1)_X \times U(1)_I$ and to maintain Higgs-doublet in 4D spacetime. The parity assignment is written in 32 dimensional spinor basis of $SO(12)$ such as

$$\begin{aligned} SO(12) \supset SU(3) \times SU(2)_L \times U(1)_Y \times U(1)_X \times U(1)_I, \\ 32 = (3, 2)^{(+)}(1, -1, 1) + (\bar{3}, 2)^{(-)}(-1, 1, -1) \\ + (3, 1)^{(-)}(4, 1, -1) + (\bar{3}, 1)^{(+)}(-4, -1, 1) \\ + (3, 1)^{(+)}(-2, -3, -1) + (\bar{3}, 1)^{(-)}(2, 3, 1) \\ + (1, 2)^{(+)}(3, -3, -1) + (1, 2)^{(-)}(-3, 3, 1) \\ + (1, 1)^{(-)}(6, -1, 1) + (1, 1)^{(+)}(-6, 1, -1) \\ + (1, 1)^{(-)}(0, -5, 1) + (1, 1)^{(+)}(0, 5, -1), \end{aligned} \quad (3.4)$$

where for example, $(+, -)$ means that the parities $(P_1, \hat{P}_2)$ of the associated components are (even, odd). We find the gauge symmetry in 4D spacetime by surveying parity assignment

for the gauge field. The parity assignments of the gauge field under $A_\mu \rightarrow P_1 A_\mu P_1 (\hat{P}_2 A_\mu \hat{P}_2)$ are

$$\begin{aligned}
66 = & (8, 1)^{++}(0, 0, 0) + (1, 3)^{++}(0, 0, 0) + (1, 1)^{++}(0, 0, 0) \\
& + (1, 1)^{++}(0, 0, 0) + (1, 1)^{++}(0, 0, 0) \\
& + \left[(3, 2)^{(-+)}(-5, 0, 0) + (\bar{3}, 2)^{(-+)}(5, 0, 0) \right. \\
& + (3, 2)^{(--)}(1, 4, 0) + (\bar{3}, 2)^{(--)}(-1, -4, 0) \\
& + (3, 1)^{(+ -)}(4, -4, 0) + (\bar{3}, 1)^{(+ -)}(-4, 4, 0) \\
& + (3, 1)^{(+ -)}(-2, 2, 2) + (\bar{3}, 1)^{(+ -)}(2, -2, -2) \\
& \quad \underline{+ (3, 1)^{++}(-2, 2, -2) + (\bar{3}, 1)^{++}(2, -2, 2)} \\
& + \underline{(1, 2)^{(--)}(3, 2, 2) + (1, 2)^{(--)}(-3, -2, -2)} \\
& + \underline{(1, 2)^{(-+)}(3, 2, -2) + (1, 2)^{(-+)}(-3, -2, 2)} \\
& \left. + (1, 1)^{(+ -)}(6, 4, 0) + (1, 1)^{(+ -)}(-6, -4, 0) \right]. \tag{3.5}
\end{aligned}$$

The components with an underline are originated from $10(2)$ and $10(-2)$ of $SO(10) \times U(1)_I$, which do not satisfy constraint equations (2.20), and hence these components do not remain in 4D spacetime. Thus we have the gauge fields with $(+, +)$ parity components without an underline in 4D spacetime, and the gauge symmetry is $SU(3) \times SU(2)_L \times U(1)_Y \times U(1)_X \times U(1)_I$.

The scalar particle contents in 4D spacetime are determined by the parity assignments, under $\Phi_{1,2} \rightarrow -P_1 \Phi_{1,2} P_1$ and $\hat{P}_2 \Phi_{1,2} \hat{P}_2$:

$$\begin{aligned}
66 = & (8, 1)^{(-+)}(0, 0, 0) + (1, 3)^{(-+)}(0, 0, 0) + (1, 1)^{(-+)}(0, 0, 0) \\
& + (1, 1)^{(-+)}(0, 0, 0) + (1, 1)^{(-+)}(0, 0, 0) \\
& + \left[(3, 2)^{(+ +)}(-5, 0, 0) + (\bar{3}, 2)^{(+ +)}(5, 0, 0) \right. \\
& + (3, 2)^{(+ -)}(1, 4, 0) + (\bar{3}, 2)^{(+ -)}(-1, -4, 0) \\
& + (3, 1)^{(--)}(4, -4, 0) + (\bar{3}, 1)^{(--)}(-4, 4, 0) \\
& + (3, 1)^{(--)}(-2, 2, 2) + (\bar{3}, 1)^{(--)}(2, -2, -2) \\
& \quad \underline{+ (3, 1)^{(-+)}(-2, 2, -2) + (\bar{3}, 1)^{(-+)}(2, -2, 2)} \\
& \left. \right].
\end{aligned}$$

$$\begin{aligned}
& + \underline{(1,2)^{+-}(3,2,2)} + \underline{(1,2)^{+-}(-3,-2,-2)} \\
& + \underline{(1,2)^{++}(3,2,-2)} + \underline{(1,2)^{++}(-3,-2,2)} \\
& + (1,1)^{--}(6,4,0) + (1,1)^{--}(-6,-4,0) \Big].
\end{aligned} \tag{3.6}$$

Note that the relative sign for the parity assignment of P_1 is different from (3.5), and that the only underlined parts satisfy the constraint equations (2.21). Thus the scalar components in 4D spacetime are $(1,2)(3,2,-2)$ and $(1,2)(-3,-2,2)$.

We specify the massless fermion contents in 4D spacetime, by surveying the parity assignments for each components of fermion fields. We introduce two types of left-handed Weyl fermions that belong to 32 representation of $SO(12)$, which have parity assignments $\psi^{(\hat{P}_2)} \rightarrow \gamma_5 P_1 \psi^{(\hat{P}_2)} (\hat{P}_2 \psi^{(\hat{P}_2)})$ and $\psi^{(-\hat{P}_2)} \rightarrow \gamma_5 P_1 \psi^{(-\hat{P}_2)} (-\hat{P}_2 \psi^{(-\hat{P}_2)})$, respectively. They have the parity assignments as

$$\begin{aligned}
32_L^{(\hat{P}_2)} &= \underline{(3,2)^{--}(1,-1,1)_L} + \underline{(\bar{3},2)^{--}(-1,1,-1)_L} \\
&+ \underline{(\bar{3},1)^{+-}(-4,-1,1)_L} + \underline{(3,1)^{+-}(4,1,-1)_L} \\
&+ \underline{(\bar{3},1)^{++}(2,3,1)_L} + \underline{(3,1)^{++}(-2,-3,-1)_L} \\
&+ \underline{(1,2)^{+-}(-3,3,1)_L} + \underline{(1,2)^{+-}(3,-3,-1)_L} \\
&+ \underline{(1,1)^{+-}(6,-1,1)_L} + \underline{(1,1)^{+-}(-6,1,-1)_L} \\
&+ \underline{(1,1)^{++}(0,-5,1)_L} + \underline{(1,1)^{++}(0,5,-1)_L}, \\
32_R^{(\hat{P}_2)} &= \underline{(3,2)^{+-}(1,-1,1)_R} + \underline{(\bar{3},2)^{+-}(-1,1,-1)_R} \\
&+ \underline{(\bar{3},1)^{--}(-4,-1,1)_R} + \underline{(3,1)^{--}(4,1,-1)_R} \\
&+ \underline{(\bar{3},1)^{+-}(2,3,1)_R} + \underline{(3,1)^{+-}(-2,-3,-1)_R} \\
&+ \underline{(1,2)^{++}(-3,3,1)_R} + \underline{(1,2)^{++}(3,-3,-1)_R} \\
&+ \underline{(1,1)^{--}(6,-1,1)_R} + \underline{(1,1)^{--}(-6,1,-1)_R} \\
&+ \underline{(1,1)^{+-}(0,-5,1)_R} + \underline{(1,1)^{+-}(0,5,-1)_R}, \\
32_L^{(-\hat{P}_2)} &= \underline{(3,2)^{+-}(1,-1,1)_L} + \underline{(\bar{3},2)^{+-}(-1,1,-1)_L} \\
&+ \underline{(\bar{3},1)^{++}(-4,-1,1)_L} + \underline{(3,1)^{++}(4,1,-1)_L}
\end{aligned}$$

