Research Article

Physical Quantities of Reissner-Nordström Spacetime with Arbitrary Function and Regularized Procedure

Gamal G. L. Nashed^{1,2}

¹ Center for Theoretical Physics, British University in Egypt, Sherouk City 11837, P.O. Box 43, Cairo, Egypt

² Mathematics Department, Faculty of Science, Ain Shams University, Cairo 11566, Egypt

Correspondence should be addressed to Gamal G. L. Nashed; nashed@bue.edu.eg

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We use the covariant teleparallel approach to compute the total energy of a *spherically symmetric frame with an arbitrary function*, that is, $\mathfrak{F}(r)$. We show how the total energy is always effected by the inertia. When use is made of the pure gauge connection, teleparallel gravity always yields the physically relevant result. We also calculate the total conserved charge and show how inertia spoils the physics in the time coordinate direction. Therefore, a regularized expression is employed to get a plausible value of energy. Finally, we use the Euclidean continuation method, in the context of TEGR, to calculate the energy, Hawking temperature, entropy, and first law of thermodynamics.

1. Introduction

Teleparallel theories are interesting for several reasons: first of all, general relativity (GR) can be considered as a particular theory of teleparallelism, and thus it could be viewed from a different viewpoint that can lead to the same results [1]. Second, one can define an energy-momentum tensor for the gravitational field that is a true tensor under all general coordinate transformations but not under local Lorentz transformation. This is the reason why teleparallelism was reconsidered by Møller when he was studying the problem of defining an energy-momentum tensor for the gravitational field [2, 3], but he did not succeed in defining a true tensor that is invariant under general coordinate transformations. The idea was taken over by Pellegrini and Plebański that constructed the general Lagrangian for these theories [4]. The third reason why these theories are interesting is that they can be seen as gauge theories of the translation group (not the full Poincaré group), and, thus, they give an alternative interpretation of GR [5–16].

Geometrically, teleparallel models are described by the Weitzenböck spacetime that is characterized by vanishing curvature and nonzero torsion. The tetrad (or a frame) is the basic field variable which can be treated as a gauge potential corresponding to the group of local translations.

Then, torsion is interpreted as the corresponding gauge field strength. As a result, a gravitational teleparallel Lagrangian is straightforwardly constructed from the quadratic torsion invariants.

Mathematically, there are infinitely many tetrads since a reference frame of an observer can obviously be constructed in an infinitely many ways. In particular, from a given tetrad field h_i^μ , we can obtain a continuous family of tetrads by performing the local Lorentz transformation $h_i'^\alpha = \Lambda^\alpha_\beta h_i^\beta$, where the elements of the Lorentz matrix $\Lambda^\alpha_\beta(x)$ are arbitrary functions of the spacetime coordinates [17].

An important difference between Einstein theory and teleparallel theories is that it is possible to distinguish gravitational field from inertia [18]. With the purpose of getting a deeper insight into the covariant teleparallel formalism, Lucas et al. [17] reanalyze the computation of the total energy of two models. Obukhov et al. [19] computed the energy and momentum transported by exact plane gravitational-wave solutions of Einstein equations using the teleparallel equivalent formulation of Einstein's theory.

The existence of symmetries in a physical system means that there exist one or more transformations that leave the physical system invariant. Since physical systems can be fully described by their Lagrangians/Hamiltonians, it is natural to expect the symmetry transformations to be canonical

ones. The number of symmetry transformations that can be generated from the system depends on how many conserved quantities of the associated system are considered. It has been known that conserved quantities of a physical system are linked to the generator of the associated symmetry transformations [20–22].

A number of methods for calculating energy and conserved charges that provide reliable result, are used in the present time. Using the Lagrange-Noether approach, one can derive the conserved currents that arise from the invariance of the classical action under transformations of fields. However, in Riemannian geometry, one cannot find symmetries that can be used to generate the conserved energy-momentum currents. Only one can speak about the energy of asymptotically flat spacetime. Earlier analysis of this problem can be found in details in [23, 24]. In the framework of the TEGR, it is possible to make definite statements about the energy and momentum of the gravitational field. This fact constitutes the major motivation for considering this theory. In the $3 + 1$ formulation of the TEGR [25], and by imposing Schwinger's time gauge condition [26] for the tetrad field by fixing $h_{(a)}^0 = 0$ which implies $h^{(0)}_a = 0$, it is found that the Hamiltonian and vector constraints contain each one a divergence in the form of scalar and vector densities, respectively, that can be identified with the energy and momentum densities of the gravitational field [27, 28]. Gibbons et al. [29] compute the energy of Kerr-AdS black holes first indirectly through integrating the first law and then using the conformal definition of Ashtekar et al. [30, 31], and they indicate that both of them agree. In addition to the Regge-Teitelboim method [32], some other various definitions are the following: the approach of Abbott and Deser [33–35], the spinor definition [36, 37] based on the electric Weyl tensor, covariant phase space methods [38–41], cohomological techniques [42, 43], the KBL approach [44–46], Noether methods [47–52], and the “counter term subtraction method” [53, 54] (and more references on this in [40], for example) improved surface integrals [55] and regularization of the Euclidean action [56, 57].

Recently, the definition of conserved currents and charges for models with “quasi-invariant” Lagrangians is applied for Kerr-NUT spacetimes [58]. Within the framework of MAG, a stationary axially symmetric exact solution of the vacuum field equations is obtained for a specific gravitational Lagrangian ([59] and the references therein). In the teleparallel gravity, Nester et al. [60] have considered the quasilocal center of mass (COM). They have used the covariant Hamiltonian formalism, in which quasilocal quantities are given by the Hamiltonian boundary term, along with the covariant asymptotic Hamiltonian boundary expressions. Consideration of the COM not only gives the most restrictive asymptotic conditions on the variables but also gives strong constraints on the acceptable expressions [61]. A relation between spinor Lagrangian and teleparallel theory is established [61]. In metric-affine generalization of teleparallelism, it has been shown that there is an inconsistency in the coupling of spinors. Many discussions have been given for this in [62]. *It's the aim of the present study to calculate*

energy, conserved charges, and thermodynamical quantities of a spherically symmetric frame with an arbitrary function. This calculation is done using two different procedures. In the first one, only Riemannian connection is considered, $\tilde{\Gamma}^\beta_\alpha$, while in the second procedure, Weitzenböck connection Γ^β_α is taken into account besides the Riemannian connection.

