Research Article

DeWitt-Virasoro Construction in Tensor Representations

Partha Mukhopadhyay

The Institute of Mathematical Sciences, C.I.T. Campus, Taramani, Chennai 600113, India

Correspondence should be addressed to Partha Mukhopadhyay, parthamu@imsc.res.in

Received 5 June 2012; Accepted 17 August 2012

Academic Editor: Richard J. Szabo

Copyright © 2012 Partha Mukhopadhyay. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We generalize the DeWitt-Virasoro (DWV) construction of arXiv:0912.3987 [hep-th] to tensor representations of higher ranks. A rank- n tensor state, which is by itself coordinate invariant, is expanded in terms of position eigenstates that transform as tensors of the same rank. The representation of the momentum operator in these basis states is then obtained by generalizing DeWitt's argument (1952). Such a representation is written in terms of certain bivector of parallel displacement and its covariant derivatives. With this machinery at hand we find tensor representations of the DWV generators defined in the previous work. The results differ from those in spin-zero representation by additional terms involving the spin connection. However, we show that the DWV algebra found earlier as a scalar expectation value remains the same, as required by consistency, as all the additional contributions conspire to cancel in various ways. In particular, vanishing of the anomaly term requires the same condition of Ricci-flatness for the background.

1. Introduction and Summary

With the motivation of understanding exact conformal invariance of string worldsheet theory in certain pp-wave background in Hamiltonian framework [1–3], a particular approach to study the latter was considered in [4, 5]. The simplest case of a bosonic string moving in an arbitrary metric-background was considered. The basic idea was to find a way to describe the string quantum mechanics in a coordinate independent way following the same line of argument that was used by DeWitt [6, 7] for particle (two-dimensional nonlinear sigma model has been studied in lagrangian framework in [8–18]. See [19–30] for earlier studies using hamiltonian framework. DeWitt's argument was used earlier in string theory in [31]).

The construction of [4, 5], hereafter called the DeWitt-Virasoro (DWV) construction, goes as follows. We first rewrite the worldsheet theory in terms of the Fourier modes in such a way that it can be interpreted as a single particle moving in an infinite-dimensional

curved background subject to certain potential. Since the chosen degrees of freedom are nonlocal on the worldsheet the classical conformal invariance is not directly visible. However, the description possesses some more structure which look unusual from the point of view of a particle dynamics but captures the underlying worldsheet. This enables us to write down the classical Virasoro generators in terms of the phase-space variables of the infinite-dimensional/particle description. Then implementing DeWitt's argument [6, 7] naturally leads to a quantum background independent definition for such generators, hereafter called DeWitt-Virasoro generators. The machinery available through [6] enables one to calculate the matrix element of an arbitrary operator constructed out of the DWV generators between any two scalar states. The result written in position space is manifestly generally covariant. Using this tool we computed the DWV algebra in spin-zero representation. The result is given by the Witt algebra with certain additional anomalous terms that vanish for Ricci-flat backgrounds.

In this work, we will generalize the above construction to higher rank tensor representations in a certain sense. Before we delve into the details of this generalization we first try to motivate and explain what we intend to achieve. String theory contains infinite number of higher rank tensor states. Around flat background such states are constructed by applying suitable creation operators on the ground state (which can be solved exactly). State-operator mapping of this sort should hold for any CFT representing an exact classical background in string theory and the set of all possible excited states, with suitable restriction, should correspond to the fluctuation of the tensor fields around that particular background (background independence restricted to conformal backgrounds has been studied in the context of classical closed string field theory (CSFT) [32] in [33–36]. Construction of CSFT for a nonconformal background was discussed in [37]). However, as we will argue, state-operator mapping, at least in this form, is not applicable in our generally covariant description. What appears to be more relevant in this case are the Schrödinger wavefunctions of the tensor states in position representation. In an arbitrary background these wavefunctions should correspond to the fluctuations of the tensor fields in spacetime. One would like to properly formulate this connection in a precise way. For example, one would like to understand how such wavefunctions should be explicitly constructed in order to be able to define the *string Hilbert space*.

Before we attempt to answer these questions, in this work, we would like to understand certain computations that are entirely guided by the tensorial property of the wavefunctions and therefore can be carried out without knowing the explicit construction of those. More precisely, our goal is to understand how to compute coordinate invariant matrix elements involving arbitrary tensor wavefunctions. To this end, we describe a framework where the tensor wavefunctions are encoded into the states in a special way. This is done by introducing the position eigenstates which themselves are higher rank tensors. The resulting analysis can be viewed as a tensor generalization of the work in [6]. In that work the momentum operator in position space was obtained by demanding that the canonical commutation relations and the hermiticity conditions hold true as expectation values between two position eigenstates of spin zero. The basic ingredient that goes into such an analysis is the orthonormality and completeness conditions for such basis states. A natural generalization of the orthonormality condition for higher rank basis states is given in terms of bivector of parallel displacement. A bivector is a bilocal object which carries a vector index associated with each of the two points in the argument. Given two such points in spacetime the parallel displacement bivector, contracted with another vector at one of the points, gives the same vector parallel transported to the other point along a given path. An example is the bivector of geodetic parallel transport as discussed in [38] where the path is the unique geodesic passing through the two points. Our analysis is sensitive to the expansion of the bivector only to

first-order in separation and therefore is independent of the choice of path. It turns out that the desired representation of the momentum operator can be written in terms of such bivectors and their covariant derivatives. We use this to find the representation of the DWV generators as defined in our previous work. These expressions contain all the terms that are present in the scalar representation. Moreover, they receive additional contributions involving the spin connection. These are calculated by explicitly writing the parallel displacement bivector in terms of the vielbeins.

The DWV algebra was computed in [4, 5] in spin-zero representation. Given that the calculation was done between two arbitrary scalar states, one expects the result to be true as operator equations and therefore is independent of the representation considered (I thank Romesh K. Kaul for emphasizing this point). Since in this work we have found the tensor representation from first principle, it is a good idea to check explicitly if this is true. We do this following the same basic method as described in [4, 5]. As expected, the computation gets more complicated than the previous one because of the additional contributions. There arise three kinds of additional terms which depend on Riemann tensor, covariant derivative of spin connection, and higher power of spin connection. It turns out that these terms are organized in such a way that they cancel each other due to various identities in general relativity reproducing indeed the same result for the DWV algebra.

The rest of the paper is organized as follows. The tensor representation has been constructed in Section 2. We discuss in what sense the existing framework in [4–6] is inadequate for the present purpose in Section 2.1. The new framework is described in Section 2.2. Tensor representation of the momentum operator has been discussed in Section 2.3. We use the results of Section 2 to calculate the DWV generators and algebra in Section 3. Various details of our technical derivations have been given in several appendices.

2. Construction of Tensor Representations

2.1. Scope of the Existing Framework

As mentioned before, the analysis of [4, 5] was done using an infinite-dimensional language. We have summarized the relevant details of it in Appendix A. All our discussion in this work will be done using the same language. Below we will establish that the scope of the existing framework in [4–6] is as follows

$$\begin{aligned} \text{Nontrivial invariant matrix elements can be constructed} \\ \text{only in scalar representation.} \end{aligned} \tag{2.1}$$

Therefore, a new framework needs to be developed in order to construct the desired tensor representations.

We first discuss the kind of questions that are relevant for our present purpose. We define a tensor state to be one whose wavefunction is a tensor in the infinite-dimensional sense. More precisely, the latter can be obtained either by multiplying a tensor field with a scalar wavefunction or by applying covariant derivatives on it. Given all possible tensor states, one would next like to understand how to compute the general coordinate invariant matrix elements of the DWV generators between such states. In position space such a matrix element should be given by an invariant integral where the integrand is constructed out of the tensor wavefunctions defined above.

Recall that in flat space the vacuum state can be explicitly solved and the tensor states are obtained by applying suitable creation operators on the vacuum state (Notice that the Schrödinger wavefunctions [39] of all such states are of the type that we are considering here. As mentioned in Section 1, the aim of this work is not to find a background independent generalization, if it exists, of this procedure. Rather here we want to understand certain computations that are entirely guided by the tensorial properties of the wavefunctions). This is expected from the point of view of ordinary quantum mechanics. However, as we will demonstrate with an example below, this procedure is not suitable for our generally covariant construction.

Using the existing framework of [4–6], one can calculate invariant matrix elements of the following form:

$$\langle \chi | \hat{\mathcal{O}} | \psi \rangle, \quad (2.2)$$

where $|\chi\rangle$ and $|\psi\rangle$ are two arbitrary scalar states and $\hat{\mathcal{O}}$ is a scalar operator which can in general depend on momentum. For example, the scalar expectation value of the DWV algebra as computed in [4, 5] is of the above form. Let us now consider the following two vector wavefunctions:

$$\chi^i(x) = a^i(x)\chi(x), \quad \psi^i(x) = a^i(x)\psi(x), \quad (2.3)$$

where $\chi(x)$ and $\psi(x)$ are the wavefunctions (see Appendix A for our notations) of the states in (2.2), respectively. Usually the corresponding states will be defined as follows:

$$|\chi^i\rangle = a^i(\hat{x})|\chi\rangle = \int dw \chi^i(x)|x\rangle, \quad |\psi^i\rangle = a^i(\hat{x})|\psi\rangle = \int dw \psi^i(x)|x\rangle, \quad (2.4)$$

respectively. However, notice that the matrix element $\langle \chi_i | \hat{\mathcal{O}} | \psi^i \rangle$ is not coordinate invariant unless $\hat{\mathcal{O}}$ is a scalar field independent of momentum. This is simply because under a general coordinate transformation (GCT) this matrix element transforms as (GCT has been introduced in our quantum theory in Appendix A below (A.17). In Appendix D we have discussed how GCT is realized as a unitary transformation in tensor representations)

$$\langle \chi_i | \hat{\mathcal{O}} | \psi^i \rangle \longrightarrow \langle \chi_j | \lambda_i^j(\hat{x}) \hat{\mathcal{O}} \lambda_k^i(\hat{x}) | \psi^k \rangle. \quad (2.5)$$

Since $\lambda_i^j(\hat{x})$ does not commute with $\hat{\mathcal{O}}$ if the latter depends on momentum, the above matrix element is not invariant in general. Proceeding in the similar way one can argue that the only nontrivial (i.e., involving momentum operator) invariant matrix elements that can be constructed are of the type (2.2). This establishes (2.1).

2.2. A New Framework

Given the above discussion, here we introduce a new framework where a tensor state is not created by applying operators on a scalar state, rather it is related to the corresponding

wavefunction in a special way. The resulting construction will enable us to compute invariant matrix elements in tensor representations.

In this framework a rank- n tensor state $|\psi_{(n)}\rangle$ is by itself GCT invariant. It is expanded in position space basis in the following way:

$$|\psi_{(n)}\rangle = \int dw \psi^{i_1 i_2 \dots i_n}(x) |_{i_1 i_2 \dots i_n}; x\rangle, \quad (2.6)$$

where $\psi^{i_1 i_2 \dots i_n}(x)$ is the corresponding rank- n tensor wavefunction, which is constructed following the general definition given in the previous section. We have introduced the rank- n position eigenstates $|_{i_1 \dots i_n}; x\rangle$ satisfying the same eigenvalue equation as in (A.15). Under a GCT $\psi^{i_1 i_2 \dots i_n}(x)$ and $|_{i_1 i_2 \dots i_n}; x\rangle$ transform as a contravariant and a covariant tensor of rank n , respectively (see Appendix D). The orthonormality and completeness conditions of the basis states are given by

$$\langle_{i_1 \dots i_n}; x |_{j_1 \dots j_{\tilde{n}}}; \tilde{x}\rangle = \delta_{n, \tilde{n}} \Delta_{i_1 \dots i_n j_1 \dots j_{\tilde{n}}}(x, \tilde{x}) \delta(x, \tilde{x}), \quad (2.7)$$

$$\int dw |_{i_1 \dots i_n}; x\rangle g^{i_1 j_1}(x) \dots g^{i_n j_n}(x) \langle_{j_1 \dots j_n}; x| = 1, \quad (2.8)$$

where,

$$\Delta_{i_1 \dots i_n j_1 \dots j_n}(x, \tilde{x}) = \Delta_{i_1 j_1}(x, \tilde{x}) \dots \Delta_{i_n j_n}(x, \tilde{x}). \quad (2.9)$$

Equations (2.7), (2.8) are the tensor generalizations of (A.16). Before discussing what $\Delta_{ij}(x, \tilde{x})$ is, we will first introduce a bitensor in the sense of [38] in our notations. A bitensor $t_{i_1 \dots i_n j_1 \dots j_{\tilde{n}}}(x, \tilde{x})$ is a tensorial object with two sets of indices $\{i_1, \dots, i_n\}$ and $\{j_1, \dots, j_{\tilde{n}}\}$ associated with the points x and $\tilde{x}$, respectively such that it transforms as a tensor of rank n and $\tilde{n}$, respectively at the two points separately. Consider

$$t_{i_1 \dots i_n j_1 \dots j_{\tilde{n}}}(x, \tilde{x}) \longrightarrow t'_{i_1 \dots i_n j_1 \dots j_{\tilde{n}}}(\tilde{x}', \tilde{x}') = \lambda_{i_1}^{k_1}(x) \dots \lambda_{i_n}^{k_n}(x) \lambda_{j_1}^{l_1}(\tilde{x}) \dots \lambda_{j_{\tilde{n}}}^{l_{\tilde{n}}}(\tilde{x}) t_{k_1 \dots k_n l_1 \dots l_{\tilde{n}}}(x, \tilde{x}). \quad (2.10)$$

In this sense $\delta(x, \tilde{x})$ is a biscalar and $\Delta_{ij}(x, \tilde{x})$ is a bivector. The latter is called a bivector of parallel displacement which we have defined and explained the properties of in Appendix B.

Certain comments regarding (2.7) are in order. From the discussion of Appendix B it is clear that in the coincidence limit the bivector approaches the metric

$$\lim_{x \rightarrow \tilde{x}} \Delta_{ij}(x, \tilde{x}) \longrightarrow g_{ij}(\tilde{x}). \quad (2.11)$$

Therefore, it might appear unnecessary to use it in (2.7) due to the presence of the delta function. The reason for this is as follows. First of all, notice that the expression in (2.7) makes it manifest that the relevant matrix element is a bitensor. Secondly, the tensor representation of the momentum operator discussed in the next section will involve derivatives. As we will argue, such a quantity will be sensitive to the expansion of the bivector up to first-order in separation.

Since we are using a complex basis, the Hermitian conjugate of a basis state is given by

$$(|_{i_1 \dots i_n}; x\rangle)^\dagger = \langle \bar{i}_1 \dots \bar{i}_n; x|, \quad (2.12)$$

and that of any operator $\mathcal{O}$ is defined to be (For example, for $\mathcal{O} = x^i$, using (A.12) on the left hand side of (2.14) one gets, $x^{\bar{i}} \delta_{n, \tilde{n}} \Delta_{i_1 \dots i_n j_1 \dots j_n}(x, \tilde{x}) \delta(x, \tilde{x})$. Then using the reality property in (A.15) and,

$$\Delta_{ij}(x, \tilde{x})^* = \Delta_{\bar{i}\bar{j}}(x, \tilde{x}), \quad (2.13)$$

one gets the same result from the right hand side of (2.14)),

$$\langle i_1 \dots i_n; x | \mathcal{O}^\dagger | j_1 \dots j_n; \tilde{x} \rangle = \langle \bar{j}_1 \dots \bar{j}_n; \tilde{x} | \mathcal{O} | \bar{i}_1 \dots \bar{i}_n; x \rangle^*. \quad (2.14)$$

This completes the description of the present framework using which all the calculations of this paper will be done.