$$\begin{aligned}
& + \underline{(\bar{3}, 1)^{(+ -)}} (2, 3, 1)_L + (3, 1)^{(+ -)} (-2, -3, -1)_L \\
& + \underline{(1, 2)^{(- -)}} (-3, 3, 1)_L + (1, 2)^{(- -)} (3, -3, -1)_L \\
& + \underline{(1, 1)^{(+ +)}} (6, -1, 1)_L + (1, 1)^{(+ +)} (-6, 1, -1)_L \\
& + \underline{(1, 1)^{(+ -)}} (0, -5, 1)_L + (1, 1)^{(+ -)} (0, 5, -1)_L, \\
32_R^{(-\hat{P}_2)} & = (3, 2)^{(+ +)} (1, -1, 1)_R + \underline{(\bar{3}, 2)^{(+ +)}} (-1, 1, -1)_R \\
& + \underline{(\bar{3}, 1)^{(- +)}} (-4, -1, 1)_R + \underline{(3, 1)^{(- +)}} (4, 1, -1)_R \\
& + \underline{(\bar{3}, 1)^{(- +)}} (2, 3, 1)_R + \underline{(3, 1)^{(- +)}} (-2, -3, -1)_R \\
& + (1, 2)^{(+ -)} (-3, 3, 1)_R + \underline{(1, 2)^{(+ -)}} (3, -3, -1)_R \\
& + (1, 1)^{(- +)} (6, -1, 1)_R + \underline{(1, 1)^{(- +)}} (-6, 1, -1)_R \\
& + (1, 1)^{(- -)} (0, -5, 1)_R + \underline{(1, 1)^{(- -)}} (0, 5, -1)_R,
\end{aligned} \tag{3.7}$$

where $L(R)$ means the left-handedness (right-handedness) of fermions in 4D spacetime, and the underlined parts correspond to the components which satisfy constraint equations (2.23). Note the relative sign for parity assignment of P_1 between left-handed fermion and right-handed fermion and that of $\hat{P}_2$ between $32^{(\hat{P}_2)}$ and $32^{(-\hat{P}_2)}$. The difference between $32^{(\hat{P}_2)}$ and $32^{(-\hat{P}_2)}$ is allowed because of the bilinear form of the fermion sector. We thus find that the massless fermion components in 4D spacetime are one generation of SM-fermions with right-handed neutrino: $\{(3, 2)(1, -1, 1)_L, (3, 1)(4, 1, -1)_R, (3, 1)(-2, -3, -1)_R, (1, 2)(-3, 3, 1)_L, (1, 1)(-6, 1, -1)_R, (1, 1)(0, 5, -1)_R\}$.

3.1.2. The Higgs Sector of the Model

We analyze the Higgs-sector of our model. The Higgs-sector L_{Higgs} is the last two terms of (2.22)

$$\begin{aligned}
L_{\text{Higgs}} & = -\frac{1}{2g^2} \text{Tr} \left[D'_\mu \Phi_1(x) D'^\mu \Phi_1(x) + D'_\mu \Phi_2(x) D'^\mu \Phi_2(x) \right] \\
& - \frac{1}{2g^2 R^2} \text{Tr} [(\Phi_3 + [\Phi_1(x), \Phi_2(x)])(\Phi_3 + [\Phi_1(x), \Phi_2(x)])],
\end{aligned} \tag{3.8}$$

where the first term of RHS is the kinetic term of Higgs and the second term gives the Higgs potential. We rewrite the Higgs-sector in terms of genuine Higgs field in order to analyze it.

We first note that the Φ_i s are written as

$$\Phi_i = i\phi_i = i\phi_i^a Q_a, \quad (3.9)$$

where Q_a s are generators of gauge group $SO(12)$, since Φ_i s are originated from gauge fields $A_a = iA_a^a Q_a$; for the gauge group generator we assume the normalization $\text{Tr}(Q_a Q_b) = -2\delta_{ab}$. Note that we assumed the $-i\Phi_3$ as the generator of $U(1)_I$ embedded in $SO(12)$,

$$-i\Phi_3 = Q_I. \quad (3.10)$$

We change the notation of the scalar fields according to (2.29) such that,

$$\phi_+ = \frac{1}{2}(\phi_1 + i\phi_2), \quad \phi_- = \frac{1}{2}(\phi_1 - i\phi_2), \quad (3.11)$$

in order to express solutions of the constraint equations (2.21) clearly. The constraint equations (2.21) then rewritten as

$$[Q_I, \phi_+] = \phi_+, \quad [Q_I, \phi_-] = -\phi_-. \quad (3.12)$$

The kinetic term L_{KE} and potential $V(\phi)$ term are rewritten in terms of ϕ_+ and ϕ_- :

$$L_{KE} = -\frac{1}{g^2} \text{Tr} \left[D'_\mu \phi_+(x) D'^\mu \phi_-(x) \right], \quad (3.13)$$

$$V = -\frac{1}{2g^2 R^2} \text{Tr} \left[Q_I^2 - 4Q_I [\phi_+, \phi_-] + 4[\phi_+, \phi_-] [\phi_+, \phi_-] \right], \quad (3.14)$$

where covariant derivative D'_μ is $D'_\mu \phi_\pm = \partial_\mu \phi_\pm - [A_\mu, \phi_\pm]$.

Next, we change the notation of $SO(12)$ generators Q_a according to decomposition (3.5) such that

$$\begin{aligned} Q_G = \Big\{ & Q_i, Q_a, Q_Y, Q, Q_I, Q_{ax(-500)}, Q^{ax(500)} \\ & Q_{ax(140)}, Q^{ax(-1-40)}, Q_{a(4-40)}, Q^{a(-440)} \\ & Q_{a(-22-2)}, Q^{a(2-22)}, Q_{a(-222)}, Q^{a(2-2-2)} \\ & Q_{x(322)}, Q^{x(-3-2-2)}, Q_{x(32-2)}, Q^{x(-3-22)} \\ & Q_{(640)}, Q_{(-6-40)} \Big\}, \end{aligned} \quad (3.15)$$

Table 1: Commutation relations of $Q^{x(-3-22)}$, $Q_{x(32-2)}$, Q_α , Q_Y , Q , and Q_I .

$[Q^{x(-3-22)}, Q_{y(32-2)}] = -\sqrt{\frac{3}{10}}\delta_y^x Q_Y + -\sqrt{\frac{1}{5}}\delta_y^x Q + \delta_y^x Q_I + \frac{1}{\sqrt{2}}(\sigma_\alpha^*)_y^x Q_\alpha$	
$[Q_\alpha, Q_x] = -\frac{1}{\sqrt{2}}(\sigma_\alpha)_x^y Q_y,$	$[Q_\alpha, Q^x] = \frac{1}{\sqrt{2}}(\sigma_\alpha^*)_y^x Q^y,$
$[Q_x, Q_y] = 0,$	$[Q_Y, Q^x] = -\sqrt{\frac{3}{10}}Q^x,$
$[Q, Q^x] = -\sqrt{\frac{1}{5}}Q^x,$	$[Q_I, Q^x] = Q^x,$

where the order of generators corresponds to (3.5), index $i = 1-8$ corresponds to SU(3) adjoint rep, index $\alpha = 1-3$ corresponds to SU(2) adjoint rep, index $a = 1-3$ corresponds to SU(3)-triplet, and index $x = 1, 2$ corresponds to SU(2)-doublet. We write $\phi_\pm$ in terms of the genuine Higgs field ϕ_x which belongs to $(1, 2)(3, 2, -2)$, such that

$$\begin{aligned}\phi_+ &= \phi_x Q^{x(-3-22)}, \\ \phi_- &= \phi^x Q_{x(32-2)},\end{aligned}\tag{3.16}$$

where $\phi^x = (\phi_x)^\dagger$. We also write gauge field $A_\mu(x)$ in terms of Q s in (3.38) as

$$A_\mu(x) = i\left(A_\mu^i Q_i + A_\mu^\alpha Q_\alpha + B_\mu Q_Y + C_\mu Q + E_\mu Q_I\right).\tag{3.17}$$

We need commutation relations of $Q^{x(-3-22)}$, $Q_{x(32-2)}$, Q_α , Q_Y , Q , and Q_I in order to analyze the Higgs sector; we summarized them in Table 1.

Finally, we obtain the Higgs sector with genuine Higgs field by substituting (3.16)–(3.17) into (3.13) and (3.14) and rescaling the fields $\phi \rightarrow (g/\sqrt{2})\phi$ and $A_\mu \rightarrow (g/R)A_\mu$, and the couplings $(\sqrt{2}/R)g = g_2$ and $\sqrt{6/(5R^2)}g = g_Y$,

$$L_{\text{Higgs}} = |D_\mu \phi_x|^2 - V(\phi),\tag{3.18}$$

where the covariant derivative $D_\mu \phi_x$ and potential $V(\phi)$ are

$$D_\mu \phi_x = \partial_\mu \phi_x + ig_2 \frac{1}{2}(\sigma_\alpha)_x^y A_{\alpha\mu} \phi_y + ig_Y \frac{1}{2} B_\mu \phi_x + i\sqrt{\frac{1}{5}}g C_\mu \phi_x - ig E_\mu \phi_x,\tag{3.19}$$

$$V = -\frac{2}{R^2} \phi^x \phi_x + \frac{3g^2}{2R^2} (\phi^x \phi_x)^2,\tag{3.20}$$

respectively. Notice that we omitted the constant term in the Higgs potential. We note that the $SU(2)_L \times U(1)_Y$ part of the Higgs sector has the same form as the SM Higgs sector. Therefore

we obtain the electroweak symmetry breaking $SU(2)_L \times U(1)_Y \rightarrow U(1)_{EM}$. The Higgs field ϕ^x acquires vacuum expectation value (VEV) as

$$\begin{aligned}\langle \phi \rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \\ v &= \sqrt{\frac{4}{3}} \frac{1}{g'},\end{aligned}\tag{3.21}$$

and W boson mass m_W and Higgs mass m_H are given in terms of radius R

$$\begin{aligned}m_W &= g_2 \frac{v}{2} = \sqrt{\frac{2}{3}} \frac{1}{R}, \\ m_H &= \sqrt{3} \frac{g' v}{R} = \sqrt{4} \frac{1}{R}.\end{aligned}\tag{3.22}$$

The ratio between m_W and m_H is predicted

$$\frac{m_H}{m_W} = \sqrt{6}.\tag{3.23}$$

We thus find $m_H \sim 196$ GeV in this model. The Weinberg angle is given by

$$\sin^2 \theta_W = \frac{g_Y^2}{g_2^2 + g_Y^2} = \frac{3}{8},\tag{3.24}$$

which is same as $SU(5)$ GUT case. The prediction for the Weinberg angle at tree level is not consistent with the electroweak measurements. One should also take into account quantum corrections including contributions from the KK modes. It is, however, beyond the scope of this paper.