In Section 2, we use the language of exterior forms to give an outline of the teleparallel approach. In Section 3, we calculate the total energy of the spherically symmetric model and show that it is affected by inertia. Therefore, we use the covariant formalism and show that the Weitzenböck connection acts as a regularizing tool and always provides the physically meaningful result. In Section 4, we calculate the total conserved charge and show how inertia affects the final result. In Section 5, we use the regularized expression of conserved charge to recalculate the total charge, and plausible result is obtained. In Section 6, we calculate the energy using the Euclidean continuation method and also calculate Hawking temperature, entropy, and first law of thermodynamics. The final section is devoted for the main results and discussion.

Notation. The Latin indices $i, j, \dots$ for local holonomic spacetime coordinates and the Greek indices $\alpha, \beta, \dots$ label (co)frame components are used. Particular frame components are denoted by hats, $\hat{0}$, $\hat{1}$, and so forth. As usual, the exterior product is denoted by $\wedge$, while the interior product of a vector ξ and a p -form Ψ is denoted by $\xi \lrcorner \Psi$. The vector basis dual to the frame 1-forms ϑ^α is denoted by e_α , and they satisfy $e_\alpha \lrcorner \vartheta^\beta = \delta_\alpha^\beta$. Using local coordinates x^i , we have $\vartheta^\alpha = h^\alpha_i dx^i$ and $e_\alpha = h^\alpha_i \partial_i$, where h^α_i and h^α_i are the covariant and contravariant components of the tetrad field. We define the volume 4-form by $\eta = \vartheta^0 \wedge \vartheta^1 \wedge \vartheta^2 \wedge \vartheta^3$. Furthermore, with the help of the interior product, we define

$$\eta_\alpha = e_\alpha \lrcorner \eta = \frac{1}{3!} \epsilon_{\alpha\beta\gamma\delta} \vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta, \quad (1)$$

where $\epsilon_{\alpha\beta\gamma\delta}$ is completely antisymmetric with $\epsilon_{0123} = 1$. Consider

$$\begin{aligned} \eta_{\alpha\beta} &= e_\beta \lrcorner \eta_\alpha = \frac{1}{2!} \epsilon_{\alpha\beta\gamma\delta} \vartheta^\gamma \wedge \vartheta^\delta, \\ \eta_{\alpha\beta\gamma} &= e_\gamma \lrcorner \eta_{\alpha\beta} = \frac{1}{1!} \epsilon_{\alpha\beta\gamma\delta} \vartheta^\delta, \end{aligned} \quad (2)$$

which are bases for 3-, 2-, and 1-forms, respectively. Finally,

$$\eta_{\alpha\beta\mu\nu} = e_\nu \lrcorner \eta_{\alpha\beta\mu} = e_\nu \lrcorner e_\mu \lrcorner e_\beta \lrcorner e_\alpha \lrcorner \eta = \epsilon_{\alpha\beta\gamma\delta} \quad (3)$$

is the Levi-Civita tensor density. The η -forms satisfy the useful identities:

$$\begin{aligned} \vartheta^\beta \wedge \eta_\alpha &= \delta_\alpha^\beta \eta, & \vartheta^\beta \wedge \eta_{\mu\nu} &= \delta_\nu^\beta \eta_\mu - \delta_\mu^\beta \eta_\nu, \\ \vartheta^\beta \wedge \eta_{\alpha\mu\nu} &= \delta_\alpha^\beta \eta_{\mu\nu} + \delta_\mu^\beta \eta_{\nu\alpha} + \delta_\nu^\beta \eta_{\alpha\mu}, \\ \vartheta^\beta \wedge \eta_{\alpha\gamma\mu\nu} &= \delta_\nu^\beta \eta_{\alpha\gamma\mu} - \delta_\mu^\beta \eta_{\alpha\gamma\nu} + \delta_\gamma^\beta \eta_{\alpha\mu\nu} - \delta_\alpha^\beta \eta_{\gamma\mu\nu}. \end{aligned} \quad (4)$$

The line element $ds^2 = g_{\alpha\beta} \vartheta^\alpha \otimes \vartheta^\beta$ is defined by the spacetime metric $g_{\alpha\beta}$.

2. Brief Review of Teleparallel Gravity and Conserved Currents

Teleparallel geometry can be viewed as a gauge theory of translation [63–68]. The coframe ϑ^α plays the role of the gauge translational potential of the gravitational field. GR can be reformulated as the teleparallel theory. Geometrically, teleparallel gravity can be considered as a special case of the metric-affine gravity in which ϑ^α and the local Lorentz connection are subject to the distant parallelism constraint $R_\alpha{}^\beta = 0$ [69–79]. In this geometry, the torsion 2-form

$$\begin{aligned} T^\alpha &= D\vartheta^\alpha = d\vartheta^\alpha + \Gamma_\beta{}^\alpha \wedge \vartheta^\beta = \frac{1}{2} T_{\mu\nu}{}^\alpha \vartheta^\mu \wedge \vartheta^\nu \\ &= \frac{1}{2} T_{ij}{}^\alpha dx^i \wedge dx^j \end{aligned} \quad (5)$$

arises as the gravitational gauge field strength, $\Gamma_\alpha{}^\beta$ being the Weitzenböck 1-form connection, d is the exterior derivative, and D is the exterior covariant derivative. The torsion T^α can be decomposed into three irreducible pieces [17]: the tensor part, the trace, and the axial trace given, respectively, by

$$\begin{aligned} (1)T^\alpha &= T^\alpha - (2)T^\alpha - (3)T^\alpha, \quad \text{with} \\ (2)T^\alpha &= \frac{1}{3} \vartheta^\alpha \wedge T, \quad \text{where } T = (e_\beta \rfloor T^\beta), \\ e_\alpha \rfloor T &= T_{\mu\alpha}{}^\mu, \quad \text{polar trace of torsion} \\ (3)T^\alpha &= \frac{1}{3} e^\alpha \rfloor P, \quad \text{with } P = (\vartheta^\beta \wedge T_\beta), \\ e_\alpha \rfloor P &= T^{\mu\nu\lambda} \eta_{\mu\nu\lambda\alpha}, \quad \text{axial trace of torsion.} \end{aligned} \quad (6)$$