2.3. Tensor Representation of Momentum Operator

Here we will discuss the rank- n tensor representation of the momentum operator in position space. We will actually directly work with the shifted momentum operators as defined in (A.21) [40]. The derivation of the representation of $\hat{\pi}_k$ has been given in Appendix C. The result is given by (in $\hbar = \alpha' = 1$ unit)

$$\begin{aligned} \langle i_1 \dots i_n; x | \hat{\pi}_k | j_1 \dots j_n; \tilde{x} \rangle &= -i\delta_{n, \tilde{n}} [\Delta_{i_1 \dots i_n j_1 \dots j_n}(x, \tilde{x}) \partial_k \delta(x, \tilde{x}) \\ &\quad + F_{i_1 \dots i_n j_1 \dots j_n k}(x, \tilde{x}) \delta(x, \tilde{x})], \\ &= -i\delta_{n, \tilde{n}} \left[-\Delta_{i_1 \dots i_n j_1 \dots j_n}(x, \tilde{x}) (\tilde{\partial}_k + \gamma_k(x)) \delta(x, \tilde{x}) \right. \\ &\quad \left. + F_{i_1 \dots i_n j_1 \dots j_n k}(x, \tilde{x}) \delta(x, \tilde{x}) \right], \end{aligned} \quad (2.15)$$

where

$$F_{i_1 \dots i_n j_1 \dots j_n k}(x, \tilde{x}) = -\nabla_k \Delta_{i_1 \dots i_n j_1 \dots j_n}(x, \tilde{x}), \quad (2.16)$$

where ∇_k is the covariant derivative. We first notice that the above expressions display the expected tensorial properties of the relevant matrix element. The two equations in (2.15) are related by the second identity in (C.4). Because of the presence of delta functions in (2.15) the complete functional dependence of $\Delta_{i_1 \dots i_n j_1 \dots j_n}(x, \tilde{x})$ and $F_{i_1 \dots i_n j_1 \dots j_n k}(x, \tilde{x})$ are not needed. The following expressions, which can be obtained (see Appendix C) by expanding these

quantities to the desired order, will be used for all practical purposes. Consider the following

$$\begin{aligned}
\langle i_1 \dots i_n; x | \hat{\pi}_k | j_1 \dots j_n; \tilde{x} \rangle &= -i\delta_{n,\tilde{n}} [g_{i_1 \dots i_n j_1 \dots j_n}(\tilde{x}) \partial_k \delta(x, \tilde{x}) \\
&\quad - (\omega_{i_1 \dots i_n j_1 \dots j_n k}(x) + \gamma_{i_1 \dots i_n j_1 \dots j_n k}(x)) \delta(x, \tilde{x})], \\
&= -i\delta_{n,\tilde{n}} \left[-g_{i_1 \dots i_n j_1 \dots j_n}(x) (\tilde{\partial}_k + \gamma_k(x)) \delta(x, \tilde{x}) \right. \\
&\quad \left. - (\omega_{i_1 \dots i_n j_1 \dots j_n k}(x) - \gamma_{j_1 \dots j_n i_1 \dots i_n k}(x)) \delta(x, \tilde{x}) \right].
\end{aligned} \tag{2.17}$$

Various tensors appearing in the above equations are given by

$$\begin{aligned}
g_{i_1 \dots i_n j_1 \dots j_n} &= g_{i_1 j_1} \dots g_{i_n j_n}, \\
\omega_{i_1 \dots i_n j_1 \dots j_n k} &= \omega_{k i_1 j_1} g_{i_2 j_2} \dots g_{i_n j_n} + g_{i_1 j_1} \omega_{k i_2 j_2} g_{i_3 j_3} \dots g_{i_n j_n} + \dots, \\
\omega_{kij} &= \nabla_k e_i^{\hat{i}} \eta_{ij}^{\hat{j}} e_j^{\hat{j}} = \omega_{kij}^{\hat{i}} e_i^{\hat{i}} e_j^{\hat{j}}, \\
&= -\omega_{kji},
\end{aligned} \tag{2.18}$$

$$\begin{aligned}
\gamma_{i_1 \dots i_n j_1 \dots j_n k} &= \gamma_{k i_1 j_1} g_{i_2 j_2} \dots g_{i_n j_n} + g_{i_1 j_1} \gamma_{k i_2 j_2} g_{i_3 j_3} \dots g_{i_n j_n} + \dots, \\
\gamma_{kij} &= \gamma_{ki}^{\hat{i}} g_{ij}^{\hat{j}},
\end{aligned} \tag{2.19}$$

where $e_i^{\hat{i}}$, $\eta_{ij}^{\hat{j}}$, and $\omega_{kij}^{\hat{i}}$ are the infinite-dimensional analogues of the vielbein, the Minkowski metric, and the spin connection see (B.10) and (B.12).

Notice that in the first terms of both the expressions in (2.17) the argument of $g_{i_1 \dots i_n j_1 \dots j_n}$ and the variable with respect to which the derivative applies on the delta function are different. This is how we choose the argument to expand $\Delta_{i_1 \dots i_n j_1 \dots j_n}(x, \tilde{x})$ around. Because of this choice the derivative can pass through the factor $g_{i_1 \dots i_n j_1 \dots j_n}$ freely. This form is useful for performing various manipulations that will be used extensively in our calculation of the DWV algebra.

3. DWV Construction

The right and left moving DWV generators $\hat{L}_{(i)}$, and $\hat{\tilde{L}}_{(i)}$ respectively are defined in the same way as done in [4, 5],

$$4\hat{L}_{(i)} = \hat{K}_{(i)} - \hat{Z}_{(i)} + \hat{V}_{(i)}, \quad 4\hat{\tilde{L}}_{(i)} = \hat{K}_{(\tilde{i})} + \hat{Z}_{(\tilde{i})} + \hat{V}_{(\tilde{i})}, \tag{3.1}$$

where

$$\begin{aligned}
\hat{K}_{(i)} &= \hat{\pi}_k^* g^{kl+i}(\hat{x}) \hat{\pi}_l, \\
\hat{Z}_{(i)} &= \hat{Z}_{(i)}^L + \hat{Z}_{(i)}^R, \\
\hat{V}_{(i)} &= g_{kl}(\hat{x}) a^k(\hat{x}) a^{l+i}(\hat{x}),
\end{aligned} \tag{3.2}$$

where

$$\hat{Z}_{(i)}^L = \hat{\pi}_k^* a^{k+i}(\hat{x}), \quad \hat{Z}_{(i)}^R = a^{k+i}(\hat{x}) \hat{\pi}_k. \quad (3.3)$$

Notice that both $\hat{Z}_{(i)}^L$, and $\hat{Z}_{(i)}^R$ are coordinate invariant and both have the right classical limit. The relative coefficient between these two terms in the second equation of (3.2) is fixed by the hermiticity property of the DWV generators,

$$\hat{L}_{(i)}^\dagger = \hat{L}_{(\bar{i})}, \quad \hat{\bar{L}}_{(i)}^\dagger = \hat{\bar{L}}_{(\bar{i})}. \quad (3.4)$$

Given (2.17), one can compute the matrix elements of $\hat{K}_{(i)}$, $\hat{Z}_{(i)}^L$, and $\hat{Z}_{(i)}^R$ between two arbitrary tensor states. The manipulations involved in such calculations have been demonstrated in deriving (C.15). The final results are given by (notice that according to (2.15) π_k connects tensor states of the same rank only)

$$\begin{aligned} \langle \chi_{(n)} | \hat{K}_{(i)} | \psi_{(n)} \rangle &= (-i)^2 \int dw \left[\bar{\chi}^{i_1 \dots i_n} \nabla_{(i)}^2 \psi_{i_1 \dots i_n} + \nabla^{k+i} \bar{\chi}^{i_1 \dots i_n} \omega_{i_1 \dots i_n j_1 \dots j_n k} \psi^{j_1 \dots j_n} \right. \\ &\quad \left. - \bar{\chi}^{i_1 \dots i_n} \omega_{i_1 \dots i_n j_1 \dots j_n k} \nabla^{k+i} \psi^{j_1 \dots j_n} \right. \\ &\quad \left. + \bar{\chi}^{i_1 \dots i_n} \omega_{i_1 \dots i_n k_1 \dots k_n k} \omega^{k_1 \dots k_n}_{j_1 \dots j_n} \psi^{j_1 \dots j_n} \right], \end{aligned} \quad (3.5)$$

$$\langle \chi_{(n)} | \hat{Z}_{(i)}^L | \psi_{(n)} \rangle = i \int dw \left(\nabla_k \bar{\chi}_{i_1 \dots i_n} + \bar{\chi}^{j_1 \dots j_n} \omega_{j_1 \dots j_n i_1 \dots i_n k} \right) a^{k+i} \psi^{i_1 \dots i_n}, \quad (3.6)$$

$$\langle \chi_{(n)} | \hat{Z}_{(i)}^R | \psi_{(n)} \rangle = -i \int dw \bar{\chi}^{i_1 \dots i_n} a^{k+i} \left(\nabla_k \psi_{i_1 \dots i_n} - \omega_{i_1 \dots i_n j_1 \dots j_n k} \psi^{j_1 \dots j_n} \right), \quad (3.7)$$

where $\bar{\chi}^{i_1 \dots i_n} = \chi^{* \bar{i}_1 \dots \bar{i}_n}$ and $\nabla_{(i)}^2 = \nabla_k \nabla^{k+i}$ is the *shifted Laplace operator* [4, 5]. Notice that only the first terms in the above expressions were found in [4, 5] for scalar representation. The spin connection terms, which originate from the F -term in (2.15), are the additional contributions due to tensor generalization.

In [4, 5] the DWV algebra was calculated as expectation value between two spin-zero states. The result, written in operator form, is given by

$$\begin{aligned} [\hat{L}_{(i)}, \hat{L}_{(j)}] &= (i-j) \hat{L}_{(i+j)} + \hat{A}_{(i)(j)}^R, \\ [\hat{\bar{L}}_{(i)}, \hat{\bar{L}}_{(j)}] &= (i-j) \hat{\bar{L}}_{(i+j)} + \hat{A}_{(i)(j)}^L, \\ [\hat{L}_{(i)}, \hat{\bar{L}}_{(j)}] &= \hat{A}_{(i)(j)}. \end{aligned} \quad (3.8)$$

The anomaly terms are given by

$$\begin{aligned}\hat{A}_{(i)(j)}^R &= 0, \\ \hat{A}_{(i)(j)}^L &= 0, \\ \hat{A}_{(i)(j)} &= \frac{1}{8} \left(\hat{\pi}^{*k+i} r_{kl}(\hat{x}) a^{l+\bar{j}}(\hat{x}) - a^{k+i}(\hat{x}) r_{kl}(\hat{x}) \hat{\pi}^{l+\bar{j}} \right),\end{aligned}\tag{3.9}$$

where $r_{ij}(x)$ is the Ricci tensor in the infinite-dimensional spacetime, which, according to the general map (A.5), is related to the same in physical spacetime, namely, $R_{\mu\nu}(X)$ in the following way:

$$r_{ij}(x) \sim 2\pi\delta(0) \oint \frac{d\sigma}{2\pi} R_{\mu\nu}(X(\sigma)) e^{i(m+n)\sigma}.\tag{3.10}$$

The DWV algebra, as operator equations, should be valid independent of the representation considered. Therefore, the result in (3.8) is correct up to possible additional operator terms whose expectation value vanishes in the calculation of [4, 5]. However, since the calculation was done between two arbitrary states, it is expected that such terms are not present.

Notice that we have found the tensor representations in the previous section from first principle. This construction does not *a priori* know about the DWV algebra. Consistency would therefore require the results in (3.8), (3.9) to hold true as expectation value between two arbitrary tensor states of any rank. Following the basic method of [4, 5], we have checked this consistency condition to affirmative in Appendix E.

Appendices

A. Summary of Previous Work

All the analysis in this paper has been done using the infinite-dimensional/particle language introduced in [4, 5]. Here, we will briefly review the basic results of that work. The matter part of the conformally gauge fixed worldsheet lagrangian takes the following form in this language:

$$L(x, \dot{x}) = \frac{1}{2} g_{ij}(x) \left[\dot{x}^i \dot{x}^j - a^i(x) a^j(x) \right],\tag{A.1}$$

where the infinite-dimensional spacetime index is give by

$$i = \{\mu, m\},\tag{A.2}$$

$\mu = 0 \cdots D-1$ and $m \in \mathbb{Z}$ being the physical spacetime index and the string-mode-number, respectively (throughout the paper, we make the following type of index identifications: $i = \{\mu, m\}$, $j = \{\nu, n\}$, $k = \{\kappa, q\}$). The coordinate x^i is implicitly dependent on the worldsheet

time coordinate τ . A dot indicates derivative with respect to τ . The map between the infinite-dimensional and worldsheet languages is given by

$$\begin{aligned} x^i &= \oint \frac{d\sigma}{2\pi} X^\mu(\sigma) e^{-im\sigma}, \\ g_{ij}(x) &= \oint \frac{d\sigma}{2\pi} G_{\mu\nu}(X(\sigma)) e^{i(m+n)\sigma}, \\ a^i(x) &= \oint \frac{d\sigma}{2\pi} \partial X^\mu(\sigma) e^{-im\sigma}, \end{aligned} \quad (\text{A.3})$$

where $G_{\mu\nu}$ is the metric in physical spacetime, $\oint \equiv \int_0^{2\pi}$ and $\partial \equiv \partial_\sigma$. Using the above map an arbitrary field in the infinite-dimensional spacetime can be mapped to a nonlocal worldsheet operator. A class of examples, which includes the classical Virasoro generators and their quantum version, is given by a multi-indexed object $u_{i_2 j_2 \dots}^{i_1 j_1 \dots}(x)$ constructed out of the metric, its inverse, their derivatives, and $a^i(x)$ (but not its derivatives) such that $u_{i_2 j_2 \dots}^{i_1 j_1 \dots}(x)$ cannot be factored into pieces which are not contracted with each other. In this case one can construct a local worldsheet operator $U_{\mu_2 \nu_2 \dots}^{\mu_1 \nu_1 \dots}(X(\sigma))$ simply by performing the following replacements in the expression of $u_{i_2 j_2 \dots}^{i_1 j_1 \dots}(x)$:

$$g_{ij}(x) \longrightarrow G_{\mu\nu}(X(\sigma)), \quad g^{ij}(x) \longrightarrow G^{\mu\nu}(X(\sigma)), \quad \partial_i \longrightarrow \partial_\mu, \quad a^i(x) \longrightarrow \partial X^\mu(\sigma). \quad (\text{A.4})$$

The two objects $u_{i_2 j_2 \dots}^{i_1 j_1 \dots}(x)$ and $U_{\mu_2 \nu_2 \dots}^{\mu_1 \nu_1 \dots}(X(\sigma))$ are related to each other by the following general rule:

$$u_{i_2 j_2 \dots}^{i_1 j_1 \dots}(x) \sim [2\pi\delta(0)]^N \oint \frac{d\sigma}{2\pi} U_{\mu_2 \nu_2 \dots}^{\mu_1 \nu_1 \dots}(X(\sigma)) e^{i(m_2+n_2+\dots)\sigma - i(m_1+n_1+\dots)\sigma}, \quad (\text{A.5})$$

where N is the number of traces in u (e.g., γ_k as defined in (A.22) has a single trace and therefore is divergent:

$$\gamma_k(x) = 2\pi\delta(0) \oint \frac{d\sigma}{2\pi} \Gamma_\kappa(X(\sigma)) e^{iq\sigma}, \quad (\text{A.6})$$

where the expression for $\Gamma_\kappa(X)$ can be read out from (A.22) using the replacement (A.4). The Ricci tensor is another example of a single-trace field (see (3.10)), and the argument of the Dirac delta function $\delta(0)$ appearing on the right hand side is the worldsheet space direction

$$\delta(0) = \lim_{\sigma \rightarrow \sigma'} \delta(\sigma - \sigma') = \lim_{\sigma \rightarrow \sigma'} \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{in(\sigma - \sigma')}. \quad (\text{A.7})$$