In principle, one-loop power divergences in the Higgs potential would reappear since the operator linear in F_{ab} is allowed, where $\{a, b\}$ denote extraspatial components [40]. Such an operator would have the form

$$F_{\theta\phi}^\alpha(x),\tag{3.25}$$

where α corresponds to the index of the $U(1)$ generator remaining in 4D. This operator is potentially dangerous since its coefficient can be divergent. We can readily avoid this by requiring parity invariance on S^2/Z_2 as in the T^2/Z_2 case [41].

First, consider the parity transformation $\theta \rightarrow \pi - \theta$. The parity conditions for the fields are defined as

$$\begin{aligned}
 A_\mu(x, \theta, \phi) &\longrightarrow A_\mu(x, \pi - \theta, \phi), \\
 A_\theta(x, \theta, \phi) &\longrightarrow -A_\theta(x, \pi - \theta, \phi), \\
 A_\phi(x, \theta, \phi) &\longrightarrow A_\phi(x, \pi)(x, \pi - \theta, \phi), \\
 \Psi(x, \theta, \phi) &\longrightarrow \pm \Gamma^4 \Psi(x, \pi - \theta, \phi),
 \end{aligned} \tag{3.26}$$

where $\Gamma^4 = \gamma_5 \otimes \sigma_1$. It is easy to see that the action in 6D, (2.4), is invariant under such a parity transformation.

Secondly, we check the consistency between the orbifold boundary conditions on S^2/Z_2 , (2.10)–(2.12), and the parity conditions, (3.26). By performing the parity transformation on both sides of the orbifold boundary conditions, (2.10)–(2.12), we obtain

$$\begin{aligned}
 A_\mu(x, \theta, -\phi) &= P_1 A_\mu(x, \pi - \theta, \phi) P_1, \\
 -A_\theta(x, \theta, -\phi) &= P_1 A_\theta(x, \pi - \theta, \phi) P_1, \\
 A_\phi(x, \theta, -\phi) &= -P_1 A_\phi(x, \pi - \theta, \phi) P_1, \\
 \pm \Gamma^4 \Psi(x, \theta, -\phi) &= \pm \gamma_5 P_1 (\pm \Gamma^4) \Psi(x, \pi - \theta, \phi), \\
 A_\mu(x, \pi - \theta, \phi + 2\pi) &= \hat{P}_2 A_\mu(x, \pi - \theta, \phi) P_2, \\
 -A_\theta(x, \pi - \theta, \phi + 2\pi) &= \hat{P}_2 A_\theta(x, \pi - \theta, \phi) P_2, \\
 A_\phi(x, \pi - \theta, \phi + 2\pi) &= -\hat{P}_2 A_\phi(x, \pi - \theta, \phi) P_2, \\
 \pm \Gamma^4 \Psi(x, \pi - \theta, \phi + 2\pi) &= \pm \gamma_5 \hat{P}_2 (\pm \Gamma^4) \Psi(x, \pi - \theta, \phi).
 \end{aligned} \tag{3.27}$$

Since (2.10)–(2.12) hold for any θ and ϕ and Γ^4 commutes with γ_5 , we find that the orbifold boundary conditions still hold under the parity transformation with the identification of $\theta = \pi - \theta'$. In other words, the orbifold boundary conditions, (2.10)–(2.12), are parity invariant.

Finally, we find that under the parity, the operator $F_{\theta\phi}^\alpha$ transforms to $-F_{\theta\phi}^\alpha$. Therefore, this operator is forbidden by parity invariance of the action. An explicit calculation of one-loop corrections to the Higgs potential to show that this operator vanishes, however, is beyond the scope of this paper.

3.2. The E_6 Model without Symmetry Condition

Here we show a model based on a gauge group $G = E_6$ with a representation **27** for a fermion, under the scheme without symmetry condition [22].

3.2.1. Gauge Group Reduction

We consider the following gauge group reduction

$$\begin{aligned}
 E_6 &\supset SO(10) \times U(1)_I \\
 &\supset SU(5) \times U(1)_X \times U(1)_I \\
 &\supset SU(3) \times SU(2) \times U(1)_Y \times U(1)_X \times U(1)_I.
 \end{aligned} \tag{3.28}$$

The background gauge field in (2.35) is chosen to belong to the $U(1)_I$ group. This choice is needed in order to obtain chiral SM fermions in 4D spacetime to be discussed later. There are two other symmetry reduction schemes. One can prove that the results in those two schemes are effectively the same as the one considered here once we require the correct $U(1)$ combinations for the hypercharge and the background field.

We then impose the parity assignments with respect to the fixed points, (2.10)–(2.15). The parity assignments for the fundamental representation of E_6 is chosen to be

$$\begin{aligned}
 \mathbf{27} = & (1,2)(-3,-2,-2)^{(+,+)} + (1,2)(3,2,-2)^{(-,-)} + (1,2)(-3,3,1)^{(+,-)} \\
 & + (1,1)(6,-1,1)^{(+,+)} + (1,1)(0,-5,1)^{(-,-)} + (1,1)(0,0,4)^{(-,+)} \\
 & + (3,2)(1,-1,1)^{(-,+)} + (3,1)(-2,2,-2)^{(+,-)} + \left(\bar{3},1\right)(-4,-1,1)^{(+,+)} \\
 & + \left(\bar{3},1\right)(2,3,1)^{(+,+)} + \left(\bar{3},1\right)(2,-2,-2)^{(-,+)},
 \end{aligned} \tag{3.29}$$

where, for example, $(+,-)$ means that the parities under P_1 and P_2 are (even, odd). By the requirement of consistency, we find that the components of A_μ in the adjoint representation have the parities under $A_\mu \rightarrow P_1 A_\mu P_1$ ($P_2 A_\mu P_2$) as follows:

$$\begin{aligned}
 \mathbf{78}|_{A_\mu} = & \underline{(8,1)(0,0,0)^{(+,+)} + (1,3)(0,0,0)^{(+,+)}} \\
 & + \underline{(1,1)(0,0,0)^{(+,+)} + (1,1)(0,0,0)^{(+,+)} + (1,1)(0,0,0)^{(+,+)}} \\
 & + (3,2)(-5,0,0)^{(-,+)} + \left(\bar{3},2\right)(5,0,0)^{(-,+)} \\
 & + (3,2)(1,4,0)^{(+,-)} + \left(\bar{3},2\right)(-1,-4,0)^{(+,-)} \\
 & + (3,1)(4,-4,0)^{(-,-)} + \left(\bar{3},1\right)(-4,4,0)^{(-,-)} \\
 & + (1,1)(-6,-4,0)^{(-,-)} + (1,1)(6,4,0)^{(-,-)} \\
 & + (3,2)(1,-1,-3)^{(+,+)} + \left(\bar{3},2\right)(-1,1,3)^{(+,+)} \\
 & + (3,1)(4,1,3)^{(-,+)} + \left(\bar{3},1\right)(-4,-1,-3)^{(-,+)} \\
 & + (3,1)(-2,-3,3)^{(+,-)} + \left(\bar{3},1\right)(2,3,-3)^{(+,-)}
 \end{aligned}$$

$$\begin{aligned}
& + (1, 2)(-3, 3, -3)^{(-,-)} + (1, 2)(3, -3, 3)^{(-,-)} \\
& + (1, 1)(-6, 1, 3)^{(-,+)} + (1, 1)(6, -1, -3)^{(-,+)} \\
& + (1, 1)(0, -5, -3)^{(+,-)} + (1, 1)(0, 5, 3)^{(+,-)},
\end{aligned} \tag{3.30}$$

where the underlined components correspond to the adjoint representations of $SU(3) \times SU(2) \times U(1)_Y \times U(1)_X \times U(1)_I$, respectively. We note that the components with parity $(+, +)$ can have massless zero modes in 4D spacetime. Such components include the adjoint representations of $SU(3) \times SU(2) \times U(1)^3$, $(3, 2)(1, -1, -3)$ and its conjugate. The latter components seem problematic. Yet they do not appear in the low-energy spectrum due to nonzero $U(1)_I$ charge. The zero modes of these components will get masses from the background field as in (2.41).