The Lagrangian of the teleparallel equivalent of general relativity (TEGR) has the form (The effect of adding the non-Riemannian parity odd pseudoscalar curvature to the Hilbert-Einstein-Cartan scalar curvature was studied by many authors (cf. [80–84] and the references therein.))

$$V = -\frac{1}{2\kappa} T^\alpha \wedge * \left((1)T_\alpha - 2(2)T_\alpha - \frac{1}{2}(3)T_\alpha \right). \quad (7)$$

$\kappa = 8\pi G/c^3$, G is the Newtonian constant, c is the speed of light, and $*$ denotes the Hodge duality in the metric $g_{\alpha\beta}$ which is assumed to be flat Minkowski metric $g_{\alpha\beta} = o_{\alpha\beta} = \text{diag}(+1, -1, -1, -1)$ that is used to raise and lower local frame (Greek) indices.

The variation of the total action with respect to the coframe which gives the field equations in the form

$$D\tilde{H}_\alpha - E_\alpha = \Sigma_\alpha, \quad \text{where } \Sigma_\alpha = \frac{\delta L_{\text{matter}}}{\delta \vartheta^\alpha} \quad (8)$$

is the symmetric energy-momentum tensor 3-form of matter which is considered as the source. In accordance with the general Lagrange-Noether scheme [55, 69], one derives from (7) the translational momentum 2-form and the canonical

energy-momentum 3-form (In the absence of Weitzenböck connection, the contortion $K^\alpha{}_\beta$ will be identical with Riemannian connection.)

$$\tilde{H}_\alpha = -\frac{\partial V}{\partial T^\alpha} = \frac{1}{\kappa} * \left((1)T_\alpha - 2(2)T_\alpha - \frac{1}{2}(3)T_\alpha \right) \quad (9)$$

$$= \frac{1}{2\kappa} \tilde{\Gamma}^{\beta\gamma} \wedge \eta_{\alpha\beta\gamma},$$

$$\begin{aligned} E_\alpha &= \frac{\partial V}{\partial \vartheta^\alpha} = e_\alpha \rfloor V + (e_\alpha \rfloor T^\beta) \wedge \tilde{H}_\beta \\ &= \frac{1}{2} \left[(e_\alpha \rfloor d\vartheta^\beta) \wedge \tilde{H}_\beta - d\vartheta^\beta \wedge (e_\alpha \rfloor \tilde{H}_\beta) \right], \end{aligned} \quad (10)$$

where $K^{\beta\gamma}$ is the contortion 1-form which is related to the torsion through the relation

$$T^\alpha = K^\alpha{}_\beta \wedge \vartheta^\beta. \quad (11)$$

Due to geometric identities [17], the Lagrangian (7) can be recast as

$$V = -\frac{1}{2} T^\alpha \wedge \tilde{H}_\alpha. \quad (12)$$

The presence of the connection field $\Gamma_\alpha{}^\beta$ plays an important regularizing role due to the following.

First. The theory becomes explicitly covariant under the local Lorentz transformations of the coframe; that is, the Lagrangian (7) is invariant under the change of variables

$$\vartheta'^\alpha = \Lambda^\alpha{}_\beta \vartheta^\beta, \quad \Gamma'^\beta{}_\alpha = (\Lambda^{-1})^\mu{}_\alpha \Gamma^\beta{}_\mu \Lambda^\beta{}_\nu + \Lambda^\beta{}_\gamma d(\Lambda^{-1})^\gamma{}_\alpha. \quad (13)$$

Due to the noncovariant transformation law of $\Gamma_\alpha{}^\beta$, see (13), if a connection vanishes in a given frame, it will not vanish in any other frame related to the first by a local Lorentz transformation. When $\Gamma_\alpha{}^\beta = 0$ for all frames, which is the tetrad gravity, the Lagrangian (7) is only quasi-invariant; that is, it changes by a total divergence.

Second. $\Gamma_\alpha{}^\beta$ plays an important role in the teleparallel framework. This role represents the inertial effects which arise from the choice of the reference system [17]. This means that when one changes the reference system, that is, uses different tetrad which reproduced the same metric, the effect of the inertia will change. The contributions of this inertia in many cases lead to unphysical results for the total energy of the system. Therefore, the role of the teleparallel connection is to separate the inertial contribution from the truly gravitational one. Since the teleparallel curvature is zero, the connection is a “pure gauge,” that is,

$$\Gamma_\alpha{}^\beta \stackrel{\text{def.}}{=} (\Lambda^\beta{}_\gamma)^{-1} d(\Lambda^\gamma{}_\alpha). \quad (14)$$

The Weitzenböck connection always has the form (14). The relation between Weitzenböck and Riemannian connections has the form [17, 19]

$$\Gamma_\alpha{}^\beta = \tilde{\Gamma}_\alpha{}^\beta - K_\alpha{}^\beta. \quad (15)$$

The teleparallel model (7) belongs to the class of quasi-invariant theories. In fact, one can verify that under a change of the coframe $\vartheta'^\alpha = (\Lambda(x)^\alpha_\beta) \vartheta^\beta$, the Lagrangian changes by a total derivative:

$$\bar{V}(\vartheta') = \bar{V}(\vartheta) - \frac{1}{2\kappa} d \left[(\Lambda^{-1})^\alpha_\beta d\Lambda^\beta_\gamma \wedge \eta^\gamma_\alpha \right]. \quad (16)$$