See [4, 5] for further discussion on the appearance of such $\delta(0)$ -factors. In the infinite-dimensional language the problem at hand possesses certain shift properties which can be written as

$$u_{j_1 j_2 \dots}^{i_1 + i i_2 \dots} = u_{j_1 j_2 \dots}^{i_1 i_2 + i \dots} = u_{j_1 - i j_2 \dots}^{i_1 i_2 \dots} = u_{j_1 j_2 - i \dots}^{i_1 i_2 \dots} = \dots . \quad (\text{A.8})$$

A shift in the infinite-dimensional index is defined to be $i + j = \{\mu, m + n\}$ (notice that we choose to denote the physical spacetime index corresponding to $i + j$ by the one associated with the first index (i.e., i) appearing in the shift. We will follow this convention in all our expressions). It is obvious that (A.8) is a direct consequence of (A.5). Last equation in (A.3) implies

$$\partial_j a^k = i(j) \delta_j^k, \quad (\text{A.9})$$

where given the spacetime index i as in (throughout the paper we make the following type of index identifications: $i = \{\mu, m\}$, $j = \{\nu, n\}$, $k = \{\kappa, q\}$), we have defined $(i) = m$. An important identity which was proved in [4, 5] and will be used crucially in deriving the results in Appendix E is given by

$$a^{k+i} \partial_k u_{i_2 j_2 \dots}^{i_1 j_1 \dots} = i \{ (i + i_1 + j_1 + \dots) - (i_2 + j_2 + \dots) \} u_{i_2 j_2 \dots}^{i_1 + i j_1 \dots}. \quad (\text{A.10})$$

To quantize the theory we first define the conjugate momentum $p_i = \partial L / \partial \dot{x}^i$ and then impose (in $\hbar = \alpha' = 1$ unit),

$$[\hat{x}^i, \hat{p}_j] = i \delta_j^i, \quad (\text{A.11})$$

where we have used a hat to denote a quantum operator. The hermiticity properties are given by

$$(\hat{x}^i)^\dagger = \hat{x}^{\bar{i}}, \quad \hat{p}_i^\dagger = \hat{p}_{\bar{i}}, \quad (\text{A.12})$$

where given (A.2) we have defined

$$\bar{i} = \{\mu, -m\}. \quad (\text{A.13})$$

In [6, 7] it was shown how to construct the position space representation of this quantum theory such that general covariance is manifest. An arbitrary scalar state $|\psi\rangle$ is expanded in position basis in the following way (throughout this work we will suppress the τ dependence of the Schrödinger states to reduce clutter):

$$|\psi\rangle = \int dw \psi(x) |x\rangle, \quad (\text{A.14})$$

where $dw = dx\sqrt{g}$ ($g = |\det g_{ij}|$) is the invariant measure. The position eigenstate $|x\rangle$ satisfies

$$\hat{x}^i|x\rangle = x^i|x\rangle, \quad \left(x^i\right)^* = x^{\bar{i}} \quad (\text{A.15})$$

and the following orthonormality and completeness conditions,

$$\langle x | \tilde{x} \rangle = \delta(x, \tilde{x}), \quad \int dw |x\rangle \langle x| = 1, \quad (\text{A.16})$$

where

$$\delta(x, \tilde{x}) = (g(x)g(\tilde{x}))^{-(1/4)} \delta(x - \tilde{x}), \quad (\text{A.17})$$

$\delta(x - \tilde{x})$ being the Dirac delta function.

The general coordinate transformation (GCT) is understood in the quantum theory in the following way. The eigenvalue x^i of the position operator is the coordinate of a point $\mathcal{P}$ on the manifold $\mathcal{M}$, where the particle-dynamics is taking place. A GCT is given by a set of functions $f = \{f^i(x) \mid f^i(x)^* = f^{\bar{i}}(x)\}$ such that

$$\begin{aligned} \hat{x}^i &\longrightarrow \hat{x}^{\prime i} = f^i(\hat{x}), & \hat{p}_i &\longrightarrow \hat{p}'_i = \frac{1}{2} \left(\lambda_i^j(\hat{x}) \hat{p}_j + \hat{p}_j \lambda_i^j(\hat{x}) \right), \\ x^i &\longrightarrow x^{\prime i} = f^i(x), \end{aligned} \quad (\text{A.18})$$

where

$$\lambda_j^i(x) = \frac{\partial x^{\prime i}}{\partial x^j}, \quad \lambda_i^j(x) = \frac{\partial x^j}{\partial x^{\prime i}} \quad (\text{A.19})$$

are the Jacobian matrix for the transformation and its inverse, respectively. The original system $(\hat{x}^i, \hat{p}_i, x^i)$ and the transformed one $(\hat{x}^{\prime i}, \hat{p}'_i, x^{\prime i})$ give equivalent descriptions of the quantum theory such that x^i and $x^{\prime i}$ are associated to the same point $\mathcal{P}$ in $\mathcal{M}$. Both the wavefunction and the position eigenstate in (A.14) are scalar under GCT,

$$\psi(x) \longrightarrow \psi'(x') = \psi(x), \quad |x\rangle \longrightarrow |x'\rangle' = |x\rangle. \quad (\text{A.20})$$

The DWV generators are defined in (3.1), (3.2), (3.3) in terms of certain shifted momentum operators,

$$\hat{\pi}_k = \hat{p}_k + \frac{i}{2} \gamma_k(\hat{x}), \quad \hat{\pi}_k^* = \hat{p}_k - \frac{i}{2} \gamma_k(\hat{x}), \quad (\text{A.21})$$

where γ_k are the contracted Christoffel symbols,

$$\gamma_k = \gamma_{ki}^i, \quad \gamma_{jk}^i = \frac{1}{2} g^{il} (\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk}). \quad (\text{A.22})$$

Position space representation of $\hat{\pi}_k$ is found to be

$$\langle x | \hat{\pi}_k | \tilde{x} \rangle = -i \partial_k \delta(x, \tilde{x}). \quad (\text{A.23})$$

Using either the definitions in (A.21) or the above representation, one can establish the following transformations under GCT,

$$\hat{\pi}_k \longrightarrow \hat{\pi}'_k = \lambda^l_k(\hat{x}) \hat{\pi}_l, \quad \hat{\pi}^*_k \longrightarrow \hat{\pi}^{\star'}_k = \hat{\pi}^*_l \lambda^l_k(\hat{x}), \quad (\text{A.24})$$

which show that the DWV generators are scalar operators. Using this basic ingredient one calculates the DWV algebra in spin-zero representation. The result is as given in (3.8), (3.9).

B. Bivector of Parallel Displacement

Here we will define and discuss the properties of the parallel displacement bivector $\Delta_{ij}(x, \tilde{x})$. As explained in Section 1, such a bivector contracted with another vector at one of the points, gives the same vector parallel transported to the other point along a given path. However, as remarked below (2.11), our analysis is sensitive to the expansion of $\Delta_{ij}(x, \tilde{x})$ only to first-order in $(x - \tilde{x})^i$. Such contributions are independent of the path chosen. To make the discussion valid for arbitrary separation and for definiteness we choose to discuss the bivector of geodetic parallel displacement [38] where the aforementioned path is a geodesic. We will first define this object (following [38]) for a particle in the physical spacetime using our notation and elaborate, to some extent, on the explicit representation in terms of the vielbeins. Then we will consider the string-case and describe how it is constructed in the infinite-dimensional language.

Let us consider two nearby points X^μ and $\tilde{X}^\mu$ in the physical spacetime such that a unique geodesic passes through them. The world-function $\Sigma(X, \tilde{X})$, which is a biscalar, is given by [41] (the proportionality constant is usually taken as $\pm(1/2)$ with the $\pm$ sign corresponding to space-like and time-like separations respectively. However, for our purpose such details are not needed)

$$\Sigma(X, \tilde{X}) \propto S(X, \tilde{X})^2, \quad (\text{B.1})$$

where $S(X, \tilde{X}) = S(\tilde{X}, X)$ is the biscalar of geodetic interval. The bivector of geodetic parallel displacement $D_{\mu\nu}(X, \tilde{X})$ satisfies the following defining equations [38]:

$$\begin{aligned} \nabla^\rho \Sigma(X, \tilde{X}) \nabla_\rho D_{\mu\nu}(X, \tilde{X}) &= 0, \\ \tilde{\nabla}^\rho \Sigma(X, \tilde{X}) \tilde{\nabla}_\rho D_{\mu\nu}(X, \tilde{X}) &= 0, \\ \lim_{X \rightarrow \tilde{X}} D_{\mu\nu}(X, \tilde{X}) &= G_{\mu\nu}(\tilde{X}), \end{aligned} \quad (\text{B.2})$$

where ∇_μ and $\tilde{\nabla}_\mu$ denote the covariant derivatives with respect to X^μ and $\tilde{X}^\mu$ respectively. It turns out that $\nabla^\rho \Sigma(X, \tilde{X})$ and $\tilde{\nabla}^\rho \Sigma(X, \tilde{X})$ are directed along (or opposite to) the tangent

vectors to the geodesic at X and $\tilde{X}$, respectively [41]. From this one infers the following geometrical interpretation of $D_{\mu\nu}(X, \tilde{X})$:

$$D_{\mu}^{\nu}(X, \tilde{X})V_{\nu}(\tilde{X}) = V_{\mu}(\tilde{X} \rightarrow X), \quad D_{\nu}^{\mu}(X, \tilde{X})V_{\mu}(X) = V_{\nu}(X \rightarrow \tilde{X}), \quad (\text{B.3})$$

where $V_{\mu}(X)$ is an arbitrary vector and $V_{\mu}(X \rightarrow \tilde{X})$ denotes the vector $V_{\mu}(X)$ parallel transported from X to $\tilde{X}$. Therefore, $D_{\mu\nu}(X, \tilde{X})$ parallel displaces a vector along the geodesic. As argued in [38], $D_{\mu\nu}(X, \tilde{X})$ is unique. Therefore, (B.3) can be taken as an alternative definition. The following properties follow from this geometrical interpretation:

$$\begin{aligned} D_{\mu\nu}(X, \tilde{X}) &= D_{\nu\mu}(\tilde{X}, X), \\ D_{\mu\rho}(X, \tilde{X})D_{\nu}^{\rho}(X, \tilde{X}) &= G_{\mu\nu}(X), \\ D_{\rho\mu}(X, \tilde{X})D_{\nu}^{\rho}(X, \tilde{X}) &= G_{\mu\nu}(\tilde{X}). \end{aligned} \quad (\text{B.4})$$

For our calculations we will use the explicit representation of $D_{\mu\nu}(X, \tilde{X})$ in terms of the vielbeins $E_{\mu}^{\hat{\mu}}(X)$, where $\hat{\mu}$ is the local Lorentz index. The representation is given by

$$\begin{aligned} D_{\mu\nu}(X, \tilde{X}) &= E_{\mu}^{\hat{\mu}}(\tilde{X} \rightarrow X)\eta_{\hat{\mu}\hat{\nu}}E_{\nu}^{\hat{\nu}}(\tilde{X}), \\ &= E_{\mu}^{\hat{\mu}}(X)\eta_{\hat{\mu}\hat{\nu}}E_{\nu}^{\hat{\nu}}(X \rightarrow \tilde{X}), \end{aligned} \quad (\text{B.5})$$

where the parallel transport of the vielbeins is defined by vanishing of the following covariant derivative:

$$\nabla_{\mu}E_{\nu}^{\hat{\mu}} \equiv \partial_{\mu}E_{\nu}^{\hat{\mu}} - \Gamma_{\mu\nu}^{\kappa}E_{\kappa}^{\hat{\mu}} = -\Omega_{\mu}^{\hat{\mu}}{}_{\hat{\nu}}E_{\nu}^{\hat{\nu}}, \quad (\text{B.6})$$

along the geodesic, such that the Lorentz index remains unaffected. In the last step above we have written the result in terms of the spin connection Ω_{μ} , using the fact that the *total covariant derivative* of the vielbein vanishes.

The simplest way to justify the representation (B.5), is to show that it leads to the same geometrical interpretation in (B.3). According to (B.6) the Lorentz component $V^{\hat{\mu}} = E_{\mu}^{\hat{\mu}}V^{\mu}$ of a vector V^{μ} remains constant under parallel transport. Therefore, to parallel transport a vector it is sufficient to transport only the vielbein,

$$\begin{aligned} V^{\mu}(\tilde{X} \rightarrow X) &= E^{\hat{\mu}\mu}(\tilde{X} \rightarrow X)\eta_{\hat{\mu}\hat{\nu}}V^{\hat{\nu}}(\tilde{X}), \\ &= E^{\hat{\mu}\mu}(\tilde{X} \rightarrow X)\eta_{\hat{\mu}\hat{\nu}}E_{\nu}^{\hat{\nu}}(\tilde{X})V^{\nu}(\tilde{X}). \end{aligned} \quad (\text{B.7})$$

Combining the last equation above and the first equation in (B.3) one identifies the first equation in (B.5). The second equation in (B.3) can also be obtained in a similar way. These

two equations make the first property in (B.4) manifestly true. It is also easy to prove the other two properties in (B.4) by recalling the fact that a vielbein remains so under a parallel transport [41]. So we must have

$$E_{\mu}^{\hat{\mu}}(\tilde{X} \longrightarrow X) E_{\hat{\mu}\nu}(\tilde{X} \longrightarrow X) = G_{\mu\nu}(X). \quad (\text{B.8})$$

Notice that the representations in (B.5) is unique up to a single Lorentz transformation which may be taken to be the one relating the objects, say, at X in the two expressions. But according to the definition of covariant derivative in (B.6), a Lorentz transformation matrix is invariant under parallel transport and therefore the same transformation relates the objects at $\tilde{X}$. However, when $D_{\mu\nu}(X, \tilde{X})$ acts on a vector this ambiguity gets canceled by the same in defining the Lorentz component of that vector.