3.2.2. Scalar Field Contents in 4D Spacetime

With the parity assignments with respect to the fixed points, (2.11) and (2.14), we have for the A_θ and A_ϕ fields

$$\begin{aligned}
78|_{A_{\theta\phi}} = & (8, 1)(0, 0, 0)^{(-,-)} + (1, 3)(0, 0, 0)^{(-,-)} \\
& + (1, 1)(0, 0, 0)^{(-,-)} + (1, 1)(0, 0, 0)^{(-,-)} + (1, 1)(0, 0, 0)^{(-,-)} \\
& + (3, 2)(-5, 0, 0)^{(+,-)} + (\bar{3}, 2)(5, 0, 0)^{(+,-)} \\
& + (3, 2)(1, 4, 0)^{(-,+)} + (\bar{3}, 2)(-1, -4, 0)^{(-,+)} \\
& + (3, 1)(4, -4, 0)^{(+,+)} + (\bar{3}, 1)(-4, 4, 0)^{(+,+)} \\
& + (1, 1)(-6, -4, 0)^{(+,+)} + (1, 1)(6, 4, 0)^{(+,+)} \\
& + (3, 2)(1, -1, -3)^{(-,-)} + (\bar{3}, 2)(-1, 1, 3)^{(-,-)} \\
& + (3, 1)(4, 1, 3)^{(+,-)} + (\bar{3}, 1)(-4, -1, -3)^{(+,-)} \\
& + (3, 1)(-2, -3, 3)^{(-,+)} + (\bar{3}, 1)(2, 3, -3)^{(-,+)} \\
& + (1, 2)(-3, 3, -3)^{(+,+)} + (1, 2)(3, -3, 3)^{(+,+)} \\
& + (1, 1)(-6, 1, 3)^{(+,-)} + (1, 1)(6, -1, -3)^{(+,-)} \\
& + (1, 1)(0, -5, -3)^{(-,+)} + (1, 1)(0, 5, 3)^{(-,+)} .
\end{aligned} \tag{3.31}$$

Components with $(+, -)$ or $(-, +)$ parity do not have KK modes since they are odd under $\phi \rightarrow \phi + 2\pi$ and the KK modes of gauge field are specified by integer angular momentum quantum numbers ℓ and m on the two spheres. In the S^1/Z_2 case, the translation group on S^1 is $U(1)$

and any quantum number is allowed. After orbifolding, we obtain the quantum numbers allowed by parity and they can be nonintegers. On the other hand, the translation group on S^2 is $SU(2)$ and only integer quantum numbers are allowed because they correspond to quantized angular momenta. We then concentrate on the components which have either $(+, +)$ or $(-, -)$ parity and nonzero $U(1)_I$ charges as the candidate for the Higgs field. These include $\{(1, 2)(3, -3, 3) + \text{h.c.}\}$ and $\{(3, 2)(1, -1, -3) + \text{h.c.}\}$ with parities $(+, +)$ and $(-, -)$, respectively. The representations $(1, 2)(-3, 3, -3)$ and $(1, 2)(3, -3, 3)$ have the correct quantum numbers for the SM Higgs doublet. Therefore, we identify the $(1, 1)$ mode of these components as the SM Higgs fields in 4D spacetime.

3.2.3. Chiral Fermion Contents in 4D Spacetime

In our model, we choose the fermions as the Weyl fermions Ψ_- belonging to the **27** representation of E_6 . The **27** representation is decomposed as in (3.29) under the group reduction, (3.28). In this decomposition, we find that our choice of the background gauge field of $U(1)_I$ is suitable for obtaining massless fermions since all such components have $U(1)_I$ charge 1. In the fundamental representation, the $U(1)_I$ generator is

$$Q_I = \frac{1}{6} \text{diag}(-2, -2, -2, -2, 1, 1, 1, 1, 4, 1, 1, 1, 1, 1, 1, -2, -2, -2, 1, 1, 1, 1, 1, -2, -2, -2), \quad (3.32)$$

according to the decomposition equation (3.29). By identifying $Q = 3Q_I$, we readily obtain the condition

$$Q\Psi_- = \frac{1}{2}\Psi_-. \quad (3.33)$$

Therefore, the chiral fermions ψ_L in 4D spacetime have zero modes.

Next, we consider the parity assignments for the fermions with respect to the fixed points of S^2/Z_2 . The boundary conditions are given by (2.12) and (2.15). It turns out that four **27** fermion copies with different boundary conditions are needed in order to obtain an entire generation of massless SM fermions. They are denoted by $\Psi^{(1,2,3,4)}$ with the following parity assignments:

$$\begin{aligned} \Psi_{\pm}^{(i)}(x, \pi - \theta, -\phi) &= \xi \gamma_5 P_1 \Psi_{\pm}^{(i)}(x, \theta, \phi), \\ \Psi_{\pm}^{(i)}(x, \pi - \theta, 2\pi - \phi) &= \eta \gamma_5 P_2 \Psi_{\pm}^{(i)}(x, \theta, \phi), \end{aligned} \quad (3.34)$$

where γ_5 is the chirality operator, and $(\xi, \eta) = (+, +), (-, -), (+, -)$ and $(-, +)$ for $i = 1, 2, 3, 4$, respectively. From these fermions we find that $\psi_{1,2,3,4}$ have the parity assignments

$$\begin{aligned}
27_{\psi_L^{(1)}} &= (1, 2)(-3, -2, -2)^{(-, -)} + (1, 2)(3, 2, -2)^{(+, +)} + (1, 2)(-3, 3, 1)^{(-, +)} \\
&\quad + (1, 1)(6, -1, 1)^{(-, -)} + \underline{(1, 1)(0, -5, 1)^{(+, +)}} + (1, 1)(0, 0, 4)^{(+, -)} \\
&\quad + (3, 2)(1, -1, 1)^{(+, -)} + (3, 1)(-2, 2, -2)^{(-, +)} + (\bar{3}, 1)(-4, -1, 1)^{(-, -)} \\
&\quad + (\bar{3}, 1)(2, 3, 1)^{(-, -)} + (\bar{3}, 1)(2, -2, -2)^{(+, -)}, \\
27_{\psi_L^{(2)}} &= (1, 2)(-3, -2, -2)^{(+, +)} + (1, 2)(3, 2, -2)^{(-, -)} + (1, 2)(-3, 3, 1)^{(+, -)} \\
&\quad + \underline{(1, 1)(6, -1, 1)^{(+, +)}} + (1, 1)(0, -5, 1)^{(-, -)} + (1, 1)(0, 0, 4)^{(-, +)} \\
&\quad + (3, 2)(1, -1, 1)^{(-, +)} + (3, 1)(-2, 2, -2)^{(+, -)} + (\bar{3}, 1)(-4, -1, 1)^{(+, +)} \\
&\quad + \underline{(\bar{3}, 1)(2, 3, 1)^{(+, +)}} + (\bar{3}, 1)(2, -2, -2)^{(-, +)}, \\
27_{\psi_L^{(3)}} &= (1, 2)(-3, -2, -2)^{(-, +)} + (1, 2)(3, 2, -2)^{(+, -)} + (1, 2)(-3, 3, 1)^{(-, -)} \\
&\quad + (1, 1)(6, -1, 1)^{(+, +)} + (1, 1)(0, -5, 1)^{(+, -)} + (1, 1)(0, 0, 4)^{(+, +)} \\
&\quad + \underline{(3, 2)(1, -1, 1)^{(+, +)}} + (3, 1)(-2, 2, -2)^{(-, -)} + (\bar{3}, 1)(-4, -1, 1)^{(-, +)} \\
&\quad + (\bar{3}, 1)(2, 3, 1)^{(-, +)} + (\bar{3}, 1)(2, -2, -2)^{(+, +)}, \\
27_{\psi_L^{(4)}} &= (1, 2)(-3, -2, -2)^{(+, -)} + (1, 2)(3, 2, -2)^{(-, +)} + \underline{(1, 2)(-3, 3, 1)^{(+, +)}} \\
&\quad + (1, 1)(6, -1, 1)^{(+, -)} + (1, 1)(0, -5, 1)^{(-, +)} + (1, 1)(0, 0, 4)^{(-, -)} \\
&\quad + (3, 2)(1, -1, 1)^{(-, -)} + (3, 1)(-2, 2, -2)^{(+, +)} + (\bar{3}, 1)(-4, -1, 1)^{(+, -)} \\
&\quad + (\bar{3}, 1)(2, 3, 1)^{(+, -)} + (\bar{3}, 1)(2, -2, -2)^{(-, -)},
\end{aligned} \tag{3.35}$$

where the underlined components have even parities and $U(1)_I$ charge 1. One can readily identify one generation of SM fermions, including a right-handed neutrino, as the zero modes of these components.