In addition to (16), it is easy to verify that (9) and (10) change like

$$\begin{aligned} \bar{H}'_\alpha(\vartheta') &= (\Lambda^{-1})^\beta_\alpha \bar{H}_\beta(\vartheta) \\ &\quad - \frac{1}{2\kappa} \left[(\Lambda^{-1})^\beta_\alpha (\Lambda^{-1})^\gamma_\nu d\Lambda^\nu_\mu \wedge \eta^\mu_\beta \right], \\ \bar{E}'_\alpha(\vartheta') &= (\Lambda^{-1})^\beta_\alpha \bar{E}_\beta(\vartheta) + d(\Lambda^{-1})^\beta_\alpha \wedge \bar{H}_\beta(\vartheta) \\ &\quad - \frac{1}{2\kappa} d \left[(\Lambda^{-1})^\beta_\alpha (\Lambda^{-1})^\gamma_\nu d\Lambda^\nu_\mu \wedge \eta^\mu_\beta \right]. \end{aligned} \quad (17)$$

The total conserved charge in the tetrad gravity theory is given by [61]

$$\mathcal{Q}(\xi, \vartheta) = \int_{\partial S} \xi^\alpha \bar{H}_\alpha = \frac{1}{2\kappa} \int_{\partial S} \xi^\alpha \bar{\Gamma}^{\beta\gamma} \wedge \eta_{\alpha\beta\gamma}, \quad (18)$$

where $\xi^\alpha = \xi^i \vartheta'^\alpha$. Equation (18) is obviously reference frame dependent like Freud's superpotential for the Einstein pseudotensor [85]. Under a local Lorentz transformation, (18) transforms as

$$\begin{aligned} \bar{\mathcal{Q}}(\xi, \vartheta') &= \bar{\mathcal{Q}}(\xi, \vartheta) - \frac{1}{2\kappa} \int_{\partial S} \xi^\alpha (\Lambda^{-1})^\gamma_\nu d\Lambda^\nu_\mu \wedge \eta^\mu_\alpha \\ &= \frac{1}{2\kappa} \int_{\partial S} \xi^\alpha (\bar{\Gamma}^{\beta\gamma} - \Gamma^{\beta\gamma}) \wedge \eta_{\alpha\beta\gamma}, \end{aligned} \quad (19)$$

where Γ^ν_μ is given by (14). The vector $\xi = \xi^i \partial_i$ does not depend on the choice of the frame, while its components, that is, ξ^α , transform as a vector. Equation (19) which depends on a tensorial object (the difference between 2 connections) is more satisfactory. In this study, we call it regularization but in the quasilocal literature it is described as a choice of [86].

3. Reissner-Nordström Spacetime with Arbitrary Function

In the spherical local coordinates (t, r, θ, ϕ) , a spherically symmetric spacetime with arbitrary function is described by [87]

$$h^\alpha_i = \begin{pmatrix} \frac{\beta}{\alpha_1} & \beta_1 \alpha_1 \sin \theta \cos \phi & \beta_1 r \sin \theta \sin \phi & \beta_1 r \alpha_1 \sin \theta \cos \theta \\ -\frac{\beta_1}{\alpha_1} f(\theta, \phi) & -\alpha_1 \sin \theta \cos \phi [1 + \beta_2] & -r [\sin \theta \sin \phi \beta_2 + \cos \theta \cos \phi] & -r \sin \theta [\cos \theta \beta_2 + \sin \phi] \\ -\frac{\beta_1}{\alpha_1} f_1(\theta, \phi) & -\alpha_1 \sin \theta [\sin \phi + \cos \phi \beta_3] & -r \sin \phi [\sin \theta \beta_3 + \cos \theta] & -r \sin \theta [\cos \theta \beta_3 + \cos \phi] \\ \frac{\beta_1}{\alpha_1} f_2(\theta, \phi) & -\alpha_1 \cos \theta [1 + \tan \theta \cos \phi \beta_4] & -r \sin \theta [\sin \phi \beta_4 + 1] & r \sin \theta \cos \theta \beta_4 \end{pmatrix}, \quad (20)$$

where $\alpha_1, \beta, \beta_1, \beta_2, \beta_3, \beta_4, f(r, \theta), f_1(r, \theta)$ and $f_2(r, \theta)$ have the form

$$\begin{aligned} \alpha_1 &= \left(1 - \frac{2M}{r} + \frac{q^2}{r^2} \right)^{-(1/2)}, \\ \beta &= \left\{ \frac{(r\mathfrak{F}' - 1)^2}{1 - (2c_1/r) + (c_2^2/r^2)} \right\}^{1/2}, \\ \beta_1 &= \sqrt{\beta^2 - 1}, \quad \beta_2 = [\beta - 1] f(\theta, \phi), \\ \beta_3 &= [\beta - 1] f_1(\theta, \phi), \quad \beta_4 = [\beta - 1] f_2(\theta, \phi), \\ f(\theta, \phi) &= (\sin^2 \theta \cos^2 \phi + \cos \theta \sin \phi [\sin \theta \cos \phi - 1]), \\ f_1(\theta, \phi) &= (\cos \theta \cos \phi + \sin \theta \sin \phi [\sin \theta \cos \phi + \cos \theta \sin \phi]), \\ f_2(\theta, \phi) &= \sin \theta (\sin \theta \sin \phi - \cos \phi \cos \theta), \end{aligned} \quad (21)$$

where c_1 and c_2 are two constants and M and q are the mass and the electric charge parameters, respectively. The metric associated with tetrad (20) has the Reissner Nordström spacetime [88, 89]. When $c_2 = 0$ and $q = 0$, one returns to the Schwarzschild spacetime [90]. Tetrad (20) has been studied extensively in [88, 89] when $q = c_1 = c_2 = 0$, but with different notation. In those studies, it is shown that the value of energy did not coincide with the ADM value [90, 91]. Some explanation related to the asymptotic behavior of $\mathfrak{F}$ has been given to explain how to make energy coincides with ADM [92]. Also, in [88, 89], it is shown that the value of energy in general depends on $\mathfrak{F}$. In the present study, it is our goal to recalculate the value of energy related to tetrad (20) using the method developed in [17]. It is of interest to note that any physical tetrad must behave as

$$h_{\alpha i} = O_{\alpha i} + e_{\alpha i} \left(\frac{1}{r} \right), \quad (22)$$

with $O_{\alpha i}$ as Minkowski's metric tensor and $e_{\alpha i}(1/r)$ as the first term in the asymptotic expansion of $g_{\mu\nu}$. Asymptotically flat

spacetimes are defined by the above unnumbered equation together with $\partial_\beta h_{\alpha i} = O(1/r^2)$.