We now generalize the above discussion to a string. We consider two nearby string-embeddings $X(\sigma)$ and $\tilde{X}(\sigma)$ in physical spacetime, both parametrized by σ , such that the pair of points on the two strings at the same value of σ are connected by a unique geodesic. This leads to the definition of the geodetic parallel displacement bivector $D_{\mu\nu}(X(\sigma), \tilde{X}(\sigma))$ for the string. The infinite-dimensional analogue of the same is given by the general map (A.5),

$$\begin{aligned} \Delta_{ij}(x, \tilde{x}) &= \oint \frac{d\sigma}{2\pi} D_{\mu\nu}(X(\sigma), \tilde{X}(\sigma)) e^{i(m+n)\sigma}, \\ &= e_{\hat{i}}^{\hat{i}}(\tilde{x} \longrightarrow x) \eta_{\hat{i}\hat{j}} e_{\hat{j}}^{\hat{j}}(\tilde{x}), \\ &= e_{\hat{i}}^{\hat{i}}(x) \eta_{\hat{i}\hat{j}} e_{\hat{j}}^{\hat{j}}(x \longrightarrow \tilde{x}), \end{aligned} \quad (\text{B.9})$$

where the infinite-dimensional analogues of the vielbeins and Minkowski metric are given by

$$\begin{aligned} e_{\hat{i}}^{\hat{i}}(x) &= \oint \frac{d\sigma}{2\pi} E_{\hat{\mu}}^{\hat{\mu}}(X(\sigma)) e^{i(m-\hat{m})\sigma}, \\ \eta_{\hat{i}\hat{j}} &= \oint \frac{d\sigma}{2\pi} \eta_{\hat{\mu}\hat{\nu}} e^{i(m+n)\sigma} = \eta_{\hat{\mu}\hat{\nu}} \delta_{m+n,0}, \end{aligned} \quad (\text{B.10})$$

respectively. This way all the relevant equations that we discussed in the context of physical spacetime have infinite-dimensional analogues. For example, the analogue of (B.6), which we use extensively in our computations, is given by

$$\nabla_{\hat{i}} e_{\hat{j}}^{\hat{i}} = \partial_{\hat{i}} e_{\hat{j}}^{\hat{i}} - \gamma_{\hat{i}\hat{j}}^k e_{\hat{k}}^{\hat{i}} = -\omega_{\hat{i}\hat{j}}^{\hat{j}} e_{\hat{j}}^{\hat{j}}, \quad (\text{B.11})$$

where,

$$w_{\hat{i}\hat{j}} = \oint \frac{d\sigma}{2\pi} \Omega_{\hat{\mu}\hat{\nu}}(X(\sigma)) e^{i(m+\hat{m}+\hat{n})\sigma}. \quad (\text{B.12})$$

C. Tensor Representation of $\hat{\pi}_k$

Here we will derive the tensor representation of $\hat{\pi}_k$ as given in (2.17). We follow the basic line of argument in [6]. In particular, we demand that the following operator equations hold as matrix elements,

$$[\hat{x}^l, \hat{\pi}_k] = i\delta_k^l, \quad [\hat{\pi}_k, \hat{\pi}_l] = 0, \quad (C.1)$$

$$\hat{\pi}_k^\dagger = \hat{\pi}_k^*. \quad (C.2)$$

Using the first commutator in (C.1) one gets the following condition:

$$(x - \tilde{x})^l \langle i_1 \dots i_n; x | \hat{\pi}_k | j_1 \dots j_n; \tilde{x} \rangle = i\delta_k^l \delta_{n,\tilde{n}} \Delta_{i_1 j_1}(x, \tilde{x}) \cdots \Delta_{i_n j_n}(x, \tilde{x}) \delta(x, \tilde{x}). \quad (C.3)$$

Using the identities,

$$\begin{aligned} (x - \tilde{x})^l \partial_k \delta(x, \tilde{x}) &= -\delta_k^l \delta(x, \tilde{x}), \\ \partial_k \delta(x, \tilde{x}) &= -(\tilde{\partial}_k + \gamma_k(x)) \delta(x, \tilde{x}), \end{aligned} \quad (C.4)$$

we write down the general solution in the form as given in (2.15) with $\Delta_{i_1 \dots i_n j_1 \dots j_n}(x, \tilde{x})$ as given in (2.7). The hermiticity condition in (C.2) is easily verified by using the expression of $F_{i_1 \dots i_n j_1 \dots j_n k}(x, \tilde{x})$ as given in (2.16). Consider the following:

$$\begin{aligned} \langle i_1 \dots i_n; x | \hat{\pi}_k^\dagger | j_1 \dots j_n; \tilde{x} \rangle &= \langle \bar{j}_1 \dots \bar{j}_n; \tilde{x} | \hat{\pi}_k | \bar{i}_1 \dots \bar{i}_n; x \rangle^*, \\ &= i\delta_{n\tilde{n}} \left[\Delta_{j_1 \dots j_n i_1 \dots i_n}(\tilde{x}, x) \tilde{\partial}_k \delta(x, \tilde{x}) + F_{\bar{j}_1 \dots \bar{j}_n \bar{i}_1 \dots \bar{i}_n k}^*(\tilde{x}, x) \delta(x, \tilde{x}) \right], \\ &= -i\delta_{n\tilde{n}} \left[\Delta_{i_1 \dots i_n j_1 \dots j_n}(x, \tilde{x}) (\partial_k + \gamma_k(x)) \delta(x, \tilde{x}) \right. \\ &\quad \left. + F_{i_1 \dots i_n j_1 \dots j_n k}(x, \tilde{x}) \delta(x, \tilde{x}) \right], \\ &= \langle i_1 \dots i_n; x | \hat{\pi}_k^* | j_1 \dots j_n; \tilde{x} \rangle, \end{aligned} \quad (C.5)$$

where in the second step we have used the reality property (2.13). In the third step we have used

$$\Delta_{i_1 \dots i_n j_1 \dots j_n}(x, \tilde{x}) = \Delta_{j_1 \dots j_n i_1 \dots i_n}(\tilde{x}, x), \quad (C.6)$$

the second equation in (C.4), and

$$\begin{aligned} F_{i_1 \dots i_n j_1 \dots j_n k}^*(x, \tilde{x}) &= F_{\bar{i}_1 \dots \bar{i}_n \bar{j}_1 \dots \bar{j}_n \bar{k}}(x, \tilde{x}), \\ F_{i_1 \dots i_n j_1 \dots j_n k}(x, \tilde{x}) \delta(x, \tilde{x}) &= -F_{j_1 \dots j_n i_1 \dots i_n k}(\tilde{x}, x) \delta(x, \tilde{x}). \end{aligned} \quad (C.7)$$

We will verify the second commutator in (C.1) between two arbitrary invariant tensor states using the expressions in (2.17) instead of the position eigenstates. This will make the expressions look more transparent. To this end we first show how one gets the results in (2.17) from (2.15). We first expand $\Delta_{ij}(x, \tilde{x})$ about $\tilde{x}$,

$$\Delta_{ij}(x, \tilde{x}) = g_{ij}(\tilde{x}) + (x - \tilde{x})^k \gamma_{ki}^j(\tilde{x}) g_{lj}(\tilde{x}) + \dots \quad (\text{C.8})$$

This is obtained by expanding the vielbein $e_i^{\hat{j}}(\tilde{x} \rightarrow x)$ in the second equation of (B.9) up to first-order. Using this one finds

$$\Delta_{i_1 \dots i_n j_1 \dots j_n}(x, \tilde{x}) \partial_k \delta(x, \tilde{x}) = g_{i_1 \dots i_n j_1 \dots j_n}(\tilde{x}) \partial_k \delta(x, \tilde{x}) - \gamma_{i_1 \dots i_n j_1 \dots j_n k}(x) \delta(x, \tilde{x}), \quad (\text{C.9})$$

where $g_{i_1 \dots i_n j_1 \dots j_n}$ and $\gamma_{i_1 \dots i_n j_1 \dots j_n k}$ are as given in (2.18) and (2.19), respectively. Interchanging $x \leftrightarrow \tilde{x}$ and $i \leftrightarrow j$ in (C.9) and using (C.6) and

$$g_{i_1 \dots i_n j_1 \dots j_n}(x) = g_{j_1 \dots j_n i_1 \dots i_n}(x), \quad (\text{C.10})$$

one establishes

$$\Delta_{i_1 \dots i_n j_1 \dots j_n}(x, \tilde{x}) \tilde{\partial}_k \delta(x, \tilde{x}) = g_{i_1 \dots i_n j_1 \dots j_n}(x) \tilde{\partial}_k \delta(x, \tilde{x}) - \gamma_{j_1 \dots j_n i_1 \dots i_n k}(x) \delta(x, \tilde{x}). \quad (\text{C.11})$$

Finally, it is straightforward to show

$$F_{i_1 \dots i_n j_1 \dots j_n k}(x, \tilde{x}) \delta(x, \tilde{x}) = \omega_{i_1 \dots i_n j_1 \dots j_n k}(x) \delta(x, \tilde{x}). \quad (\text{C.12})$$

Substituting the results (C.9), (C.11), and (C.12) in (2.15) one finds the results (2.17).

Next we will consider the matrix element of the second commutator in (C.1) between two arbitrary tensor states $|\chi_{(n)}\rangle$ and $|\psi_{(n)}\rangle$ of rank n . In order to do such calculations we need the following results:

$$\begin{aligned} \langle \chi_{(n)} | \hat{\pi}_k | j_1 \dots j_n; x \rangle &= i \left[(\nabla_k + \gamma_k(x)) \bar{\chi}_{j_1 \dots j_n}(x) + \bar{\chi}^{i_1 \dots i_n}(x) \omega_{i_1 \dots i_n j_1 \dots j_n k}(x) \right], \\ \langle i_1 \dots i_n; x | \hat{\pi}_k | \psi_{(n)} \rangle &= -i \left[\nabla_k \psi_{i_1 \dots i_n}(x) - \omega_{i_1 \dots i_n j_1 \dots j_n k}(x) \psi^{j_1 \dots j_n}(x) \right]. \end{aligned} \quad (\text{C.13})$$

Such results are derived using the expressions in (2.17) and the following relations,

$$\begin{aligned} \partial_k g_{i_1 \dots i_n j_1 \dots j_n} &= \gamma_{i_1 \dots i_n j_1 \dots j_n k} + \gamma_{j_1 \dots j_n i_1 \dots i_n k}, \\ \nabla_k \psi_{i_1 \dots i_n} &= \partial_k \psi_{i_1 \dots i_n} - \gamma_{i_1 \dots i_n j_1 \dots j_n k} \psi^{j_1 \dots j_n}. \end{aligned} \quad (\text{C.14})$$

We will now calculate $\langle \chi | \hat{\pi}_k \hat{\pi}_l | \psi \rangle$ by first inserting the completeness relation (2.8) between the two $\hat{\pi}$'s and then using the results (C.13). To get to the final expression we need to go through certain manipulations that are generally involved in the computations of the matrix

elements carried out in this paper. We give the details of the present calculation in the following series of steps:

$$\begin{aligned}
\langle \chi_{(n)} | \hat{\pi}_k \hat{\pi}_l | \psi_{(n)} \rangle &= \int d\omega \langle \chi_{(n)} | \hat{\pi}_k |_{k_1 \dots k_n}; x \rangle g^{k_1 l_1}(x) \dots g^{k_n l_n}(x) \langle_{l_1 \dots l_n}; x | \hat{\pi}_l | \psi_{(n)} \rangle, \\
&= \int d\omega \left\{ (\nabla_k + \gamma_k) \bar{\chi}_{k_1 \dots k_n} + \bar{\chi}^{i_1 \dots i_n} \omega_{i_1 \dots i_n k_1 \dots k_n k} \right\} \\
&\quad \left(\nabla_l \psi^{k_1 \dots k_n} - \omega_{l_1 \dots l_n l}^{k_1 \dots k_n} \psi^{l_1 \dots l_n} \right), \\
&= \int d\omega \left\{ (\partial_k + \gamma_k) \bar{\chi}_{k_1 \dots k_n} - \left(\gamma_{k k_1}^{k'} \bar{\chi}_{k' k_2 \dots k_n} + \gamma_{k k_2}^{k'} \bar{\chi}_{k_1 k' k_3 \dots k_n} + \dots \right) \right. \\
&\quad \left. + \bar{\chi}^{i_1 \dots i_n} \omega_{i_1 \dots i_n k_1 \dots k_n k} \right\} \left(\nabla_l \psi^{k_1 \dots k_n} - \omega_{l_1 \dots l_n l}^{k_1 \dots k_n} \psi^{l_1 \dots l_n} \right), \\
&= - \int d\omega \left[\bar{\chi}_{k_1 \dots k_n} \partial_k \left(\nabla_l \psi^{k_1 \dots k_n} - \omega_{l_1 \dots l_n l}^{k_1 \dots k_n} \psi^{l_1 \dots l_n} \right) \right. \\
&\quad + \bar{\chi}_{k_1 \dots k_n} \left\{ \gamma_{k k'}^{k_1} \left(\nabla_l \psi^{k' k_2 \dots k_n} - \omega_{l_1 \dots l_n l}^{k' k_2 \dots k_n} \psi^{l_1 \dots l_n} \right) \right. \\
&\quad \left. + \gamma_{k k'}^{k_2} \left(\nabla_l \psi^{k_1 k' k_2 \dots k_n} - \omega_{l_1 \dots l_n l}^{k_1 k' k_2 \dots k_n} \psi^{l_1 \dots l_n} \right) \dots \right\} \quad (C.15) \\
&\quad \left. - \bar{\chi}^{i_1 \dots i_n} \omega_{i_1 \dots i_n k_1 \dots k_n k} \left(\nabla_l \psi^{k_1 \dots k_n} - \omega_{l_1 \dots l_n l}^{k_1 \dots k_n} \psi^{l_1 \dots l_n} \right) \right], \\
&= - \int d\omega \left[\bar{\chi}_{k_1 \dots k_n} \nabla_k \left(\nabla_l \psi^{k_1 \dots k_n} - \omega_{l_1 \dots l_n l}^{k_1 \dots k_n} \psi^{l_1 \dots l_n} \right) \right. \\
&\quad + \bar{\chi}_{k_1 \dots k_n} \gamma_{kl}^{l'} \left(\nabla_{l'} \psi^{k_1 \dots k_n} - \omega_{l_1 \dots l_n l'}^{k_1 \dots k_n} \psi^{l_1 \dots l_n} \right) \\
&\quad \left. - \bar{\chi}^{i_1 \dots i_n} \omega_{i_1 \dots i_n k_1 \dots k_n k} \left(\nabla_l \psi^{k_1 \dots k_n} - \omega_{l_1 \dots l_n l}^{k_1 \dots k_n} \psi^{l_1 \dots l_n} \right) \right], \\
&= - \int d\omega \left[\bar{\chi}_{k_1 \dots k_n} \nabla_k \nabla_l \psi^{k_1 \dots k_n} - \bar{\chi}_{k_1 \dots k_n} \nabla_k \omega_{l_1 \dots l_n l}^{k_1 \dots k_n} \psi^{l_1 \dots l_n} \right. \\
&\quad + \bar{\chi}^{i_1 \dots i_n} \omega_{i_1 \dots i_n k_1 \dots k_n k} \omega_{l_1 \dots l_n l}^{k_1 \dots k_n} \psi^{l_1 \dots l_n} \\
&\quad + \bar{\chi}_{k_1 \dots k_n} \gamma_{kl}^{l'} \left(\nabla_{l'} \psi^{k_1 \dots k_n} - \omega_{l_1 \dots l_n l'}^{k_1 \dots k_n} \psi^{l_1 \dots l_n} \right) \\
&\quad \left. - \bar{\chi}_{k_1 \dots k_n} \left(\omega_{l_1 \dots l_n l}^{k_1 \dots k_n} \nabla_k + k \longleftrightarrow l \right) \psi^{l_1 \dots l_n} \right].
\end{aligned}$$

The last two terms in the above expression are symmetric in k and l . Therefore, they do not contribute to the commutator which is given by

$$\langle \chi_{(n)} | [\hat{\pi}_k, \hat{\pi}_l] | \psi_{(n)} \rangle = - \int d\omega V, \quad (C.16)$$

where

$$V = \bar{\chi}_{k_1 \dots k_n} [\nabla_k, \nabla_l] \psi^{k_1 \dots k_n} - \bar{\chi}_{k_1 \dots k_n} \left(\nabla_k \omega_{l_1 \dots l_n}^{k_1 \dots k_n} - k \longleftrightarrow l \right) \psi^{l_1 \dots l_n} \\ + \bar{\chi}^{i_1 \dots i_n} \left(\omega_{i_1 \dots i_n k_1 \dots k_n} \omega_{j_1 \dots j_n}^{k_1 \dots k_n} - k \longleftrightarrow l \right) \psi^{j_1 \dots j_n}. \quad (\text{C.17})$$

Below we will show that V vanishes. Using

$$[\nabla_k, \nabla_l] \psi^{k_1 \dots k_n} = r_{k'kl}^{k_1} \psi^{k'k_2 \dots k_n} + r_{k'kl}^{k_2} \psi^{k_1 k'k_3 \dots k_n} + \dots, \quad (\text{C.18})$$

r_{jkl}^i being the Riemann tensor, one may write the first term in (C.17) as

$$\bar{\chi}_{k_1 \dots k_n} [\nabla_k, \nabla_l] \psi^{k_1 \dots k_n} = \bar{\chi}^{k_1 \dots k_n} r_{k_1 \dots k_n l_1 \dots l_n kl} \psi^{l_1 \dots l_n}, \quad (\text{C.19})$$

where

$$r_{k_1 \dots k_n l_1 \dots l_n kl} = r_{k_1 l_1 kl} g_{k_2 l_2} \dots g_{k_n l_n} + g_{k_1 l_1} r_{k_2 l_2 kl} g_{k_3 l_3} \dots g_{k_n l_n} + \dots. \quad (\text{C.20})$$