A long-standing problem in the gauge-Higgs unification framework is the Yukawa couplings of the Higgs boson to the matter fields. Here we discuss about the Yukawa couplings in our model. As mentioned before, the SM Higgs is the ($\ell = 1, |m| = 1$) KK mode of the extraspatial component of the gauge field, the Yukawa term at tree level has the following form:

$$L_{\text{Yukawa}} \supset \bar{\psi}_L^{00} \Phi^{11} \psi_R^{\ell 1} + \bar{\psi}_L^{\ell 1} \Phi^{11} \psi_R^{00} + \text{h.c.}, \tag{3.36}$$

where $\psi^{\ell m}$ s are the fermionic KK modes with the $(l = 0, m = 0)$ modes appearing as the chiral fermions and Φ^{11} denotes the SM Higgs field. We here identify the left-handed fermionic zero modes as SU(2) doublets and the right-handed fermionic zero modes as SU(2) singlets, as in the SM. Therefore, the $(\ell, |m| = 1)$ modes and the $(\ell = 0, |m| = 0)$ modes mix after spontaneous symmetry breaking. One needs to diagonalize the mass terms to obtain physical eigenstates. The Yukawa couplings in our model are thus more complicated than other gauge-Higgs unification models in the sense that there is mixing between KK modes including the zero modes without a bulk mass term or fixed point localized term. However, similar mixing occurs in models on warped 5D spacetime or even in models with a flat metric if one takes into account the bulk mass term or fixed point localized term. In such cases, diagonalization is necessary.

The difficulty of obtaining a realistic fermion mass spectrum comes from the fact that the Yukawa couplings arise from gauge interactions. However, one can overcome the difficulty by introducing SM fermions localized at an orbifold fixed point and additional massive bulk fermions. The realistic Yukawa couplings would be obtained from nonlocal interactions of the fixed point localized fermions involving Wilson lines after integrating out the massive bulk fermions [41–43]. Another possible solution is to consider fermions in 6D spacetime belonging to a higher dimensional representation of the original E_6 gauge group, rendering more than one generation of SM fermions. In that case, mixing among generations will be obtained from gauge interactions and is given by Clebsch-Gordan coefficients. We expect that realistic Yukawa couplings could be obtained using these methods. A detailed analysis of this issue is beyond the scope of the paper and left for a future work.

3.2.4. Higgs Potential of the Model

Here we analyze the Higgs potential for the E_6 model. To further simplify the Higgs potential, we need to find out the algebra of the gauge group generators. Note that the E_6 generators are chosen according to the decomposition of the adjoint representation given in (3.30)

$$\begin{aligned} & \{Q_i, Q_\alpha, Q_Y, Q_X, Q_I, \\ & Q_{ax(-5,0,0)}, Q^{ax(5,0,0)}, Q_{ax(1,4,0)}, Q^{ax(-1,-4,0)}, \\ & Q_{a(4,-4,0)}, Q^{a(-4,4,0)}, Q_{(-6,-4,0)}, Q_{(6,4,0)}, \\ & Q_{ax(1,-1,-3)}, Q^{ax(-1,1,3)}, Q_{a(4,1,3)}, Q^{a(-4,-1,-3)}, \\ & Q_{a(-2,-3,3)}, Q^{a(2,3,-3)}, Q_{x(3,-3,3)}, Q^{x(-3,3,-3)}, \\ & Q_{(-6,1,3)}, Q_{(6,-1,-3)}, Q_{(0,-5,-3)}, Q_{(0,5,3)}\}, \end{aligned} \quad (3.37)$$

where the generators are listed in the corresponding order of the terms in (3.30) and the indices

$$i = 1, \dots, 8 : \text{SU}(3) \text{ adj rep index} \Rightarrow Q_i : \text{SU}(3) \text{ generators}, \quad (3.38)$$

$$\alpha = 1, 2, 3 : \text{SU}(2) \text{ adj rep index} \Rightarrow Q_\alpha : \text{SU}(2) \text{ generators}, \quad (3.39)$$

$$Q_{X,Y,I} : \text{U}(1)_{X,Y,I} \text{ generators}, \quad (3.40)$$

Table 2: Commutation relations of Q_α , $Q_{X,Y,I}$, $Q_{x(3,-3,3)}$ and $Q^{x(-3,3,-3)}$, where σ_i are the Pauli matrices.

$[Q_{x(3,-3,3)}, Q^{y(-3,3,-3)}] = \frac{1}{2}\delta_x^y Q_I - \frac{1}{2}\sqrt{\frac{3}{5}}\delta_x^y Q_X + \frac{1}{\sqrt{10}}\delta_x^y Q_Y + \frac{1}{\sqrt{6}}(\sigma_\alpha)_x^y Q_\alpha$	
$[Q_\alpha, Q_{x(3,-3,3)}] = \frac{1}{\sqrt{6}}(\sigma_\alpha)_x^y Q_{y(3,-3,3)},$	$[Q_\alpha, Q^{x(-3,3,-3)}] = -\frac{1}{\sqrt{6}}(\sigma_\alpha^*)_x^y Q^{y(-3,3,-3)},$
$[Q_{x(3,-3,3)}, Q_{y(3,-3,3)}] = 0,$	$[Q_I, Q_{x(3,-3,3)}] = \frac{1}{2}Q_{x(3,-3,3)},$
$[Q_X, Q_{x(3,-3,3)}] = -\frac{1}{2}\sqrt{\frac{3}{5}}Q_{x(3,-3,3)},$	$[Q_Y, Q_{x(3,-3,3)}] = \frac{1}{\sqrt{10}}Q_{x(3,-3,3)},$

$$x = 1, 2 : \text{SU}(2) \text{ doublet index}, \quad (3.41)$$

$$a = 1, 2, 3 : \text{SU}(3) \text{ color index}. \quad (3.42)$$

Here we take the normalization for generators, $\text{Tr}[QQ^\dagger] = 2$ which is taken from [24]. The Higgs fields are in the representations of $(1, 2)(3, -3, 3)$ and $(1, 2)(-3, 3, -3)$. We write

$$\Phi(x) = \phi^x Q_{x(3,-3,3)} \quad (\Phi^\dagger(x) = \phi_x Q^{x(-3,3,-3)}). \quad (3.43)$$

Likewise, the gauge field $A_\mu(x)$ in terms of the Q 's in (3.38) is

$$A_\mu(x) = A_\mu^i Q_i + A_\mu^\alpha Q_\alpha + B_\mu Q_Y + C_\mu Q_X + E_\mu Q_I. \quad (3.44)$$

The commutation relations between the generators Q_α , $Q_{X,Y,I}$, $Q_{x(3,-3,3)}$, and $Q^{x(-3,3,-3)}$ are summarized in Table 2.

Finally, we obtain the Lagrangian associated with the Higgs field by applying (3.43) and (3.44) to (2.54) and (2.55) and carrying out the trace. Furthermore, to obtain the canonical form of kinetic terms, the Higgs field, the gauge field, and the gauge coupling need to be rescaled in the following way:

$$\begin{aligned} \phi &\longrightarrow \frac{g}{\sqrt{2}}\phi, \\ A_\mu &\longrightarrow \frac{g}{R}A_\mu, \\ \frac{g}{\sqrt{6\pi R^2}} &= g_2, \end{aligned} \quad (3.45)$$

where g_2 denotes the $\text{SU}(2)$ gauge coupling. The Higgs sector is then given by

$$\mathcal{L}_{\text{Higgs}} = |D_\mu \phi|^2 - V(\phi), \quad (3.46)$$

where

$$D_\mu \phi = \left[\partial_\mu + ig_2 \frac{\sigma_\alpha}{2} A_{\alpha\mu} + ig \frac{1}{\sqrt{40\pi R^2}} B_\mu - ig \frac{1}{2} \sqrt{\frac{3}{20\pi R^2}} C_\mu + ig \frac{1}{2\sqrt{4\pi R^2}} E_\mu \right] \phi, \quad (3.47)$$

$$V = -\frac{\chi}{8R^2} \phi^\dagger \phi + \frac{3g^2}{40\pi R^2} (\phi^\dagger \phi)^2, \quad (3.48)$$

where $\chi = 7 + 9\mu_1 + 9\mu_2$. The numerical values $\mu_{1,2}$ are given by $\mu_1 = 1 - (3/2) \ln 2$ and $\mu_2 = (3/4)(1 - 2 \ln 2)$ as in Section 2.3.4. We have omitted the constant term in the Higgs potential. Comparing the potential derived above with the standard form $\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2$ in the SM, we see that the model has a tree-level μ^2 term that is negative and proportional to R^{-2} . The negative contribution to the squared mass term comes from the interaction between background gauge field and ϕ as seen in Section 2.3.4. Moreover, the quartic coupling $\lambda = 3g^2/(40\pi R^2)$ is related to the 6D gauge coupling g and grants perturbative calculations because it is about 0.16, using the value of R to be extracted in the next section. Therefore, the order parameter in this model is controlled by a single parameter R , the compactification scale.