3.1. Calculation of Energy. The necessary component of the superpotential 2-form needed to the energy calculation takes the form (We will not write the nonvanishing components of Riemannian connection and Weitzenböck connection due to their huge form.):

$$\begin{aligned} \tilde{H}_0 = \frac{r \sin \theta}{8\pi} \left[\left(1 - \sin^2 \theta \cos^2 \phi \right. \right. \\ \left. \left. - \cos \theta \sin \phi [1 + \sin \theta \cos \phi] \right) \{\beta - 1\} \right. \\ \left. + 2 \left\{ 1 - \frac{\beta}{\alpha_1} \right\} \right] (d\theta \wedge d\phi). \end{aligned} \quad (23)$$

Computing the total energy at a fixed time in the 3-space with a spatial boundary 2-dimensional surface $\partial S = \{r = R, \theta, \phi\}$, we obtain

$$\tilde{E} = \int_{\partial S} \tilde{H}_0 = \frac{R}{3} \left\{ 2 + \beta \left(1 - \frac{3}{\alpha_1} \right) \right\}. \quad (24)$$

From (24), when $\beta = 1$, we obtain the value of energy up to $(1/R^2)$ in the form:

$$\begin{aligned} \tilde{E} = R \left\{ 1 - \frac{1}{\sqrt{1 - (2M/r) + (q^2/r^2)}} \right\} \\ \approx M - \frac{q^2}{2R} + \frac{3M^2}{2R} - \frac{3Mq^2 - 5M^3}{2R^2} + O\left(\frac{1}{R^3}\right) + \dots \end{aligned} \quad (25)$$

Generally, (24) shows that the standard energy value for Reissner Nordström spacetime depends on the arbitrary function $\mathfrak{F}(r)$ and on the two constants c_1 and c_2 . By giving a specific value of $\mathfrak{F}(r)$, we get a different value of the energy.

3.2. Regularization of Energy. To overcome the contribution of inertia to physical quantities that appeared in the preceding subsection, we are going to use the regularization framework that is based on the covariance property, that is, taking into account the Weitzenböck connection Γ_α^β provided by (14). The local Lorentz transformation introduced in [17] depends on the spatial coordinate because the energy is divergent, that is, depends on the radial coordinate r ; however, in this study, we will introduce another local Lorentz transformation because the form of energy is not in the known form. If $\beta = 1$, that is, $\mathfrak{F}(r) = \text{constant}$ and $c_1 = c_2 = q = 0$, then tetrad (20) will coincide with tetrad (4.10) studied in [17] which gives the correct value of energy as shown after (24) and also in [17]. This means that the main reason for the contribution of inertia comes from the arbitrary function $\mathfrak{F}(r)$ and also from the two constants c_1 and c_2 . Therefore, if we succeed in rewriting tetrad (20) as a composition of diagonal tetrad, local Lorentz transformation, which depends on the spatial coordinate, and another local Lorentz transformation, which depends on the arbitrary function, then we can use the resulting local Lorentz transformations as a regularizing tool that can remove inertia appearing in the previous subsection.

Tetrad (20) can be put into the following form:

$$h^\alpha_i = (\Lambda_1^\alpha_\gamma) (\Lambda_2^\gamma_\delta) h^\delta_{i_R}, \quad (26)$$

where $(\Lambda_1^\alpha_\gamma)$ is a local Lorentz transformation given by

$$(\Lambda_1^\alpha_\gamma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\ 0 & \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\ 0 & \cos \theta & -\sin \theta & 0 \end{pmatrix} \quad (27)$$

and $(\Lambda_2^\gamma_\delta)$ is another local Lorentz transformation which has the form

$$(\Lambda_2^\gamma_\delta) = \begin{pmatrix} \beta & \beta_1 \sin \theta \cos \phi & \beta_1 \sin \theta \sin \phi & \beta_1 \cos \theta \\ -\beta_1 \sin \theta \cos \phi & 1 + (1 - \beta) \sin^2 \theta \cos^2 \phi & (1 - \beta) \sin^2 \theta \sin \phi \cos \phi & (1 - \beta) \sin \theta \cos \theta \cos \phi \\ -\beta_1 \sin \theta \sin \phi & (1 - \beta) \sin^2 \theta \sin \phi \cos \phi & 1 + (1 - \beta) \sin^2 \theta \sin^2 \phi & (1 - \beta) \sin \theta \cos \theta \sin \phi \\ -\beta_1 \cos \theta & (1 - \beta) \sin \theta \cos \theta \cos \phi & (1 - \beta) \sin \theta \cos \theta \sin \phi & 1 + (1 - \beta) \cos^2 \theta \end{pmatrix}. \quad (28)$$

Equation (28) in general is a boost transformation which depends on β , and $(\Lambda_1^\alpha_\gamma)$ is a special case of Euler's angle, and finally $(h^\gamma_{i_R})$ is the square root of the Reissner-Nordström metric. The trivial case to undo the effect of β is to put it equal 1. The coframe associated with tetrad (20) has the form

$$\vartheta^\alpha = h^\alpha_i dx^i = (\Lambda_1^\alpha_\gamma) (\Lambda_2^\gamma_\delta) \vartheta^\delta_R, \quad (29)$$

where $\vartheta^\delta_R = (h^\delta_{i_R}) dx^i$ has the form

$$\begin{aligned} \vartheta^{\hat{0}}_R &= \frac{1}{\alpha_1} c dt, & \vartheta^{\hat{1}}_R &= \alpha_1 dr, \\ \vartheta^{\hat{2}}_R &= r d\theta, & \vartheta^{\hat{3}}_R &= r \sin \theta d\phi. \end{aligned} \quad (30)$$