To calculate the second term in (C.17) one first derives, following the standard argument [42],

$$\nabla_k \omega_{lij} - k \longleftrightarrow l = r_{ijkl} + d_{ijkl}, \quad (\text{C.21}) \\ d_{ijkl} = \omega_{kii'} \omega_{l'j}^i - k \longleftrightarrow l,$$

using which one may write,

$$\nabla_k \omega_{k_1 \dots k_n l_1 \dots l_n l} - k \longleftrightarrow l = r_{k_1 \dots k_n l_1 \dots l_n kl} + d_{k_1 \dots k_n l_1 \dots l_n kl}, \quad (\text{C.22})$$

where we have defined,

$$d_{i_1 \dots i_n, j_1 \dots j_n, kl} \equiv d_{i_1 j_1 kl} g_{i_2 j_2} \dots g_{i_n j_n} + g_{i_1 j_1} d_{i_2 j_2 kl} g_{i_3 j_3} \dots g_{i_n j_n} + \dots. \quad (\text{C.23})$$

We note down the following antisymmetry properties for future use:

$$r_{i_1 \dots i_n j_1 \dots j_n kl} = -r_{j_1 \dots j_n i_1 \dots i_n kl} = -r_{i_1 \dots i_n j_1 \dots j_n lk}, \quad (\text{C.24}) \\ d_{i_1 \dots i_n j_1 \dots j_n kl} = -d_{j_1 \dots j_n i_1 \dots i_n kl} = -d_{i_1 \dots i_n j_1 \dots j_n lk}.$$

The last term in (C.17) is calculated in the following way:

$$\omega_{i_1 \dots i_n k_1 \dots k_n} \omega_{j_1 \dots j_n}^{k_1 \dots k_n} - k \longleftrightarrow l = (\omega_{k i_1 k_1} g_{i_2 k_2} \dots g_{i_n k_n} + g_{i_1 k_1} \omega_{k i_2 k_2} g_{i_3 k_3} \dots g_{i_n k_n} + \dots) \\ \left(\omega_{l' j_1}^{k_1} g_{j_2}^{k_2} \dots g_{j_n}^{k_n} + g_{j_1}^{k_1} \omega_{l' j_2}^{k_2} g_{j_3}^{k_3} \dots g_{j_n}^{k_n} + \dots \right) - k \longleftrightarrow l, \quad (\text{C.25}) \\ = \mathfrak{D}_{i_1 \dots i_n, j_1 \dots j_n, kl} + \mathcal{C}_{i_1 \dots i_n, j_1 \dots j_n, kl} - k \longleftrightarrow l,$$

where the *diagonal* and the *cross* terms are

$$\begin{aligned} \mathfrak{D}_{i_1 \dots i_n, j_1 \dots j_n, kl} &= (\omega_{ki_1 k_1} \omega_l^{k_1}_{j_1}) g_{i_2 j_2} \dots g_{i_n j_n} \\ &+ g_{i_1 j_1} (\omega_{ki_2 k_2} \omega_l^{k_2}_{j_2}) g_{i_3 j_3} \dots g_{i_n j_n} + \dots, \end{aligned} \quad (C.26)$$

$$\begin{aligned} \mathcal{C}_{i_1 \dots i_n, j_1 \dots j_n, kl} &= \{(\omega_{ki_1 j_1}) (\omega_{li_2 j_2}) g_{i_3 j_3} \dots g_{i_n j_n} + k \longleftrightarrow l\} \\ &+ \{(\omega_{ki_1 j_1}) g_{i_2 j_2} (\omega_{li_3 j_3}) g_{i_4 j_4} \dots g_{i_n j_n} + k \longleftrightarrow l\} + \dots, \end{aligned} \quad (C.27)$$

respectively. The cross-term, being symmetric under $k \leftrightarrow l$, does not contribute to (C.25). Therefore,

$$\omega_{i_1 \dots i_n, k_1 \dots k_n, k} \omega_{j_1 \dots j_n, l}^{k_1 \dots k_n} - k \longleftrightarrow l = d_{i_1 \dots i_n, j_1 \dots j_n, kl}. \quad (C.28)$$

Using the results in (C.19), (C.22), and (C.28) in (C.17) one shows that V vanishes implying

$$\langle \chi_{(n)} | [\hat{\pi}_k, \hat{\pi}_l] | \psi_{(n)} \rangle = 0. \quad (C.29)$$

The above calculations show why the F -term in (2.15) or the spin connection term in (2.17) is essential. This is needed to cancel the Riemann tensor contributions which will otherwise survive in the calculation of the above commutator.

D. GCT as a Unitary Transformation

It should be possible to understand GCT, as introduced below (A.17), as a usual unitary transformation in quantum mechanics. This has been explained for spin-zero representation in [7]. Here we will generalize the argument to incorporate the tensor representations.

The rank- n tensor wavefunction and the position eigenstate in (2.6) transform under GCT in the following way:

$$\begin{aligned} \psi^{i_1 \dots i_n}(x) &\longrightarrow \psi'^{i_1 \dots i_n}(x') = \lambda_{j_1}^{i_1}(x) \dots \lambda_{j_n}^{i_n}(x) \psi^{j_1 \dots j_n}(x), \\ |i_1 i_2 \dots i_n; x\rangle &\longrightarrow |i_1 i_2 \dots i_n; x'\rangle' = \lambda_{i_1}^{j_1}(x) \lambda_{i_2}^{j_2}(x) \dots \lambda_{i_n}^{j_n}(x) |j_1 j_2 \dots j_n; x\rangle. \end{aligned} \quad (D.1)$$

Using (A.18) and (D.1) one finds, as expected,

$$\hat{x}^i |i_1 \dots i_n; x'\rangle' = x'^i |i_1 \dots i_n; x'\rangle'. \quad (D.2)$$

Let us now consider the following unitary operator:

$$\begin{aligned} \hat{U} &= e^{(i/2)\hat{V}}, \\ \hat{V} &= \hat{\pi}_i^* v^i(\hat{x}) + v^i(\hat{x}) \hat{\pi}_i, \end{aligned} \quad (D.3)$$

where the $\hat{\pi}$ -operators are defined in (A.21) and $v^i(x)$ is a vector field such that

$$v^i(x)^* = \bar{v}^i(x). \quad (\text{D.4})$$

The above definition implies that $\hat{V}$ is both Hermitian and GCT invariant. The latter can be checked easily by using the transformation property in (A.24). At the infinitesimal level $\hat{U}$ produces the following coordinate transformation:

$$\hat{x}^i(\hat{x}) = \hat{U} \hat{x}^i \hat{U}^\dagger = \hat{x}^i + v^i(\hat{x}). \quad (\text{D.5})$$

The question that one would like to ask is how the state $\hat{U}|_{i_1 \dots i_n}; x\rangle$ is related to $|_{i_1 \dots i_n}; x\rangle'$. Notice that both of these states have the same eigenvalue for $\hat{x}^i$, namely x^i .

To answer the above question, we first write the infinitesimal version of (D.1) in the following way:

$$|_{i_1 \dots i_n}; x - v(x)\rangle = \left(\delta_{i_1}^{j_1} \delta_{i_2}^{j_2} \dots \delta_{i_n}^{j_n} + \nabla_{i_1} v^{j_1} \delta_{i_2}^{j_2} \dots \delta_{i_n}^{j_n} + \delta_{i_1}^{j_1} \nabla_{i_2} v^{j_2} \delta_{i_3}^{j_3} \dots \delta_{i_n}^{j_n} + \dots \right) |_{j_1 \dots j_n}; x\rangle', \quad (\text{D.6})$$

where the state appearing on the left hand side has a covariant Taylor expansion in the following sense:

$$\langle_{i_1 \dots i_n}; x - v(x) | \psi_{(n)}\rangle = \psi_{i_1 \dots i_n}(x) - v^k(x) \nabla_k \psi_{i_1 \dots i_n}(x), \quad (\text{D.7})$$

where $|\psi_{(n)}\rangle$ is an arbitrary rank- n tensor state as in (2.6). Below we will show that the following relation is true,

$$\hat{U}|_{i_1 \dots i_n}; x\rangle = N_{i_1 \dots i_n}^{j_1 \dots j_n}(x) |_{j_1 \dots j_n}; x - v(x)\rangle, \quad (\text{D.8})$$

where the normalization is given by:

$$\begin{aligned} N_{i_1 \dots i_n j_1 \dots j_n} &= g_{i_1 \dots i_n j_1 \dots j_n} + \delta N_{i_1 \dots i_n j_1 \dots j_n}, \\ \delta N_{i_1 \dots i_n j_1 \dots j_n} &= -\frac{1}{2} \nabla_k v^k g_{i_1 \dots i_n j_1 \dots j_n} + \omega_{i_1 \dots i_n j_1 \dots j_n k} v^k. \end{aligned} \quad (\text{D.9})$$

To prove (D.8), (D.9) we will first calculate $\langle_{i_1 \dots i_n}; x | \hat{U}^\dagger | \psi_{(n)}\rangle$ independently by using (D.8) and (D.3) and then compare the results. Using (D.8) one gets

$$\begin{aligned} \langle_{i_1 \dots i_n}; x | \hat{U}^\dagger | \psi_{(n)}\rangle &= N_{i_1 \dots i_n}^* \bar{j}_1 \dots \bar{j}_n(x) \left(\psi_{j_1 \dots j_n}(x) - v^k(x) \nabla_k \psi_{j_1 \dots j_n}(x) \right), \\ &= \left[1 - v^k(x) \nabla_k \right] \psi_{i_1 \dots i_n}(x) + \delta N_{i_1 \dots i_n}^* \bar{j}_1 \dots \bar{j}_n \psi_{j_1 \dots j_n}(x), \end{aligned} \quad (\text{D.10})$$

where in the second step we have used the first equation in (D.9). On the other hand using (D.3) one finds

$$\left\langle_{i_1 \dots i_n} x \left| \widehat{U}^\dagger \right| \psi_{(n)} \right\rangle = \left[1 - \frac{1}{2} \nabla_k v^k(x) - v^k(x) \nabla_k \right] \psi_{i_1 \dots i_n}(x) + \omega_{i_1 \dots i_n j_1 \dots j_n k}(x) v^k(x) \psi^{j_1 \dots j_n}(x), \quad (\text{D.11})$$

where we have used the representation (2.17). Finally, comparing (D.10) and (D.11), we arrive at the second equation in (D.9).

E. Derivation of DWV Algebra

Here we will reproduce the results in (3.8), (3.9) in tensor representation. This will be done by showing that the results hold as matrix elements between two arbitrary tensor states $|\chi\rangle$ and $|\psi\rangle$ of rank n (To simplify our notations we remove the subscript (n) from the states in this appendix.) We begin by quoting certain results that will be needed as we go along.

$$\begin{aligned} \left\langle \chi \left| \widehat{K}_{(i)} \right|_{i_1 \dots i_n} ; x \right\rangle &= (-i)^2 \left[\nabla_{(i)}^2 \bar{\chi}_{i_1 \dots i_n} + 2 \nabla^{k+i} \bar{\chi}^{j_1 \dots j_n} \omega_{j_1 \dots j_n i_1 \dots i_n k} \right. \\ &\quad \left. + \bar{\chi}^{j_1 \dots j_n} \nabla^{k+i} \omega_{j_1 \dots j_n i_1 \dots i_n k} + \bar{\chi}^{j_1 \dots j_n} \omega_{j_1 \dots j_n k_1 \dots k_n k} \omega^{k_1 \dots k_n}_{i_1 \dots i_n}{}^{k+i} \right], \end{aligned} \quad (\text{E.1})$$

$$\begin{aligned} \left\langle_{i_1 \dots i_n} x \left| \widehat{K}_{(i)} \right| \psi \right\rangle &= (-i)^2 \left[\nabla_{(i)}^2 \psi_{i_1 \dots i_n} - \nabla^{k+i} \omega_{i_1 \dots i_n j_1 \dots j_n k} \psi^{j_1 \dots j_n} \right. \\ &\quad \left. - 2 \omega_{i_1 \dots i_n j_1 \dots j_n k} \nabla^{k+i} \psi^{j_1 \dots j_n} + \omega_{i_1 \dots i_n k_1 \dots k_n k} \omega^{k_1 \dots k_n}_{j_1 \dots j_n}{}^{k+i} \psi^{j_1 \dots j_n} \right], \end{aligned}$$

$$\left\langle \chi \left| \widehat{Z}_{(i)}^L \right|_{i_1 \dots i_n} ; x \right\rangle = -i \left(-\nabla_k \bar{\chi}_{i_1 \dots i_n} - \bar{\chi}^{j_1 \dots j_n} \omega_{j_1 \dots j_n i_1 \dots i_n k} \right) a^{k+i}, \quad (\text{E.2})$$

$$\left\langle_{i_1 \dots i_n} x \left| \widehat{Z}_{(i)}^L \right| \psi \right\rangle = -i \left[\nabla_k \left(a^{k+i} \psi_{i_1 \dots i_n} \right) - \omega_{i_1 \dots i_n j_1 \dots j_n k} a^{k+i} \psi^{j_1 \dots j_n} \right],$$

$$\left\langle \chi \left| \widehat{Z}_{(i)}^R \right|_{i_1 \dots i_n} ; x \right\rangle = -i \left[-\nabla_k \left(\bar{\chi}_{i_1 \dots i_n} a^{k+i} \right) - \bar{\chi}^{j_1 \dots j_n} \omega_{j_1 \dots j_n i_1 \dots i_n k} a^{k+i} \right], \quad (\text{E.3})$$

$$\left\langle_{i_1 \dots i_n} x \left| \widehat{Z}_{(i)}^R \right| \psi \right\rangle = -i a^{k+i} \left[\nabla_k \psi_{i_1 \dots i_n} - \omega_{i_1 \dots i_n j_1 \dots j_n k} \psi^{j_1 \dots j_n} \right].$$

To proceed with the computation of DWV algebra, we first introduce certain notations. We break each of the relevant commutators into several terms in the following way:

$$\begin{aligned} x \left\langle \left[\widehat{L}_{(i)}, \widehat{L}_{(j)} \right] \right\rangle_\psi &= \frac{1}{16} \left[T^{K_{(i)} K_{(j)}} + T^{Z_{(i)} Z_{(j)}} - \left(T^{K_{(i)} Z_{(j)}} - i \longleftrightarrow j \right) \right. \\ &\quad \left. + \left(T^{K_{(i)} V_{(j)}} - i \longleftrightarrow j \right) - \left(T^{Z_{(i)} V_{(j)}} - i \longleftrightarrow j \right) \right], \\ x \left\langle \left[\widehat{\bar{L}}_{(i)}, \widehat{\bar{L}}_{(j)} \right] \right\rangle_\psi &= \frac{1}{16} \left[T^{K_{(\bar{i})} K_{(\bar{j})}} + T^{Z_{(\bar{i})} Z_{(\bar{j})}} + \left(T^{K_{(\bar{i})} Z_{(\bar{j})}} - \bar{i} \longleftrightarrow \bar{j} \right) \right. \\ &\quad \left. + \left(T^{K_{(\bar{i})} V_{(\bar{j})}} - \bar{i} \longleftrightarrow \bar{j} \right) + \left(\widehat{T}^{Z_{(\bar{i})} V_{(\bar{j})}} - \bar{i} \longleftrightarrow \bar{j} \right) \right], \end{aligned}$$

$$\begin{aligned}
{}_x \left\langle \left[\hat{L}_{(i)}, \hat{L}_{(j)} \right] \right\rangle_\psi &= \frac{1}{16} \left[T^{K_{(i)} K_{(j)}} - T^{Z_{(i)} Z_{(j)}} + \left(T^{K_{(i)} Z_{(j)}} + i \longleftrightarrow \bar{j} \right) \right. \\
&\quad \left. + \left(T^{K_{(i)} V_{(j)}} - i \longleftrightarrow \bar{j} \right) - \left(T^{Z_{(i)} V_{(j)}} + i \longleftrightarrow \bar{j} \right) \right],
\end{aligned} \tag{E.4}$$

where ${}_x \langle \cdots \rangle_\psi = \langle \chi | \cdots | \psi \rangle$ and

$$T^{AB} = {}_x \left\langle \left[\hat{A}, \hat{B} \right] \right\rangle_\psi. \tag{E.5}$$