In fact, the $(1,1)$ mode of the $\{(3,2)(1,-1,-3) + \text{h.c.}\}$ representation also has a negative squared mass term because it has the same Q_I charge as the $\{(1,2)(3,-3,3) + \text{h.c.}\}$ representation. Therefore, it would induce not only electroweak symmetry breaking but also color symmetry breaking. This undesirable feature can be cured by adding brane terms

$$\frac{\alpha}{R^2 \sin^2 \theta} F_{\theta\phi}^a F^{a\theta\phi} \delta\left(\theta - \frac{\pi}{2}\right) [\delta(\phi) + \delta(\phi - \pi)], \quad (3.49)$$

where a denotes the group index of the $\{(3,2)(1,-1,-3) + \text{h.c.}\}$ representation. These brane terms preserve the Z'_2 symmetry which corresponds to the symmetry under the transformation $(\phi \rightarrow \phi + \pi)$. With an appropriate choice of the dimensionless constant α , the squared mass of the $(1,1)$ can be lifted to become positive and sufficiently large. We need to forbid a similar brane term for the $SU(2)$ doublet component, and it can be achieved by imposing some additional discrete symmetry. However, here we simply assume that such a brane term for the $SU(2)$ doublet component does not exist.

Due to a negative mass term, the Higgs potential in (3.48) can induce the spontaneous symmetry breakdown: $SU(2) \times U(1)_Y \rightarrow U(1)_{EM}$ in the SM. The Higgs field acquires a vacuum expectation value (VEV):

$$\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \quad \text{with } v = \sqrt{\frac{5\pi\chi}{3}} \frac{1}{g} \simeq \frac{4.6}{g}. \quad (3.50)$$

One immediately finds that the W boson mass:

$$m_W = \frac{g_2}{2} v = \frac{1}{6} \sqrt{\frac{5\chi}{2}} \frac{1}{R} \simeq \frac{0.53}{R}, \quad (3.51)$$

from which the compactification scale $R^{-1} \simeq 152 \text{ GeV}$ is inferred. Moreover, the Higgs boson mass at the tree level is

$$m_H = \sqrt{\frac{3}{20\pi}} \frac{gv}{R} = 3 \sqrt{\frac{2}{5}} m_W = \frac{\sqrt{\chi}}{2} \frac{1}{R}, \quad (3.52)$$

which is about 152 GeV , numerically very close to the compactification scale. Since the hypercharge of the Higgs field is $1/2$, the $U(1)_Y$ gauge coupling is derived from (3.47) as

$$g_Y = \frac{g}{\sqrt{10\pi R^2}}. \quad (3.53)$$

The Weinberg angle is thus given by

$$\sin^2 \theta_W = \frac{g_Y^2}{g_2^2 + g_Y^2} = \frac{3}{8}, \quad (3.54)$$

and the Z boson mass

$$m_Z = \frac{m_W}{\cos \theta_W} = m_W \sqrt{\frac{8}{5}}, \quad (3.55)$$

both at the tree level. These relations are the same as the $SU(5)$ GUT at the unification scale. This is not surprising because this part only depends on the group structure. Again, this Weinberg angle is not consistent with experimental measurements, and we need to take into account quantum corrections.

We can repeat the discussion in Section 3.1.2 about the one-loop power divergence in the Higgs potential associated with the linear operator F_{ab} . The operator $F_{\theta\phi}^\alpha$ transform to $-F_{\theta\phi}^\alpha$ under the parity transformation $\theta \rightarrow \pi - \theta$. Hence, this operator is forbidden by parity invariance of the action. In this case, we check the consistency between the orbifold boundary

conditions on S^2/Z_2 , (2.13)–(2.15), and the parity conditions, (3.26). By performing the parity transformation on both sides of the orbifold boundary conditions, (2.13)–(2.15), we obtain

$$\begin{aligned}
A_\mu(x, \theta, -\phi) &= P_1 A_\mu(x, \pi - \theta, \phi) P_1, \\
-A_\theta(x, \theta, -\phi) &= P_1 A_\theta(x, \pi - \theta, \phi) P_1, \\
A_\phi(x, \theta, -\phi) &= -P_1 A_\phi(x, \pi - \theta, \phi) P_1, \\
\pm \Gamma^4 \Psi(x, \theta, -\phi) &= \pm \gamma_5 P_1 (\pm \Gamma^4) \Psi(x, \pi - \theta, \phi), \\
A_\mu(x, \theta, 2\pi - \phi) &= P_2 A_\mu(x, \pi - \theta, \phi) P_2, \\
-A_\theta(x, \theta, 2\pi - \phi) &= P_2 A_\theta(x, \pi - \theta, \phi) P_2, \\
A_\phi(x, \theta, 2\pi - \phi) &= -P_2 A_\phi(x, \pi - \theta, \phi) P_2, \\
\pm \Gamma^4 \Psi(x, \theta, 2\pi - \phi) &= \pm \gamma_5 P_2 (\pm \Gamma^4) \Psi(x, \pi - \theta, \phi).
\end{aligned} \tag{3.56}$$

Since (2.13)–(2.15) hold for any θ and ϕ and Γ^4 commutes with γ_5 , we find that the orbifold boundary conditions still hold under the parity transformation with the identification of $\theta = \pi - \theta'$. In other words, the orbifold boundary conditions, (2.13)–(2.15), are parity invariant.

3.2.5. KK Mode Spectrum of Each Field

Since we did not impose symmetry condition, we have KK modes for each field in this model. Here we show KK mass spectrum under the existence of background field for our E_6 model. The masses are basically controlled by the compactification radius R of the two spheres. They receive two kinds of contributions: one arising from the angular momentum in the S^2 space and the other coming from the interactions with the background field.

The KK masses for fermions have been given in [35, 36, 38]. We give them in terms of our notation here:

$$M_{\ell m}^{\text{KK}}(\psi_L) = \frac{1}{R} \sqrt{\ell(\ell+1) - \frac{4q^2-1}{4}}, \tag{3.57}$$

where q is proportional to the $U(1)_I$ charge of a fermion and determined by the action of $Q = 3Q_I$ on fermions as $Q\Psi = q\Psi = 3q_I\Psi$. Note that the mass does not depend on the quantum number m . The lightest KK mass, corresponding to $\ell = 1$ and $q_I = 1/6$, is about 214 GeV at the tree level. The range of ℓ is

$$\frac{2q \pm 1}{2} \leq \ell \quad (+ : \text{ for } \psi_{R(L)} \text{ in } \Psi_{+(-)}, - : \text{ for } \psi_{L(R)} \text{ in } \Psi_{-(+)}). \tag{3.58}$$

We thus can have zero mode for $Q\Psi = \pm(1/2)\Psi$, where this condition is given in (2.43).

For the 4D gauge field A_μ , its kinetic term, and KK mass term are obtained from the terms:

$$L = \int d\Omega \text{Tr} \left[-\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2R^2} F_{\mu\theta} F^\mu{}_\theta + \frac{1}{2R^2 \sin^2 \theta} F_{\mu\phi} F^\mu{}_\phi \right]. \quad (3.59)$$

Taking terms quadratic in A_μ , we get

$$L_{\text{quad}} = \int d\Omega \text{Tr} \left[-\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu) (\partial^\mu A^\nu - \partial^\nu A^\mu) + \frac{1}{2R^2} \partial_\theta A_\mu \partial_\theta A^\mu \right. \\ \left. + \frac{1}{2R^2 \sin^2 \theta} \partial_\phi A_\mu \partial_\phi A^\mu - \frac{1}{2R^2} [A_\mu, \tilde{A}_\phi^B] [A^\mu, \tilde{A}_\phi^B] \right], \quad (3.60)$$

where $\tilde{A}_\phi^B$ is the background gauge field given in (2.35). The KK expansion of A_μ is

$$A_\mu = \sum_{\ell m} A_\mu^{\ell m}(x) Y_{\ell m}^\pm(\theta, \phi), \quad (3.61)$$

where $Y_{\ell m}^\pm(\theta, \phi)$ are the linear combinations of spherical harmonics satisfying the boundary condition $Y_{\ell m}^\pm(\pi - \theta, -\phi) = \pm Y_{\ell m}^\pm(\theta, \phi)$. Their explicit forms are [35]

$$Y_{\ell m}^+(\theta, \phi) \equiv \frac{(i)^{\ell+m}}{\sqrt{2}} [Y_{\ell m}(\theta, \phi) + (-1)^\ell Y_{\ell -m}(\theta, \phi)] \quad \text{for } m \neq 0, \\ Y_{\ell m}^-(\theta, \phi) \equiv \frac{(i)^{\ell+m+1}}{\sqrt{2}} [Y_{\ell m}(\theta, \phi) - (-1)^\ell Y_{\ell -m}(\theta, \phi)] \quad \text{for } m \neq 0, \quad (3.62) \\ Y_{\ell 0}^{+(-)}(\theta) \equiv \begin{cases} Y_{\ell 0}(\theta) & \text{for } m = 0, \quad \ell = \text{even (odd)} \\ 0 & \text{for } m = 0, \quad \ell = \text{odd (even)}. \end{cases}$$