The most appropriate form of local Lorentz transformation which is used in the calculation of the Weitzenböck connection, given by (14), is given by

$$(\Lambda)^\alpha_\beta = (\Lambda_1)^\alpha_\gamma (\Lambda_2^{-1})^\gamma_\delta (\Lambda_1^{-1})^\delta_\beta, \quad \text{where } (\Lambda_1)^\alpha_\gamma, (\Lambda_2)^\alpha_\gamma \quad (31)$$

are defined in (27) and (28). This transformation is the only suitable one that can serve to remove the inertia and leave only the physical quantities as will be shown below. It is interesting to note that the use of the above equation is to return to the fact that Lagrangian (7) is not invariant under local Lorentz transformation as shown by (16). Due to this noninvariance, there exists an extra term which appears as shown by (17). To remove such term, the most suitable *connection* which help for such purpose is constructed out from the above equation. Calculating the necessary components of the superpotential 2-form, we finally get

$$\tilde{H}_0 = \frac{r\beta \sin \theta (1 - (1/\alpha_1))}{4\pi} (d\theta \wedge d\phi). \quad (32)$$

Using (32) to compute the total energy at a fixed time in the 3-space, with a spatial boundary 2-dimensional surface $\partial S = \{r = R, \theta, \phi\}$, we finally obtain the asymptotic value of the energy up to $(1/R^2)$ in the form

$$\begin{aligned} \tilde{E} &= R\beta \left\{ 1 - \frac{1}{\sqrt{1 - (2M/R) + (q^2/R^2)}} \right\} \\ &\approx M - \frac{q^2}{2R} + \frac{3M^2}{2R} + \frac{c_1 M}{R} + O\left(\frac{1}{R^2}\right) + \dots, \end{aligned} \quad (33)$$

which is the value of energy of Reissner Nordström spacetime up to order $O(1/R)$ [88, 89, 93, 94]. Equation (33) shows in a clear way that whatever the value of β , there will be no contribution to $O(1/R^0)$. The contribution begins from $O(1/R)$ as is clear from (33) when $\mathfrak{F} \sim \text{const.}$ or $\mathfrak{F} \sim 1/R$, and so forth.

4. Total Conserved Charge of Reissner-Nordström Spacetime with Arbitrary Function

We are going to calculate the necessary quantities needed to compute the conserved charge. The nonvanishing components of Riemannian connection of tetrad (20), up to $O(1/r^3)$ when $\beta = 1$, have the form

$$\begin{aligned} \tilde{\Gamma}_0^{\hat{1}} &\approx -\left(M \sin \theta \cos \phi \left(1 + 2 \sin \phi \cos \theta \right. \right. \\ &\quad \left. \left. \times [1 - \sin \theta \cos \phi] + 2 \cos^2 \phi \sin^2 \theta \right) \right. \\ &\quad \left. \times (r^2)^{-1} \right) dt + O\left(\frac{1}{r^3}\right), \\ \tilde{\Gamma}_0^{\hat{2}} &\approx \left(M \sin \theta \left(2 \sin \theta \cos \theta \cos \phi \sin^2 \phi \right. \right. \\ &\quad \left. \left. - \sin \phi [1 - 2 \cos^2 \phi \sin^2 \theta] \right. \right. \\ &\quad \left. \left. + 2 \cos \theta \cos^2 \phi \right) \times (r^2)^{-1} \right) dt + O\left(\frac{1}{r^3}\right), \end{aligned}$$

$$\begin{aligned} \tilde{\Gamma}_0^{\hat{3}} &\approx -\left(M \left(\cos \theta [1 - 2 \cos^2 \phi \sin^2 \theta] \right. \right. \\ &\quad \left. \left. + 2 \sin \theta \sin \phi \cos \phi \sin^2 \theta \right) \right. \\ &\quad \left. \times (r^2)^{-1} \right) dt + O\left(\frac{1}{r^3}\right). \end{aligned} \quad (34)$$

The components of the vector ξ^α have the form

$$\begin{aligned} \xi^0 &= -\frac{\zeta_0 \sqrt{r^2 - 2Mr + q^2}}{\sqrt{r^2 - 2c_1 r + c_2^2}}, \\ \xi^1 &= -\left(\left(\zeta_0 (\cos \theta \sin \phi [1 - \sin \theta \cos \phi] - \cos^2 \phi \sin^2 \theta) \right. \right. \\ &\quad \left. \left. \times (r^2 - 2Mr + q^2) \sqrt{2c_1 r - c_2^2} \right) \right. \\ &\quad \left. \times \left(r \sqrt{r^2 - 2Mr + q^2} \sqrt{r^2 - 2c_1 r + c_2^2} \right)^{-1} \right), \\ \xi^2 &= -\left(\left(\zeta_0 (\cos \theta [\cos \phi + \sin \theta \sin^2 \phi] \right. \right. \\ &\quad \left. \left. + \sin \phi \cos \phi \sin^2 \theta) \right) \right. \\ &\quad \left. \times (r^2 - 2Mr + q^2) \sqrt{2c_1 r - c_2^2} \right) \\ &\quad \times \left(r \sqrt{r^2 - 2Mr + q^2} \sqrt{r^2 - 2c_1 r + c_2^2} \right)^{-1}, \\ \xi^3 &= \left(\zeta_0 (\sin \theta [\cos \theta \cos \phi - \sin \theta \sin \phi]) \right. \\ &\quad \left. \times (r^2 - 2Mr + q^2) \sqrt{2c_1 r - c_2^2} \right) \\ &\quad \times \left(r \sqrt{r^2 - 2Mr + q^2} \sqrt{r^2 - 2c_1 r + c_2^2} \right)^{-1}. \end{aligned} \quad (35)$$