Terms involving the operator $\hat{Z}_{(i)}$ are further divided into components in the following way:

$$\begin{aligned}
T^{Z_{(i)} Z_{(j)}} &= T^{Z_{(i)}^L Z_{(j)}^L} + T^{Z_{(i)}^R Z_{(j)}^R} + \left(T^{Z_{(i)}^L Z_{(j)}^R} - i \longleftrightarrow j \right), \\
T^{K_{(i)} Z_{(j)}} &= T^{K_{(i)} Z_{(j)}^L} + T^{K_{(i)} Z_{(j)}^R}, \\
T^{Z_{(i)} V_{(j)}} &= T^{Z_{(i)}^L V_{(j)}} + T^{Z_{(i)}^R V_{(j)}}.
\end{aligned} \tag{E.6}$$

Each of the commutator terms receives contributions at different orders in $\omega_{i_1 \cdots i_n j_1 \cdots j_n k}$,

$$T^{AB} = (-i)^r \left[\overline{O}_{\chi\psi}^{AB} + \overline{L}_{\chi\psi}^{AB} + \overline{Q}_{\chi\psi}^{AB} + \overline{C}_{\chi\psi}^{AB} \right], \tag{E.7}$$

where r is the number of $\hat{\pi}$ operator involved in the computation and $\overline{O}$, $\overline{L}$, $\overline{Q}$, and $\overline{C}$ denote the terms which are zeroth, first, second, and third order in ω , respectively. We have made ordering of χ and ψ explicit in the notations of such contributions as we will exploit certain symmetry under $\bar{\chi} \leftrightarrow \psi$ to compute a particular set of terms. Although for other terms it is not needed, we use a uniform notation. Similar contributions for a matrix element of product of two operators will be denoted by the same symbols, but without the bar (The quantity ${}_x \langle \hat{K}_{(i)} \hat{K}_{(j)} \rangle_\psi$ receives contribution at the quartic order, but we will not have to calculate it explicitly.) Consider

$${}_x \left\langle \hat{A} \hat{B} \right\rangle_\psi = (-i)^r \left[O_{\chi\psi}^{AB} + L_{\chi\psi}^{AB} + Q_{\chi\psi}^{AB} + C_{\chi\psi}^{AB} \right]. \tag{E.8}$$

Below we will compute each of the terms appearing in (E.4), (E.6) separately.

$$T^{K_{(i)} K_{(j)}}$$

Using (E.1), one can explicitly compute all the terms in ${}_x \langle \hat{K}_{(i)} \hat{K}_{(j)} \rangle_\psi$. In [4, 5] it was shown that in spin-zero representation such terms are symmetric under $i \leftrightarrow j$ by using the shift property in (A.8). In the present case the wavefunctions themselves carry tensor indices and a direct application of such an argument would require the wavefunctions also to possess similar shift property. According to our general definition of tensor wavefunctions which has been explained below (2.1), these objects may in general involve derivatives of $a^i(x)$. Since

the general form of $u_{i_2 j_2 \dots}^{i_1 j_1 \dots}$ (see discussion above (A.4)), for which the shift property in (A.8) holds true, does not allow such factors, we do not want to assume the shift property for the wavefunctions. However, as will be discussed below, a slightly modified argument may be used to show that all the terms in ${}_X \langle \hat{K}_{(i)} \hat{K}_{(j)} \rangle_\psi$ are indeed symmetric in ij without assuming any special property for the wavefunctions. This would therefore imply

$$T^{K_{(i)} K_{(j)}} = 0. \quad (\text{E.9})$$

We begin with the $\mathcal{O}(\omega^0)$ -term. By using integration by parts we write it in the following form:

$$\overline{\mathcal{O}}_{X^\psi}^{K_{(i)} K_{(j)}} = \int dw \bar{\chi}^{i_1 \dots i_n} \left[\nabla_{(i)}^2, \nabla_{(j)}^2 \right] \psi_{i_1 \dots i_n}. \quad (\text{E.10})$$

We then expand the commutator in the following way:

$$\begin{aligned} \left[\nabla_{(i)}^2, \nabla_{(j)}^2 \right] &= \nabla_k \left[\nabla^{k+i}, \nabla^{l+j} \right] \nabla_l + \nabla_k \nabla_l \left[\nabla^{k+i}, \nabla^{l+j} \right] \\ &\quad + \left[\nabla^{k+i}, \nabla^{l+j} \right] \nabla_l \nabla_k + \nabla_l \left[\nabla^{k+i}, \nabla^{l+j} \right] \nabla_k. \end{aligned} \quad (\text{E.11})$$

Notice that the terms have been written in such a way that the shifts always appear within the commutators. This can be done as we may write

$$\nabla^{k+i} \nabla_k = g^{k+il} \nabla_l \nabla_k = g^{l+ik} \nabla_l \nabla_k = \nabla_k \nabla^{k+i}, \quad (\text{E.12})$$

without assuming any shift property for a tensor field on which the derivatives are acting. Then we recognize that the following is true for an arbitrary tensor $t^{i_1 \dots i_n}$, which does not necessarily possess the shift property,

$$\begin{aligned} \left[\nabla^{k+i}, \nabla^{l+j} \right] t^{i_1 \dots i_n} &= r^{i_1}_{i'} t^{k+il+j}_{i' i_2 \dots i_n} + r^{i_2}_{i'} t^{k+il+j}_{i_1 i' i_3 \dots i_n} + \dots, \\ &= r^{i_1+i+j}_{i'} t^{kl}_{i' i_2 \dots i_n} + r^{i_2+i+j}_{i'} t^{kl}_{i_1 i' i_3 \dots i_n} + \dots, \end{aligned} \quad (\text{E.13})$$

where in the second step we have used the shift property (A.8) for the Riemann tensor. This shows that the right hand side of (E.11) is symmetric under $i \leftrightarrow j$. However, the left hand side is manifestly antisymmetric. Therefore, it must vanish.

Let us now consider the linear term $\bar{L}_{\chi\psi}^{K(i)K(j)}$. Following similar argument as above many of the terms can be shown to vanish by using integration by parts and the shift property of $\omega_{i_1 \dots i_n j_1 \dots j_n k}$ and its covariant derivatives. The terms that are problematic are as follows:

$$\begin{aligned} \bar{L}_{\chi\psi}^{K(i)K(j)} = & \int dw \left[\nabla^{k+i} \bar{\chi}^{i_1 \dots i_n} \nabla^{l+j} \omega_{i_1 \dots i_n j_1 \dots j_n l} \nabla_k \psi^{j_1 \dots j_n} \right. \\ & + 2 \nabla^{k+i} \bar{\chi}^{i_1 \dots i_n} \omega_{i_1 \dots i_n j_1 \dots j_n l} \nabla_k \nabla^{l+j} \psi^{j_1 \dots j_n} \\ & - \nabla_l \bar{\chi}^{i_1 \dots i_n} \nabla^{k+i} \omega_{i_1 \dots i_n j_1 \dots j_n k} \nabla^{l+j} \psi^{j_1 \dots j_n} \\ & \left. - 2 \nabla_l \nabla^{k+i} \bar{\chi}^{i_1 \dots i_n} \omega_{i_1 \dots i_n j_1 \dots j_n k} \nabla^{l+j} \psi^{j_1 \dots j_n} \right] - i \longleftrightarrow j. \end{aligned} \quad (E.14)$$

Notice that each of the above terms is such that the shift i cannot be moved to the position of the shift j and vice versa without using the shift property of the wavefunctions. Therefore, such terms are not individually symmetric under $i \leftrightarrow j$. However, by using integrations by parts the above terms can be rearranged to give the following form:

$$\begin{aligned} \bar{L}_{\chi\psi}^{K(i)K(j)} = & \int dw \left[-\nabla^{l+j} \nabla^{k+i} \bar{\chi}^{i_1 \dots i_n} (\omega_{i_1 \dots i_n j_1 \dots j_n l} \nabla_k + k \longleftrightarrow l) \psi^{j_1 \dots j_n} \right. \\ & + \left(\nabla_k \bar{\chi}^{i_1 \dots i_n} \omega_{i_1 \dots i_n j_1 \dots j_n l} + k \longleftrightarrow l \right) \nabla^{k+i} \nabla^{l+j} \psi^{j_1 \dots j_n} \\ & + \left[\nabla^{k+i}, \nabla^{l+j} \right] \bar{\chi}^{i_1 \dots i_n} \omega_{i_1 \dots i_n j_1 \dots j_n k} \nabla_l \psi^{j_1 \dots j_n} \\ & \left. + \nabla_k \bar{\chi}^{i_1 \dots i_n} \omega_{i_1 \dots i_n j_1 \dots j_n l} \left[\nabla^{k+i}, \nabla^{l+j} \right] \psi^{j_1 \dots j_n} \right] - i \longleftrightarrow j. \end{aligned} \quad (E.15)$$

The last two terms are symmetric in i and j because of the reason discussed earlier. Therefore, they drop out in the final result when we antisymmetrize. Each of the first two terms has a factor symmetric under $k \leftrightarrow l$. Because of this such a term, combined with the counterpart with i and j interchanged, finally contains the commutator of the *shifted covariant derivatives* $[\nabla^{k+i}, \nabla^{l+j}]$. Therefore these terms also vanish following the previous argument. Similar arguments can be used to show that all the other relevant terms vanish leading to the result (E.9).

$$T^{Z(i)Z(j)}$$

Using (E.2) and manipulating various terms we first write the contributions to $_{\chi} \langle \hat{Z}_{(i)}^L \hat{Z}_{(j)}^L \rangle_{\psi}$ in the following forms:

$$O_{\chi\psi}^{Z(i)Z(j)} = \int dw \left[\nabla_l \nabla_k \bar{\chi}^{i_1 \dots i_n} a^{k+i} a^{l+j} \psi_{i_1 \dots i_n} + \nabla_k \bar{\chi}^{i_1 \dots i_n} \nabla_l a^{k+i} a^{l+j} \psi_{i_1 \dots i_n} \right], \quad (E.16)$$

$$\begin{aligned}
L_{\chi\psi}^{Z_{(i)}^L Z_{(j)}^L} &= \int dw \left[\left(\nabla_k \bar{\chi}^{i_1 \dots i_n} a^{k+i} \omega_{i_1 \dots i_n j_1 \dots j_n l} a^{l+j} \psi^{j_1 \dots j_n} + i \longleftrightarrow j \right) \right. \\
&\quad + \bar{\chi}^{i_1 \dots i_n} \nabla_l \omega_{i_1 \dots i_n j_1 \dots j_n k} a^{k+i} a^{l+j} \psi^{j_1 \dots j_n} \\
&\quad \left. + \bar{\chi}^{i_1 \dots i_n} \omega_{i_1 \dots i_n j_1 \dots j_n k} \nabla_l a^{k+i} a^{l+j} \psi^{j_1 \dots j_n} \right], \tag{E.17}
\end{aligned}$$

$$Q_{\chi\psi}^{Z_{(i)}^L Z_{(j)}^L} = \int dw \bar{\chi}^{i_1 \dots i_n} \omega_{i_1 \dots i_n k_1 \dots k_n k} \omega_{j_1 \dots j_n l}^{k_1 \dots k_n} a^{k+i} a^{l+j} \psi^{j_1 \dots j_n}. \tag{E.18}$$

Antisymmetrizing the above results in i and j leads to the following contributions to the commutator:

$$\begin{aligned}
\bar{O}_{\chi\psi}^{Z_{(i)}^L Z_{(j)}^L} &= \int dw \left[[\nabla_l, \nabla_k] \bar{\chi}^{i_1 \dots i_n} a^{k+i} a^{l+j} \psi_{i_1 \dots i_n} \right. \\
&\quad \left. + i \sum_k \left\{ (k+i) \nabla_k \bar{\chi}^{i_1 \dots i_n} a^{k+i+j} \psi_{i_1 \dots i_n} - i \longleftrightarrow j \right\} \right], \tag{E.19}
\end{aligned}$$

$$\begin{aligned}
&= \int dw \left[\bar{\chi}^{i_1 \dots i_n} r_{i_1 \dots i_n j_1 \dots j_n k l} a^{k+i} a^{l+j} \psi^{j_1 \dots j_n} + i(i-j) \nabla_k \bar{\chi}^{i_1 \dots i_n} a^{k+i+j} \psi_{i_1 \dots i_n} \right], \\
\bar{L}_{\chi\psi}^{Z_{(i)}^L Z_{(j)}^L} &= \int dw \left[\bar{\chi}^{i_1 \dots i_n} (\nabla_l \omega_{i_1 \dots i_n j_1 \dots j_n k} - \nabla_k \omega_{i_1 \dots i_n j_1 \dots j_n l}) a^{k+i} a^{l+j} \psi^{j_1 \dots j_n} \right. \\
&\quad \left. + i \sum_k \left\{ (k+i) \bar{\chi}^{i_1 \dots i_n} \omega_{i_1 \dots i_n j_1 \dots j_n k} a^{k+i+j} \psi^{j_1 \dots j_n} - i \longleftrightarrow j \right\} \right], \tag{E.20}
\end{aligned}$$

$$\begin{aligned}
&= \int dw \left[-\bar{\chi}^{i_1 \dots i_n} (r_{i_1 \dots i_n j_1 \dots j_n k l} + d_{i_1 \dots i_n j_1 \dots j_n k l}) a^{k+i} a^{l+j} \psi^{j_1 \dots j_n} \right. \\
&\quad \left. + i(i-j) \bar{\chi}^{i_1 \dots i_n} \omega_{i_1 \dots i_n j_1 \dots j_n k} a^{k+i+j} \psi^{j_1 \dots j_n} \right], \\
\bar{Q}_{\chi\psi}^{Z_{(i)}^L Z_{(j)}^L} &= \int dw \bar{\chi}^{i_1 \dots i_n} \left(\omega_{i_1 \dots i_n k_1 \dots k_n k} \omega_{j_1 \dots j_n l}^{k_1 \dots k_n} - l \longleftrightarrow k \right) a^{k+i} a^{l+j} \psi^{j_1 \dots j_n}, \\
&= \int dw \bar{\chi}^{i_1 \dots i_n} d_{i_1 \dots i_n j_1 \dots j_n k l} a^{k+i} a^{l+j} \psi^{j_1 \dots j_n}. \tag{E.21}
\end{aligned}$$

In the first steps of (E.19) and (E.20), we have evaluated the covariant derivative $\nabla_l a^{k+i}$ appearing in (E.16) using (A.9). The connection terms coming from these covariant derivatives do not contribute because of the antisymmetrization in i and j . The first terms in (E.19) and (E.20) and the term in (E.21) are evaluated using (C.19), (C.22), and (C.28), respectively. Each of these terms vanishes individually because of the shift property. However, an interesting feature of these results is the following. Even if we do not use the shift property, such terms cancel each other when we add all the above contributions to get the relevant commutator. We will see as we proceed that this sort of cancellations will be very

crucial in computing certain other terms of the DWV algebra. Using (3.6) the final result is written in the following form:

$$T^{Z_{(i)}^L Z_{(j)}^L} = -(i-j) {}_X \left\langle \hat{Z}_{(i+j)}^L \right\rangle_\psi. \quad (\text{E.22})$$