Note that we do not have KK mode functions that are odd under $\phi \rightarrow \phi + 2\pi$ since the KK modes are specified by the integer angular momentum quantum numbers ℓ and m of gauge field A_M on the two spheres. Thus, the components of A_μ and $A_{\theta, \phi}$ with $(+, -)$ or $(-, +)$ parities do not have corresponding KK modes. Applying the KK expansion and integrating about $d\Omega$, we obtain the kinetic and KK mass terms for the KK modes of A_μ

$$L_M = -\frac{1}{2} [\partial_\mu A_\nu^{\ell m}(x) - \partial_\nu A_\mu^{\ell m}(x)] [\partial^\mu A^{\ell m \nu}(x) - \partial^\nu A^{\ell m \mu}(x)] \\ + \frac{\ell(\ell+1)}{R^2} A_\mu^{\ell m}(x) A^{\ell m \mu}(x) \\ + \frac{9q_I^2}{R^2} \left[\int d\Omega \frac{(\cos \theta \pm 1)^2}{\sin^2 \theta} (Y_{\ell m}^\mp)^2 \right] A_\mu^{\ell m}(x) A^{\ell m \mu}(x), \quad (3.63)$$

where we have used $\text{Tr}[Q_i Q^i] = 2$ and $[A_\mu(x), Q_I] = q_I (A_\mu^i(x) Q_i - A_{i\mu}(x) Q^i)$. Therefore, the KK masses of A_μ are

$$M_{\ell m}^{\text{KK}}(A_\mu) = \frac{1}{R} \sqrt{\ell(\ell+1) + (m_{\ell m}^B)^2}, \quad (3.64)$$

$$(m_{\ell m}^B)^2 = 9q_I^2 \int d\Omega \frac{(\cos \theta \pm 1)^2}{\sin^2 \theta} (Y_{\ell m}^\mp)^2, \quad (3.65)$$

where $m_{\ell m}^B$ corresponds to the contribution from the background gauge field. Note that (3.64) agrees with (2.41) when $\ell = 0$. Also, since the SM gauge bosons have $q_I = 0$, their KK masses are simply $\sqrt{\ell(\ell+1)}/R$ at the tree level.

The kinetic and KK mass terms of A_θ and A_ϕ are obtained from the terms in the higher dimensional gauge sector

$$\begin{aligned} L = \frac{1}{2g^2} \int d\Omega \left\{ \left(\text{Tr}[(\partial_\mu A_\theta - i[A_\mu, A_\theta])^2] + \text{Tr}[(\partial_\mu \tilde{A}_\phi - i[A_\mu, \tilde{A}_\phi])^2] \right) \right. \\ \left. - \frac{1}{R^2} \text{Tr} \left[\left(\frac{1}{\sin \theta} \partial_\theta (\sin \theta \tilde{A}_\phi + \sin \theta \tilde{A}_\phi^B) - \frac{1}{\sin \theta} \partial_\phi A_\theta - i[A_\theta, \tilde{A}_\phi + \tilde{A}_\phi^B] \right)^2 \right] \right\}. \end{aligned} \quad (3.66)$$

The first line on the right-hand side of (3.66) corresponds to the kinetic terms, and the second line corresponds to the potential term. Applying the background gauge field (2.35), the potential becomes

$$L_V = -\frac{1}{2g^2 R^2} \int d\Omega \text{Tr} \left[\left(\frac{1}{\sin \theta} \partial_\theta (\sin \theta \tilde{A}_\phi) + Q - \frac{1}{\sin \theta} \partial_\phi A_\theta - i[A_\theta, \tilde{A}_\phi + \tilde{A}_\phi^B] \right)^2 \right]. \quad (3.67)$$

For A_θ and A_ϕ , we use the following KK expansions to obtain the KK mass terms,

$$\begin{aligned} A_\theta(x, \theta, \phi) &= \sum_{\ell m (\neq 0)} \frac{-1}{\sqrt{\ell(\ell+1)}} \left[\Phi_1^{\ell m}(x) \partial_\theta Y_{\ell m}^\pm(\theta, \phi) + \Phi_2^{\ell m}(x) \frac{1}{\sin \theta} \partial_\phi Y_{\ell m}^\pm(\theta, \phi) \right], \\ A_\phi(x, \theta, \phi) &= \sum_{\ell m (\neq 0)} \frac{1}{\sqrt{\ell(\ell+1)}} \left[\Phi_2^{\ell m}(x) \partial_\theta Y_{\ell m}^\pm(\theta, \phi) - \Phi_1^{\ell m}(x) \frac{1}{\sin \theta} \partial_\phi Y_{\ell m}^\pm(\theta, \phi) \right], \end{aligned} \quad (3.68)$$

where the factor of $1/\sqrt{\ell(\ell+1)}$ is needed for normalization. These particular forms are convenient in giving diagonalized KK mass terms [35]. Applying the KK expansions equations (3.68), we obtain the kinetic term

$$L_K = \frac{1}{2g^2} \sum_{\ell m (\neq 0)} \text{Tr} \left[\partial_\mu \Phi_1^{\ell m}(x) \partial^\mu \Phi_1^{\ell m}(x) + \partial_\mu \Phi_2^{\ell m}(x) \partial^\mu \Phi_2^{\ell m}(x) \right], \quad (3.69)$$

where only terms quadratic in $\partial_\mu \Phi$ are retained. The potential term is

$$\begin{aligned}
 L_V = & -\frac{1}{2g^2 R^2} \sum_{\{\ell m\}(\neq 0)} \int d\Omega \operatorname{Tr} \left[\left(\frac{\Phi_2^{\ell m}}{\sqrt{\ell(\ell+1)}} \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta Y_{\ell m}^\pm) + Q + \frac{\Phi_2^{\ell m}}{\sqrt{\ell(\ell+1)}} \frac{1}{\sin^2 \theta} \partial_\phi^2 Y_{\ell m}^\pm \right. \right. \\
 & - \frac{i}{\sqrt{\ell(\ell+1)\ell'(\ell'+1)}} \left[-\Phi_1^{\ell m} \partial_\theta Y_{\ell m}^\pm - \Phi_2^{\ell m} \frac{1}{\sin \theta} \partial_\phi Y_{\ell m}^\pm, \right. \\
 & \left. \left. \Phi_2^{\ell' m'} \partial_\theta Y_{\ell' m'}^\pm - \Phi_1^{\ell' m'} \frac{1}{\sin \theta} \partial_\phi Y_{\ell' m'}^\pm \right. \right. \\
 & \left. \left. + \sqrt{\ell'(\ell'+1)} A_\phi^B \right] \right)^2 \Big]. \tag{3.70}
 \end{aligned}$$

Note that these terms are not diagonal in (ℓ, m) in general. Using the relation $(1/\sin \theta) \partial_\theta (\sin \theta \partial_\theta Y_{\ell m}) + (1/\sin^2 \theta) \partial_\phi^2 Y_{\ell m} = -\ell(\ell+1) Y_{\ell m}$, the potential term is simplified as

$$\begin{aligned}
 L_V = & -\frac{1}{2g^2 R^2} \\
 & \times \sum_{\ell m(\neq 0)} \int d\Omega \operatorname{Tr} \left[\left(-\sqrt{\ell(\ell+1)} \Phi_2^{\ell m} Y_{\ell m}^\pm + Q + \frac{i}{\sqrt{\ell(\ell+1)\ell'(\ell'+1)}} [\Phi_1^{\ell m}, \Phi_2^{\ell' m'}] \right. \right. \\
 & \times \left(\partial_\theta Y_{\ell m}^\pm \partial_\theta Y_{\ell' m'}^\pm + \frac{1}{\sin^2 \theta} \partial_\phi Y_{\ell m}^\pm \partial_\phi Y_{\ell' m'}^\pm \right) \\
 & \left. \left. + \frac{i}{\sqrt{\ell(\ell+1)}} [\Phi_1^{\ell m}, \tilde{A}_\phi^B] \partial_\theta Y_{\ell m}^\pm + \frac{i}{\sqrt{\ell(\ell+1)}} [\Phi_2^{\ell m}, \tilde{A}_\phi^B] \frac{\partial_\phi Y_{\ell m}^\pm}{\sin \theta} \right)^2 \right]. \tag{3.71}
 \end{aligned}$$

To obtain the mass term, we focus on terms quadratic in $\Phi_{1,2}$:

$$\begin{aligned}
 L_M = & -\frac{1}{2g^2 R^2} \int d\Omega \operatorname{Tr} \left[\ell(\ell+1) (\Phi_2^{\ell m})^2 (Y_{\ell m}^\pm)^2 \right. \\
 & + \frac{2iQ}{\ell(\ell+1)} [\Phi_1^{\ell m}, \Phi_2^{\ell m}] \left(\partial_\theta Y_{\ell m}^\pm \partial_\theta Y_{\ell m}^\pm + \frac{1}{\sin^2 \theta} \partial_\phi Y_{\ell m}^\pm \partial_\phi Y_{\ell m}^\pm \right) \\
 & + 2i \tilde{A}_\phi^B [\Phi_1^{\ell m}, \Phi_2^{\ell m}] Y_{\ell m}^\pm \partial_\theta Y_{\ell m}^\pm - \frac{1}{\ell(\ell+1)} [\Phi_1^{\ell m}, \tilde{A}_\phi^B]^2 (\partial_\theta Y_{\ell m}^\pm)^2 \\
 & \left. - \frac{1}{\ell(\ell+1)} [\Phi_2^{\ell m}, \tilde{A}_\phi^B]^2 \frac{(\partial_\phi Y_{\ell m}^\pm)^2}{\sin^2 \theta} \right]. \tag{3.72}
 \end{aligned}$$