Finally, the nonvanishing components of the asymptotic form of the superpotential 2-form $\tilde{H}_\alpha$ up to $O(1/r)$ have the form

$$\begin{aligned} \tilde{H}_0 &= \frac{\sin \theta}{r} \left([2rc_1 + 3c_1^2 - c_2^2] \right. \\ &\quad \left. \times \{ \cos^2 \phi \sin^2 \theta + \sin \phi \cos \theta [1 + \sin \theta \cos \phi] \} \right. \\ &\quad \left. + 2q^2 - 4M(c_1 - r) - 2M^2 \right. \\ &\quad \left. - c_2^2 + c_1(3c_1 + 2r) \right) (d\theta \wedge d\phi) + \dots, \end{aligned}$$

$$\begin{aligned}
\tilde{H}_1 &= \frac{\sin \theta}{\sqrt{c_1 r}} \left([12c_1 (c_1 + r) - 8Mc_1 - 3c_2^2] \right. \\
&\quad \times \cos \phi \sin \theta \{ \cos \phi \sin \theta + \cos \theta \sin \phi \} \\
&\quad + \sin \phi \cos \theta [8Mc_1 + c_2^2 - 4c_1 (c_1 + r)] \\
&\quad \left. - 4c_1 (c_1 + r) + c_2^2 \right) (d\theta \wedge d\phi) + \dots, \\
\tilde{H}_2 &= \frac{\sin \theta}{\sqrt{c_1 r}} \left([8Mc_1 - 12c_1 (c_1 + r) - 3c_2^2] \cos \phi \sin \phi \sin^2 \theta \right. \\
&\quad + \sin \theta \cos \theta \sin^2 \phi [8Mc_1 + 3c_2^2 - 12c_1 (c_1 + r)] \\
&\quad \times \cos \phi \cos \theta [8Mc_1 - 4c_1 (c_1 + r) + c_2^2] \\
&\quad \left. \times (d\theta \wedge d\phi) + \dots \right), \\
\tilde{H}_3 &= \frac{\sin \theta}{\sqrt{c_1 r}} \left[\sin \phi [8Mc_1 - 4c_1 (c_1 + r) + c_2^2] \right. \\
&\quad - [8Mc_1 - 12c_1 (c_1 + r) + 3c_2^2] \\
&\quad \times \cos \theta (\cos \theta \sin \phi + \sin \theta \cos \phi) \left. \right] (d\theta \wedge d\phi) \\
&\quad + \dots.
\end{aligned} \tag{36}$$

Using (35) as well as (36) in (18), we finally get the total conserved charges, up to $O(1/r)$, in the following form:

$$\mathcal{Q} = \int_{\partial S} \xi^\alpha \tilde{H}_\alpha \approx \left(M + \frac{2c_1}{3} - \frac{3q^2 + 2c_2^2}{6r} \right) \zeta^0 + O\left(\frac{1}{r}\right). \tag{37}$$

For the tetrad determined by (20), the conserved charge corresponding to the diffeomorphism generated by the shift along the time coordinate has a nonusual value of the total energy of the configuration.

5. Regularization of Conserved Charge

The regularized form of the total conserved charge which can solve the above problem has the form [71]

$$\tilde{\mathcal{Q}}(\xi, \vartheta) = \frac{1}{2\kappa} \int_{\partial S} \xi^\alpha (\tilde{\Gamma}^{\beta\gamma} - \Gamma^{\beta\gamma}) \wedge \eta_{\alpha\beta\gamma}, \tag{38}$$

where $\Gamma^{\beta\gamma}$ has the form (13).

Again the most appropriate local Lorentz transformation which is used in the calculation of Weitzenböck connection has the form

$$(\Lambda)^\alpha_\beta = (\Lambda_1)^\alpha_\gamma (\Lambda_2^{-1})^\gamma_\delta (\Lambda_1^{-1})^\delta_\beta. \tag{39}$$

Using the above data, that is, Riemannian connection and Weitzenböck connection in (38), we get up to $O(1/r)$

$$\begin{aligned}
\tilde{\mathcal{Q}}(\xi, \vartheta) &= \int_{\partial S} \xi^\alpha (\tilde{\Gamma}^{\beta\gamma} - \Gamma^{\beta\gamma}) \wedge \eta_{\alpha\beta\gamma} \\
&= \left(M - \frac{q^2}{2r} \right) \zeta^0 + O\left(\frac{1}{r^2}\right).
\end{aligned} \tag{40}$$

Therefore, the last term in (38) improves the unusual coefficients in (37). It is of interest to note that the calculation of regularized total conserved charges is more accurate than the regularization of energy; this is due to the fact that the latter remove completely the contribution of inertia.

6. Thermal Properties of Reissner-Nordström Spacetime with Arbitrary Function

In GR, thermodynamical quantities are calculated using the Euclidean continuation of metric [95]. However, in TEGR, these quantities are calculated through the divergence term, which appears in the Lagrangian [90]. This term has no effect on the field equation.

Gibbons and Hawking [95] discussed the thermal properties of the Schwarzschild solution, for which the line element takes the positive-definite standard form

$$ds^2 = + \left(1 - \frac{2M}{r} \right) d\tau^2 + \left(1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 d\Omega^2, \tag{41}$$

after Euclidean continuation of the time variable, $t = -i\tau$. By using the transformation $x = 4M(1 - 2M/r)^{1/2}$, the line element squared becomes

$$ds^2 = + \left(\frac{x}{4M} \right)^2 d\tau^2 + \left(\frac{r^2}{4M^2} \right)^2 dx^2 + r^2 d\Omega^2, \tag{42}$$

which shows that τ can be regarded as an angular variable with period $8\pi M$. Now, the Euclidean section of the Schwarzschild solution is the region defined by $8\pi M \geq \tau \geq 0$ and $x > 0$, where the metric is positive definite, asymptotically flat, and nonsingular. They calculated the Euclidean action, $\hat{I}$, of GR from the surface term as

$$\hat{I} = 4\pi M^2 = \frac{\chi^2}{16\pi}, \tag{43}$$

where $\chi = 8\pi M = T^{-1}$ with T being interpreted as the absolute temperature of the Schwarzschild black hole.

For a canonical ensemble, the energy is given by [95–97]

$$E = \frac{\sum_n E_n e^{-\chi E_n}}{\sum_n e^{-\chi E_n}} = -\frac{\partial}{\partial \chi} \log Z, \tag{44}$$

where E_n is the energy in the n th state and Z is the partition function, which is in the tree approximation related to the Euclidean action of the classical solution by

$$\hat{I} = -\log Z. \quad (45)$$

Use of (43) and (45) in (44) gives

$$E = \frac{\chi}{8\pi} = M. \quad (46)$$

They also calculated the entropy of the Schwarzschild black hole to obtain

$$S = -\sum_n P_n \log P_n = \chi E + \log Z = 4\pi M^2 = \frac{1}{4} A, \quad (47)$$

where $P_n = Z^{-1} e^{-\chi E_n}$ and A is the area of the event horizon of the Schwarzschild black hole.