A similar calculation shows,

$$T^{Z_{(i)}^R Z_{(j)}^R} = -(i-j) {}_X \left\langle \hat{Z}_{(i+j)}^R \right\rangle_\psi. \quad (\text{E.23})$$

To calculate the last term in (E.6), we manipulate the terms in ${}_X \langle \hat{Z}_{(i)}^L \hat{Z}_{(j)}^R \rangle_\psi$ and ${}_X \langle \hat{Z}_{(j)}^R \hat{Z}_{(i)}^L \rangle_\psi$ to arrive at the following results for the terms in $T^{Z_{(i)}^L Z_{(j)}^R}$ at various orders:

$$\begin{aligned} \overline{O}_{X\psi}^{Z_{(i)}^L Z_{(j)}^R} &= \int dw \left[\bar{\chi}^{i_1 \dots i_n} a^{l+j} [\nabla_k, \nabla_l] \left(a^{k+i} \psi_{i_1 \dots i_n} \right) + \nabla_k \bar{\chi}^{i_1 \dots i_n} a^{l+j} \nabla_l a^{k+i} \psi_{i_1 \dots i_n} \right. \\ &\quad \left. + \bar{\chi}^{i_1 \dots i_n} \nabla_k a^{l+j} a^{k+i} \nabla_l \psi_{i_1 \dots i_n} + \bar{\chi}^{i_1 \dots i_n} \nabla_k a^{l+j} \nabla_l a^{k+i} \psi_{i_1 \dots i_n} \right], \\ &= \int dw \left[\bar{\chi}^{i_1 \dots i_n} a^{k+i} r_{kl} a^{l+j} \psi_{i_1 \dots i_n} + \bar{\chi}^{i_1 \dots i_n} a^{k+i} r_{i_1 \dots i_n j_1 \dots j_n kl} a^{l+j} \psi^{j_1 \dots j_n} \right. \\ &\quad \left. + \bar{\chi}^{i_1 \dots i_n} \nabla_k a^{l+j} \nabla_l a^{k+i} \psi_{i_1 \dots i_n} + \nabla_k \bar{\chi}^{i_1 \dots i_n} a^{l+j} \nabla_l a^{k+i} \psi_{i_1 \dots i_n} \right. \\ &\quad \left. + \bar{\chi}^{i_1 \dots i_n} \nabla_k a^{l+j} a^{k+i} \nabla_l \psi_{i_1 \dots i_n} \right], \\ \overline{L}_{X\psi}^{Z_{(i)}^L Z_{(j)}^R} &= - \int dw \bar{\chi}^{i_1 \dots i_n} \left[a^{k+i} a^{l+j} (\nabla_k \omega_{i_1 \dots i_n j_1 \dots j_n l} - k \longleftrightarrow l) \right. \\ &\quad \left. + (a^{k+i} \nabla_k a^{l+j} \omega_{i_1 \dots i_n j_1 \dots j_n l} - i \longleftrightarrow j) \right] \psi^{j_1 \dots j_n}, \\ &= \int dw \bar{\chi}^{i_1 \dots i_n} \left[-a^{k+i} a^{l+j} (r_{i_1 \dots i_n j_1 \dots j_n kl} + d_{i_1 \dots i_n j_1 \dots j_n kl}) \right. \\ &\quad \left. + i(i-j) a^{k+i+j} \omega_{i_1 \dots i_n j_1 \dots j_n k} \right] \psi^{j_1 \dots j_n}, \\ \overline{Q}_{X\psi}^{Z_{(i)}^L Z_{(j)}^R} &= \int dw \bar{\chi}^{i_1 \dots i_n} \left(\omega_{i_1 \dots i_n k_1 \dots k_n k} \omega_{j_1 \dots j_n l}^{k_1 \dots k_n} - k \longleftrightarrow l \right) a^{k+i} a^{l+j} \psi^{j_1 \dots j_n}, \\ &= \int dw \bar{\chi}^{i_1 \dots i_n} d_{i_1 \dots i_n j_1 \dots j_n kl} a^{k+i} a^{l+j} \psi^{j_1 \dots j_n}. \end{aligned} \quad (\text{E.24})$$

In this case also the terms involving the Riemann tensor and $d_{i_1 \dots i_n j_1 \dots j_n k l}$ vanish individually due to the shift property. They also cancel each other when we add all the above contributions. Finally, the relevant commutator is given by

$$\begin{aligned} T^{Z_{(i)}^L Z_{(j)}^R} = & (-i)^2 \int dw \left[\bar{\chi}^{i_1 \dots i_n} a^{k+i} r_{kl} a^{l+j} \psi_{i_1 \dots i_n} + \bar{\chi}^{i_1 \dots i_n} \nabla_k a^{l+j} \nabla_l a^{k+i} \psi_{i_1 \dots i_n} \right. \\ & + \nabla_k \bar{\chi}^{i_1 \dots i_n} a^{l+j} \nabla_l a^{k+i} \psi_{i_1 \dots i_n} + \bar{\chi}^{i_1 \dots i_n} \nabla_k a^{l+j} a^{k+i} \nabla_l \psi_{i_1 \dots i_n} \\ & \left. + i(i-j) \bar{\chi}^{i_1 \dots i_n} a^{k+i+j} \omega_{i_1 \dots i_n j_1 \dots j_n k} \psi^{j_1 \dots j_n} \right]. \end{aligned} \quad (E.25)$$

As indicated in (E.6), we need to antisymmetrize the above quantity in i and j . The first two terms drop out as they are symmetric. The final result is given by

$$T^{Z_{(i)}^L Z_{(j)}^R} - i \longleftrightarrow j = -(i-j)_\chi \left\langle \hat{Z}_{(i+j)} \right\rangle_\psi. \quad (E.26)$$

Substituting (E.22), (E.23) and (E.25) into (E.6) one gets

$$T^{Z_{(i)} Z_{(j)}} = -2(i-j)_\chi \left\langle \hat{Z}_{(i+j)} \right\rangle_\psi. \quad (E.27)$$

$T^{K_{(i)} Z_{(j)}}$

This is the most important contribution as the anomaly term arises here. After a somewhat lengthy calculation we arrive at the following expressions:

$$\begin{aligned} \bar{O}_{\chi\psi}^{K_{(i)} Z_{(j)}^L} = & \int dw \left[\nabla^{k+i} \bar{\chi}^{i_1 \dots i_n} r_{kl} a^{l+j} \psi_{i_1 \dots i_n} \right. \\ & + \nabla^{l+i} \nabla_k \bar{\chi}^{i_1 \dots i_n} \nabla_l a^{k+j} \psi_{i_1 \dots i_n} - \nabla_k \bar{\chi}^{i_1 \dots i_n} \nabla^{l+i} a^{k+j} \nabla_l \psi_{i_1 \dots i_n} \\ & - \left(\nabla^{k+i} \bar{\chi}^{i_1 \dots i_n} r_{i_1 \dots i_n j_1 \dots j_n kl} a^{l+j} \psi^{j_1 \dots j_n} + \bar{\chi} \longleftrightarrow \psi \right) \\ & \left. + \bar{\chi}^{i_1 \dots i_n} r_{i_1 \dots i_n j_1 \dots j_n kl} \nabla^{k+i} a^{l+j} \psi^{j_1 \dots j_n} \right], \end{aligned} \quad (E.28)$$

$$\begin{aligned} \bar{L}_{\chi\psi}^{K_{(i)} Z_{(j)}^L} = & - \int dw \left[\left(\bar{\chi}^{i_1 \dots i_n} \omega_{i_1 \dots i_n j_1 \dots j_n l} \nabla_k \psi^{j_1 \dots j_n} + \bar{\chi} \longleftrightarrow \psi \right) \left(\nabla^{k+i} a^{l+j} + k \longleftrightarrow l \right) \right. \\ & - \left\{ \nabla^{k+i} \bar{\chi}^{i_1 \dots i_n} (r_{i_1 \dots i_n j_1 \dots j_n kl} + d_{i_1 \dots i_n j_1 \dots j_n kl}) a^{l+j} \psi^{j_1 \dots j_n} + \bar{\chi} \longleftrightarrow \psi \right\} \\ & + 2 \bar{\chi}^{i_1 \dots i_n} r_{i_1 \dots i_n k_1 \dots k_n kl} \omega^{k_1 \dots k_n}_{j_1 \dots j_n} a^{l+j} \psi^{j_1 \dots j_n} \\ & + \bar{\chi}^{i_1 \dots i_n} \left\{ \nabla_l \left(\omega_{i_1 \dots i_n j_1 \dots j_n}^{k+i} \nabla_k a^{l+j} \right) - \nabla_l \omega_{i_1 \dots i_n j_1 \dots j_n k} \nabla^{k+i} a^{l+j} \right. \\ & \left. \left. - [\nabla_k, \nabla_l] \omega_{i_1 \dots i_n j_1 \dots j_n}^{k+i} a^{l+j} \right\} \psi^{j_1 \dots j_n} \right], \end{aligned} \quad (E.29)$$

$$\begin{aligned}
\overline{Q}_{\chi\psi}^{K(i)Z(j)} = & \int dw \left[-\nabla^{k+i} \overline{\chi}^{i_1 \dots i_n} d_{i_1 \dots i_n j_1 \dots j_n k l} a^{l+j} \psi^{j_1 \dots j_n} \right. \\
& + \overline{\chi}^{i_1 \dots i_n} (r_{i_1 \dots i_n k_1 \dots k_n k l} + d_{i_1 \dots i_n k_1 \dots k_n k l}) \omega^{k_1 \dots k_n}_{j_1 \dots j_n} a^{l+j} \psi^{j_1 \dots j_n} \\
& \left. + \overline{\chi}^{i_1 \dots i_n} \omega_{i_1 \dots i_n k_1 \dots k_n k} \omega^{k_1 \dots k_n}_{j_1 \dots j_n l} \nabla^{k+i} a^{l+j} \psi^{j_1 \dots j_n} + \overline{\chi} \longleftrightarrow \psi \right], \tag{E.30}
\end{aligned}$$

$$\begin{aligned}
\overline{C}_{\chi\psi}^{K(i)Z(j)} = & - \int dw \left[\overline{\chi}^{i_1 \dots i_n} \omega_{i_1 \dots i_n k_1 \dots k_n k} \omega^{k_1 \dots k_n}_{l_1 \dots l_n} \omega^{l_1 \dots l_n}_{j_1 \dots j_n l} a^{l+j} \psi^{j_1 \dots j_n} + \overline{\chi} \longleftrightarrow \psi \right], \\
= & - \int dw \left[\overline{\chi}^{i_1 \dots i_n} d_{i_1 \dots i_n k_1 \dots k_n k l} \omega^{k_1 \dots k_n}_{j_1 \dots j_n} a^{l+j} \psi^{j_1 \dots j_n} + \overline{\chi} \longleftrightarrow \psi \right]. \tag{E.31}
\end{aligned}$$

To get to the second step of (E.31) we have used (C.28) twice to push $\omega^{l_1 \dots l_n}_{j_1 \dots j_n l}$ to the left. In order to derive the expected result for $T^{K(i)Z(j)}$ we need cancellation of many terms to take place. This happens in two steps—some terms get canceled when we add all the above contributions to find $T^{K(i)Z(j)}$. Another set of cancellations takes place when we add the latter to $T^{K(i)Z(j)}$ to get the final result for $T^{K(i)Z(j)}$. To arrive at our final results, however, it will be easier for us to go in the reverse order. One can be easily convinced that various contributions to $T^{K(i)Z(j)}$ can be obtained by interchanging $\overline{\chi}$ and ψ in the expressions for the corresponding contributions to $T^{K(i)Z(j)}$. Therefore to get the terms in $T^{K(i)Z(j)}$ at various orders we simply symmetrize the expressions in (E.28), (E.29), (E.30), and (E.31) under $\overline{\chi} \leftrightarrow \psi$,

$$\begin{aligned}
\overline{Q}_{\chi\psi}^{K(i)Z(j)} = & \int dw \left[\nabla^{k+i} \overline{\chi}^{i_1 \dots i_n} r_{k l} a^{l+j} \psi_{i_1 \dots i_n} + \nabla^{l+i} \nabla_k \overline{\chi}^{i_1 \dots i_n} \nabla_l a^{k+j} \psi_{i_1 \dots i_n} \right. \\
& - \nabla_k \overline{\chi}^{i_1 \dots i_n} \nabla^{l+i} a^{k+j} \nabla_l \psi_{i_1 \dots i_n} \\
& \left. - 2 \nabla^{k+i} \overline{\chi}^{i_1 \dots i_n} r_{i_1 \dots i_n j_1 \dots j_n k l} a^{l+j} \psi^{j_1 \dots j_n} + \overline{\chi} \longleftrightarrow \psi \right], \\
\overline{L}_{\chi\psi}^{K(i)Z(j)} = & \int dw \left[-2 \overline{\chi}^{i_1 \dots i_n} \omega_{i_1 \dots i_n j_1 \dots j_n l} \nabla_k \psi^{j_1 \dots j_n} (\nabla^{k+i} a^{l+j} + k \longleftrightarrow l) \right. \\
& + 2 \nabla^{k+i} \overline{\chi}^{i_1 \dots i_n} (r_{i_1 \dots i_n j_1 \dots j_n k l} + d_{i_1 \dots i_n j_1 \dots j_n k l}) a^{l+j} \psi^{j_1 \dots j_n} \\
& \left. - 2 \overline{\chi}^{i_1 \dots i_n} r_{i_1 \dots i_n k_1 \dots k_n k l} \omega^{k_1 \dots k_n}_{j_1 \dots j_n} a^{l+j} \psi^{j_1 \dots j_n} + \overline{\chi} \longleftrightarrow \psi \right], \\
\overline{Q}_{\chi\psi}^{K(i)Z(j)} = & 2 \overline{Q}_{\chi\psi}^{K(i)Z(j)}, \\
\overline{C}_{\chi\psi}^{K(i)Z(j)} = & 2 \overline{C}_{\chi\psi}^{K(i)Z(j)}. \tag{E.32}
\end{aligned}$$