Here we take terms which are diagonal in (ℓ, m) for simplicity. Note that we have dropped the term proportional to $[\Phi_1, \tilde{A}_\phi^B][\Phi_2, \tilde{A}_\phi^B]$ because this term vanishes after turning the field into the linear combinations of Φ and $\Phi^\dagger$, (2.51) and (2.52):

$$\begin{aligned} \text{Tr} \left[[\Phi_1, \tilde{A}_\phi^B] [\Phi_1, \tilde{A}_\phi^B] \right] &\longrightarrow \text{Tr} \left[[(\Phi + \Phi^\dagger), Q] [(\Phi - \Phi^\dagger), Q] \right] \\ &\propto \text{Tr} \left[(\Phi - \Phi^\dagger)(\Phi + \Phi^\dagger) \right] \\ &\propto \text{Tr} [\Phi \Phi^\dagger] - \text{Tr} [\Phi^\dagger \Phi] = 0. \end{aligned} \quad (3.73)$$

Integrating the second term of (3.72) by part, we obtain

$$\begin{aligned} L_M = & -\frac{1}{2g^2 R^2} \left(\ell(\ell+1) \text{Tr} \left[(\Phi_2^{\ell m})^2 \right] + 2i \text{Tr} \left[Q [\Phi_1^{\ell m}, \Phi_2^{\ell m}] \right] \right. \\ & - 2i \text{Tr} \left[Q [\Phi_1^{\ell m}, \Phi_2^{\ell m}] \right] \int d\Omega \frac{\cos \theta \mp 1}{\sin \theta} Y_{\ell m}^\pm \partial_\theta Y_{\ell m}^\pm \\ & - \frac{1}{\ell(\ell+1)} [\Phi_1^{\ell m}, Q]^2 \int d\Omega \frac{(\cos \theta \mp 1)^2}{\sin^2 \theta} (\partial_\theta Y_{\ell m}^\pm)^2 \\ & \left. - \frac{1}{\ell(\ell+1)} [\Phi_2^{\ell m}, Q]^2 \int d\Omega \frac{(\cos \theta \mp 1)}{\sin^2 \theta} \frac{(\partial_\phi Y_{\ell m}^\pm)^2}{\sin^2 \theta} \right). \end{aligned} \quad (3.74)$$

Therefore, the KK masses depend on the $U(1)_I$ charges of the scalar fields. Note that terms in the second line to the last line of (3.74) are not diagonal in (ℓ, m) in general.

For components with zero $U(1)_I$ charge, we write $\Phi_{1(2)}(x)$ as $\phi_{1(2)}(x)Q$ where Q is the corresponding generator of E_6 in (3.30) with zero $U(1)_I$ charge. Taking the trace, we have the following kinetic and KK mass terms instead:

$$L = \sum_{\ell m (\neq 0)} \left(\partial_\mu \phi_1^{\ell m}(x) \partial^\mu \phi_1^{\ell m}(x) + \partial_\mu \phi_2^{\ell m}(x) \partial^\mu \phi_2^{\ell m}(x) + \ell(\ell+1) \phi_2^{\ell m}(x) \phi_2^{\ell m}(x) \right), \quad (3.75)$$

where we have made the substitution $\phi_i \rightarrow g\phi_i$. Note that ϕ_1 is considered as a massless Nambu-Goldstone (NG) boson in this case. For components with nonzero $U(1)_I$ charge, mass terms are not diagonal for $\Phi_{1(2)}$, and Φ_1 does not correspond to the NG boson. In this case, we need to diagonalize the mass terms and some linear combination of $\Phi_{1(2)}$ becomes the NG boson mode.

For components with nonzero $U(1)_I$ charge, we use (2.51) and (2.52) and write $\Phi(x)$ as $\phi^i(x)Q_i$ where Q_i is the corresponding generator of E_6 in (3.30) with nonzero $U(1)_I$ charge. The commutator between Q and Φ is

$$[Q, \Phi] = 3[Q_I, Q_i] \phi^i = 3q_I \phi^i, \quad (3.76)$$

where we have used $Q = 3Q_I$ as required to obtain chiral fermions in Section 3.2.3, and that q_I is a constant determined by the $U(1)_I$ charge of the corresponding component. Finally, the Lagrangian becomes

$$L = \sum_{\ell m (\neq 0)} \left\{ \partial_\mu \phi_{\ell m}^\dagger \partial^\mu \phi_{\ell m} - \frac{1}{4R^2} \left[2\ell(\ell+1) \phi_{\ell m}^\dagger \phi_{\ell m} - 12q_I \phi_{\ell m}^\dagger \phi_{\ell m} + 12q_I \phi_{\ell m}^\dagger \phi_{\ell m} \int d\Omega \frac{\cos \theta \mp 1}{\sin \theta} Y_{\ell m}^\pm \partial_\theta Y_{\ell m}^\pm + \frac{18q_I^2}{\ell(\ell+1)} \phi_{\ell m}^\dagger \phi_{\ell m} \int d\Omega \frac{(\cos \theta \mp 1)^2}{\sin^2 \theta} \left((\partial_\theta Y_{\ell m}^\pm)^2 + \frac{(\partial_\phi Y_{\ell m}^\pm)^2}{\sin^2 \theta} \right) \right] \right\}, \quad (3.77)$$

where the subscript i is omitted for simplicity. The KK masses of the complex scalar field ϕ are then

$$M_{\ell m}^{\text{KK}}(\phi) = \frac{1}{R} \sqrt{\frac{\ell(\ell+1)}{2} + (m_{\ell m}^B)^2},$$

$$\begin{aligned} (m_{\ell m}^B)^2 = & -3q_I + 3q_I \int d\Omega \frac{\cos \theta \mp 1}{\sin \theta} Y_{\ell m}^\pm \partial_\theta Y_{\ell m}^\pm \\ & + \frac{9q_I^2}{2\ell(\ell+1)} \int d\Omega \frac{(\cos \theta \mp 1)^2}{\sin^2 \theta} (\partial_\theta Y_{\ell m}^\pm)^2 \\ & + \frac{9q_I^2}{2\ell(\ell+1)} \int d\Omega \frac{(\cos \theta \mp 1)^2}{\sin^2 \theta} \frac{(\partial_\phi Y_{\ell m}^\pm)^2}{\sin^2 \theta}. \end{aligned} \quad (3.78)$$

The squared KK mass $(M_{\ell m}^{\text{KK}})^2$ is always positive except for the lowest mode ($\ell = 1, m = 1$). In fact, the squared KK mass of the $(1, 1)$ mode agrees with the coefficient of quadratic term in the Higgs potential (3.48).

4. Summary and Discussions

We have reviewed a gauge theory defined on 6D spacetime with the S^2/Z_2 topology on the extra space. Two scenarios are considered to construct a 4D theory from the 6D model. One scenario based on the $SO(12)$ gauge group requires a symmetry condition for the gauge field. The other involves the E_6 gauge group, but does not need the symmetry condition. Nontrivial boundary conditions on the extra space are imposed in both scenarios.

We explicitly give the prescriptions to identify the gauge field and the scalar field remaining in 4D spacetime after the dimensional reduction. We show that the $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_X \times U(1)_I$ gauge symmetry remains in 4D spacetime, and that the SM Higgs doublet with a suitable potential for electroweak symmetry breaking can be derived from the gauge sector in both models. The Higgs boson mass is also predicted in such

models. Our tree-level prediction of the Higgs boson mass is 196 GeV for the SO(12) model and 152 GeV for the E_6 model. These mass values are in the range of 127–600 GeV already excluded at 95% CL by recent LHC data [44, 45]. However, the mass value will become different once quantum corrections to the Higgs potential are taken into account. We expect that the Higgs boson mass in our model will become smaller than the lower limit of the exclusion region by quantum corrections. In particular, the E_6 case gives a 152 GeV Higgs boson mass at tree level that is not far from the lower limit of the exclusion region at 95%CL. However, a full analysis of the quantum corrections is beyond the scope of this paper and left as a future work. Massless fermion modes are also successfully obtained as the SM fermions by introducing appropriate field contents in 6D spacetime, with suitable parity assignments on the S^2/Z_2 extra dimension and incorporating the background gauge field. We also discuss about the massive KK modes of fermions for the scenario with the symmetry condition and the KK modes of all fields for the one without the symmetry condition. The lightest fermionic KK mode can serve as a dark matter candidate. In general, they may give rise to rich phenomena in collider experiments and implications in cosmological studies.

To make our models more realistic, there are several challenges such as eliminating the extra U(1) symmetries and constructing the realistic Yukawa couplings, which are the same as other gauge-Higgs unification models. We, however, can get Kaluza-Klein modes in our models. This suggests that we obtain the dark matter candidate in our model. Thus, it is very important to study these models further such as dark matter physics and collider physics.

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