Notice that in the process of symmetrization the last terms of (E.28) and (E.29) dropped out. We now add all the above contributions to get $T^{K(i)Z(j)}$,

$$\begin{aligned}
T^{K(i)Z(j)} = & (-i)^3 \int dw \left[\nabla^{k+i} \overline{\chi}^{i_1 \dots i_n} r_{k l} a^{l+j} \psi_{i_1 \dots i_n} + \nabla^{l+i} \nabla_k \overline{\chi}^{i_1 \dots i_n} \nabla_l a^{k+j} \psi_{i_1 \dots i_n} \right. \\
& \left. - \nabla_k \overline{\chi}^{i_1 \dots i_n} \nabla^{l+i} a^{k+j} \nabla_l \psi_{i_1 \dots i_n} - 2 \overline{\chi}^{i_1 \dots i_n} \omega_{i_1 \dots i_n j_1 \dots j_n l} \nabla_k \psi^{j_1 \dots j_n} \right]
\end{aligned}$$

$$\begin{aligned}
& \times \left(\nabla^{k+i} a^{l+j} + k \longleftrightarrow l \right) \\
& + 2 \bar{\chi}^{i_1 \dots i_n} \omega_{i_1 \dots i_n k_1 \dots k_n k} \omega_{j_1 \dots j_n l}^{k_1 \dots k_n} \nabla^{k+i} a^{l+j} \psi^{j_1 \dots j_n} + \bar{\chi} \longleftrightarrow \psi \Big], \\
& = (-i)^3 \int d\omega \left[\nabla^{k+i} \bar{\chi}^{i_1 \dots i_n} r_{kl} a^{l+j} \psi_{i_1 \dots i_n} + \bar{\chi}^{i_1 \dots i_n} a^{k+j} r_{kl} \nabla^{l+i} \psi_{i_1 \dots i_n} \right. \\
& \quad + \left(\nabla^{l+i} \nabla_k \bar{\chi}^{i_1 \dots i_n} \nabla_l a^{k+j} \psi_{i_1 \dots i_n} + \bar{\chi} \longleftrightarrow \psi \right) \\
& \quad - \left\{ \nabla_k \bar{\chi}^{i_1 \dots i_n} \nabla_l \psi_{i_1 \dots i_n} \right. \\
& \quad + 2 \left(\bar{\chi}^{i_1 \dots i_n} \omega_{i_1 \dots i_n j_1 \dots j_n l} \nabla_k \psi^{j_1 \dots j_n} - \nabla_k \bar{\chi}^{i_1 \dots i_n} \omega_{i_1 \dots i_n j_1 \dots j_n l} \psi^{j_1 \dots j_n} \right) \\
& \quad \left. \left. - 2 \bar{\chi}^{i_1 \dots i_n} \omega_{i_1 \dots i_n k_1 \dots k_n k} \omega_{j_1 \dots j_n l}^{k_1 \dots k_n} \psi^{j_1 \dots j_n} \right\} \left(\nabla^{k+i} a^{l+j} + k \longleftrightarrow l \right) \right], \tag{E.33}
\end{aligned}$$

where in the last step we have simply rearranged the terms in a different way. We will now manipulate the terms in the last three lines in certain ways. Let us first consider the terms in the following second line:

$$\begin{aligned}
& \nabla^{l+i} \nabla_k \bar{\chi}^{i_1 \dots i_n} \nabla_l a^{k+j} \psi_{i_1 \dots i_n} \\
& = i \sum_k (k+j) \nabla^{k+i+j} \nabla_k \bar{\chi}^{i_1 \dots i_n} \psi_{i_1 \dots i_n} + a^{l'} \gamma_{l'l}^{k+j} \nabla^{l+i} \nabla_k \bar{\chi}^{i_1 \dots i_n} \psi_{i_1 \dots i_n}, \\
& = i \sum_k (k+j) \nabla^{k+i+j} \nabla_k \bar{\chi}^{i_1 \dots i_n} \psi_{i_1 \dots i_n} + \frac{1}{2} a^{l'} \gamma_{l\bar{k}}^{k+i+j} g^{\tilde{k}l} (\{ \nabla_l, \nabla_k \} + [\nabla_l, \nabla_k]) \bar{\chi}^{i_1 \dots i_n} \psi_{i_1 \dots i_n}, \\
& = i \sum_k (k+j) \nabla^{k+i+j} \nabla_k \bar{\chi}^{i_1 \dots i_n} \psi_{i_1 \dots i_n} + \frac{1}{2} a^{l'} \left(\gamma_{l\bar{k}}^{k+i+j} g^{\tilde{k}l} + \gamma_{l\bar{k}}^{l+i+j} g^{\tilde{k}k} \right) \nabla_l \nabla_k \bar{\chi}^{i_1 \dots i_n} \psi_{i_1 \dots i_n} \\
& \quad + \frac{1}{2} a^{l'} \gamma_{l\bar{k}}^{k+i+j} g^{\tilde{k}l} \bar{\chi}^{i_1 \dots i_n} r_{i_1 \dots i_n j_1 \dots j_n kl} \psi^{j_1 \dots j_n}. \tag{E.34}
\end{aligned}$$

Following [4, 5] we identify the term inside the round bracket in the second line as $-\partial_{l'} g^{k+i+j}$ and then use the identity (A.10). This gives for the whole term in the second line,

$$\begin{aligned}
& - \frac{i}{2} \sum_{k,l} (k+l+i+j) g^{k+i+jl} \nabla_l \nabla_k \bar{\chi}^{i_1 \dots i_n} \psi_{i_1 \dots i_n} \\
& = - \frac{i}{2} (i+j) \nabla_{(i+j)}^2 \bar{\chi}^{i_1 \dots i_n} \psi_{i_1 \dots i_n} - \frac{i}{2} \sum_k (k) \left[\nabla^{k+i+j} \nabla_k + \nabla_k \nabla^{k+i+j} \right] \bar{\chi}^{i_1 \dots i_n} \psi_{i_1 \dots i_n}. \tag{E.35}
\end{aligned}$$

The sum over k in the second term is not a tensor contraction because of the presence of the factor (k) . For the same reason the two terms inside the square bracket are not same.

However, because of the antisymmetry property of $r_{i_1 \dots i_n j_1 \dots j_n k l}$ in (C.24) such terms give the same result once we symmetrize between $\bar{\chi}$ and ψ ,

$$\nabla^{k+i+j} \nabla_k \bar{\chi}^{i_1 \dots i_n} \psi_{i_1 \dots i_n} + \bar{\chi} \longleftrightarrow \psi = \nabla_k \nabla^{k+i+j} \bar{\chi}^{i_1 \dots i_n} \psi_{i_1 \dots i_n} + \bar{\chi} \longleftrightarrow \psi, \quad (\text{E.36})$$

where the index k is not summed over. Because of the same reason the last term in (E.34) drops out when we symmetrize between $\bar{\chi}$ and ψ . The final result for the second line in (E.33) is given by

$$\nabla^{l+i} \nabla_k \bar{\chi}^{i_1 \dots i_n} \nabla_l a^{k+j} \psi_{i_1 \dots i_n} + \bar{\chi} \longleftrightarrow \psi = \frac{i}{2} (j-i) \left(\nabla_{(i+j)}^2 \bar{\chi}^{i_1 \dots i_n} \psi_{i_1 \dots i_n} + \bar{\chi} \nabla_{(i+j)}^2 \psi_{i_1 \dots i_n} \right). \quad (\text{E.37})$$

To evaluate the terms in the last two lines of (E.33) we first derive

$$\nabla^{k+i} a^{l+j} + k \longleftrightarrow l = i(j-i) g^{k+i+jl}, \quad (\text{E.38})$$

using the same trick as described below (E.34). Using (E.37) and (E.38) in (E.33) one finally finds

$$T^{K_{(i)} Z_{(j)}} = -2(i-j)_X \left\langle \hat{K}_{(i+j)} \right\rangle_\psi +_X \left\langle \hat{\pi}^{*k+i} r_{kl}(\hat{x}) a^{l+j}(\hat{x}) - a^{k+j}(\hat{x}) r_{kl}(\hat{x}) \hat{\pi}^{l+i} \right\rangle_\psi, \quad (\text{E.39})$$

where we have used (3.5), (C.13).

The Rest

The rest of the commutators are straightforward to calculate and do not introduce anymore complications than in the case of spin-zero representation studied in [4, 5]. The relevant results are as follows:

$$\begin{aligned} T^{K_{(i)} V_{(j)}} - i &\longleftrightarrow j = -2(i-j)_X \left\langle \hat{Z}_{(i+j)} \right\rangle_\psi, \\ T^{Z_{(i)}^L V_{(j)}} = T^{Z_{(i)}^R V_{(j)}} &= -(i-j)_X \left\langle \hat{V}_{(i+j)} \right\rangle_\psi. \end{aligned} \quad (\text{E.40})$$

Finally, using the results (E.9), (E.27), (E.39), (E.40) in (E.6), and (E.4) suitably one finds the algebra as shown in (3.8), (3.9).

Acknowledgment

The author is thankful to G. Date, T. R. Govindarajan, and R. K. Kaul for illuminating discussions.

References

- [1] P. Mukhopadhyay, "On the conformal field theories for bosonic strings in pp-waves," *Journal of High Energy Physics*, no. 0811, article 034, 2008, <http://arxiv.org/abs/0807.0923>.
- [2] P. Mukhopadhyay, "Superstrings in type IIB R-R plane-wave in semi-light-cone gauge and conformal invariance," *Journal of High Energy Physics*, no. 0905, article 059, 2009, <http://arxiv.org/abs/0902.3750>.
- [3] P. Mukhopadhyay, "Dynamical supersymmetry analysis of conformal invariance for superstrings in type IIB RR plane-wave background," *Physical Review D*, vol. 80, no. 12, Article ID 126007, 2009, <http://arxiv.org/abs/0902.3750>.
- [4] P. Mukhopadhyay, "On a coordinate independent description of string worldsheet theory," . In press, <http://arxiv.org/abs/0912.3987>.
- [5] P. Mukhopadhyay, "DeWitt-Virasoro construction," *Pramana*, vol. 76, no. 3, pp. 407–420, 2011.
- [6] B. S. DeWitt, "Point transformations in quantum mechanics," *Physical Review*, vol. 85, pp. 653–661, 1952.
- [7] B. S. DeWitt, "Dynamical theory in curved spaces. I. A review of the classical and quantum action principles," *Reviews of Modern Physics*, vol. 29, pp. 377–397, 1957.
- [8] D. Friedan, "Nonlinear models in two epsilon dimensions," *Physical Review Letters*, vol. 45, no. 13, pp. 1057–1060, 1980.
- [9] D. H. Friedan, "Nonlinear models in $2 + \epsilon$ dimensions," *Annals of Physics*, vol. 163, no. 2, pp. 318–419, 1985.
- [10] L. Alvarez-Gaumé and D. Z. Freedman, "The background field method and the ultraviolet structure of the supersymmetric nonlinear σ -model," *Annals of Physics*, vol. 134, no. 1, pp. 85–109, 1981.
- [11] C. Lovelace, "Strings in curved space," *Physics Letters B*, vol. 135, no. 1–3, pp. 75–77, 1984.
- [12] E. S. Fradkin and A. A. Tseytlin, "Quantum string theory effective action," *Nuclear Physics B*, vol. 261, pp. 1–27, 1985.
- [13] E. S. Fradkin and A. A. Tseytlin, "Quantum string theory effective action," *Nuclear Physics B*, vol. 261, no. 1, pp. 1–27, 1985.
- [14] A. Sen, "Equations of motion for the heterotic string theory from the conformal invariance of the sigma model," *Physical Review Letters*, vol. 55, pp. 1846–1849, 1985.
- [15] A. Sen, "Equations of motion for the heterotic string theory from the conformal invariance of the sigma model," *Physical Review Letters*, vol. 55, no. 18, pp. 1846–1849, 1985.
- [16] C. G. Callan, D. Friedan, E. J. Martinec, and M. J. Perry, "Strings in background fields," *Nuclear Physics B*, vol. 262, no. 4, pp. 593–609, 1985.
- [17] C. G. Callan, I. R. Klebanov, and M. J. Perry, "String theory effective actions," *Nuclear Physics B*, vol. 278, no. 1, pp. 78–90, 1986.
- [18] B. E. Fridling and A. Jevicki, "Nonlinear σ -models as S-matrix generating functionals of strings," *Physics Letters B*, vol. 174, no. 1, pp. 75–80, 1986.
- [19] J. Maharana and G. Veneziano, "Strings in a background: a BRS Hamiltonian approach," *Nuclear Physics B*, vol. 283, no. 1-2, pp. 126–140, 1987.
- [20] R. Akhoury and Y. Okada, "Strings in curved space-time: virasoro algebra in the classical and quantum theory," *Physical Review D*, vol. 35, no. 6, pp. 1917–1938, 1987.
- [21] R. Akhoury and Y. Okada, "Conformal symmetry, unitarity, and the equations of motion for strings in background fields," *Nuclear Physics B*, vol. 318, no. 1, pp. 176–210, 1989.
- [22] A. K. Das and S. Roy, "Nilpotency of Q, (BRST) and background field equations," *Zeitschrift Für Physik C*, vol. 36, pp. 317–322, 1987.
- [23] A. K. Das, J. Maharana, and S. Roy, "BRST quantization of the superstring in background fields," *Physical Review D*, vol. 40, no. 12, pp. 4037–4046, 1989.
- [24] A. K. Das, J. Maharana, and S. Roy, "The Neveu-Schwarz-Ramond string in background fields: nilpotency of BRST charge," *Nuclear Physics B*, vol. 331, no. 3, pp. 573–591, 1990.
- [25] S. Fubini, J. Maharana, M. Roncadelli, and G. Veneziano, "Quantum constraint algebra for an interacting superstring," *Nuclear Physics B*, vol. 316, no. 1, pp. 36–58, 1989.
- [26] S. Alam, "Becchi-Rouet-Stora-Tyutin quantization of closed type-II superstring in curved background with torsion," *Physical Review D*, vol. 40, no. 12, pp. 4047–4055, 1989.
- [27] M. Diakonou, K. Farakos, G. Koutsoumbas, and E. Papantonopoulos, "Hamiltonian formulation and quantum constraint algebra for a closed string moving in a curved background," *Physics Letters B*, vol. 247, no. 2-3, pp. 273–279, 1990.
- [28] M. Diakonou, K. Farakos, G. Koutsoumbas, and E. Papantonopoulos, "Quantum constraint algebra for a closed bosonic string in a gravitational and dilaton background," *Physics Letters B*, vol. 240, no. 3-4, pp. 351–355, 1990.

- [29] S. Jain, G. Mandal, and S. R. Wadia, "Virasoro conditions, vertex operators, and string dynamics in curved space," *Physical Review D*, vol. 35, no. 2, pp. 778–781, 1987.
- [30] S. Jain, G. Mandal, and S. R. Wadia, "Perturbatively renormalized vertex operator, highest-weight representations of Virasoro algebra, and string dynamics in curved space," *Physical Review D*, vol. 35, no. 10, pp. 3116–3142, 1987.
- [31] A. Lawrence and E. Martinec, "String field theory in curved spacetime and the resolution of spacelike singularities," *Classical and Quantum Gravity*, vol. 13, no. 1, pp. 63–95, 1996, <http://arxiv.org/abs/hep-th/9509149>.
- [32] B. Zwiebach, "Closed string field theory: quantum action and the Batalin-Vilkovisky master equation," *Nuclear Physics B*, vol. 390, no. 1, pp. 33–152, 1993, <http://arxiv.org/abs/hep-th/9206084>.
- [33] A. Sen, "On the background independence of string field theory," *Nuclear Physics B*, vol. 345, no. 2-3, pp. 551–583, 1990.
- [34] A. Sen, "On the background independence of string field theory. II. Analysis of on-shell S -matrix elements," *Nuclear Physics B*, vol. 347, no. 1-2, pp. 270–318, 1990.
- [35] A. Sen, "On the background independence of string field theory. III. Explicit field redefinitions," *Nuclear Physics B*, vol. 391, no. 3, pp. 550–590, 1993, <http://arxiv.org/abs/hep-th/9201041>.
- [36] A. Sen and B. Zwiebach, "A proof of local background independence of classical closed string field theory," *Nuclear Physics B*, vol. 414, no. 3, pp. 649–711, 1994, <http://arxiv.org/abs/hep-th/9307088>.
- [37] B. Zwiebach, "Building string field theory around non-conformal backgrounds," *Nuclear Physics B*, vol. 480, no. 3, pp. 541–572, 1996, <http://arxiv.org/abs/hep-th/9606153>.
- [38] B. S. DeWitt and R. W. Brehme, "Radiation damping in a gravitational field," *Annals of Physics*, vol. 9, pp. 220–259, 1960.
- [39] J. Polchinski, *String Theory: An Introduction to the Bosonic String*, vol. 1, University Press, Cambridge, UK, 1998.
- [40] M. Omote and H. Sato, "Quantum mechanics of a nonlinear system," *Progress of Theoretical Physics*, vol. 47, pp. 1367–1377, 1972.
- [41] J. L. Synge, *Relativity: The General Theory*, Series in Physics, North-Holland Publishing, Amsterdam, The Netherlands, 1960.
- [42] R. M. Wald, *General Relativity*, University of Chicago Press, Chicago, Ill, USA, 1984.