Now, let us apply Hawking and Gibbons procedure to tetrad (20). The Euclidean action is given by [90]

$$\begin{aligned} \hat{I} &= -\frac{1}{2\kappa} \int \sqrt{g} \left(R - 2 {}^{(2)}T^\mu{}_{;\mu} \right) d^4x \\ &= \frac{1}{\kappa} \int \sqrt{g} {}^{(2)}T^\mu{}_{;\mu} d^4x, \end{aligned} \quad (48)$$

where R is the Riemann-Christoffel scalar curvature, which is vanishing for the Reissner Nordström spacetime, and ${}^{(2)}T^\mu$ is the basic vector field of the torsion given by (6).

Tetrad (20) after using Euclidean continuation, $t = -\sqrt{-1}\tau$, takes the following form:

$$(h^\alpha{}_i)_E = \begin{pmatrix} -\frac{\beta_5}{\alpha_1} & -\frac{\beta_1}{\alpha_1} \sin \theta \cos \phi & -\frac{\beta_1}{\alpha_1} \sin \theta \sin \phi & -\frac{\beta_1}{\alpha_1} \sin \theta \cos \theta \\ \frac{\beta_1 f_1(\theta, \phi)}{\alpha_1} & \alpha_1 \sin \theta \cos \phi [1 + \beta_6] & r [\sin \theta \sin \phi \beta_6 + \cos \theta \cos \phi] & r \sin \theta [\cos \theta \beta_6 + \sin \phi] \\ \frac{\beta_1 f_1(\theta, \phi)}{\alpha_1} & \alpha_1 \sin \theta [\sin \phi + \cos \phi \beta_7] & r \sin \phi [\sin \theta \beta_7 \cos \theta] & r \sin \theta [\cos \theta \beta_7 + \cos \phi] \\ -\frac{\beta_1}{\alpha_1} f_2(\theta, \phi) & -\alpha_1 \cos \theta [1 + \tan \theta \cos \phi \beta_8] & -r \sin \theta [\sin \phi \beta_8 + 1] & -r \sin \theta \cos \theta \beta_8 \end{pmatrix}, \quad (49)$$

where E refers to the Euclidean continuation and

$$\begin{aligned} \beta_5 &= \{2 - \beta^2\}^{1/2}, & \beta_6 &= [1 - \beta_5] f(\theta, \phi), \\ \beta_7 &= [1 - \beta_5] f_1(\theta, \phi), & \beta_8 &= [1 - \beta_5] f_2(\theta, \phi). \end{aligned} \quad (50)$$

The Euclidean line element squared of tetrad (49) takes the form

$$ds^2 = \frac{1}{\alpha_1} d\tau^2 + \alpha_1 dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (51)$$

which is the Euclidean Reissner Nordström spacetime [88, 89]. Using (6), we get the following:

$$h = r^2 \sin \theta,$$

$$\begin{aligned} {}^{(2)}T^1 &= \left((\beta + 1) (1 + \sin \theta \sin \phi [\cos \theta \cos \phi + \sin \theta \sin \phi] \right. \\ &\quad \left. + \cos \theta \sin \phi) + r\alpha_2 - \frac{2}{\alpha_1} \right) \\ &\quad \times (r\alpha_1)^{-1}, \quad \alpha_2 = \alpha'_1, \end{aligned} \quad (52)$$

where $h = \sqrt{g}$ is the determinant of tetrad (49). There are other nonvanishing components of ${}^{(2)}T^\mu$, that is, ${}^{(2)}T^0$, ${}^{(2)}T^2$, and ${}^{(2)}T^3$, but these components are not necessary in our computations. The volume integral (48),

($\theta = 0 \cdots \pi$, $\phi = 0 \cdots 2\pi$, $\tau = 0 \cdots \chi$), for tetrad (49), is calculated to give

$$\begin{aligned} \hat{I} &= \frac{1}{\kappa} \int \sqrt{g} \left({}^{(2)}T^\mu{}_{;\mu} \right) d^4x \\ &\approx \frac{2\pi r^2 (1 + 2\beta - (6/\alpha_1) - 3r\alpha_2)}{3\alpha_1 (1 + (q^2/r^2))}, \end{aligned} \quad (53)$$

where $\chi = 4\pi r/(1 - (q^2/r^2))$. The largest positive root of $g^{rr} = 0$, that is, $(r^2 - 2Mr + q^2) = 0 \Rightarrow r_\pm = M \pm \sqrt{M^2 - q^2}$. Now, one can write mass and charge in terms of horizons, that is, $M = (r_+ + r_-)/2$ and $q = \sqrt{r_- r_+}$. Equation (53) shows in a clear way that the Euclidean action depends on the arbitrary function $\mathfrak{F}$ which makes the energy also depends on this arbitrary function. Therefore, we need to regularize (48). This regularization lies in the redefinition of the nonsymmetric connection. This redefinition takes the form

$$\begin{aligned} \Gamma^i{}_{jk} &\stackrel{\text{def.}}{=} h^\alpha{}_i h^\alpha{}_{j,k} + h^\alpha{}_i \Lambda_2^{\beta\alpha} \Lambda_{2\beta\gamma, k} h^\gamma{}_j, \\ T^\mu{}_{\nu\lambda} &\stackrel{\text{def.}}{=} \Gamma^\mu{}_{\nu\lambda} - \Gamma^\mu{}_{\lambda\nu}, \quad {}^{(2)}T_\nu \stackrel{\text{def.}}{=} T^\mu{}_{\nu\mu}. \end{aligned} \quad (54)$$

Using (54) in (48), we get

$$\hat{I} = \frac{1}{\kappa} \int \sqrt{g} \left({}^{(2)}T^\mu{}_{;\mu} \right) d^4x \approx \pi r_+^2 \left(1 - \frac{q^2}{r_+^2} \right). \quad (55)$$

TABLE 1: Comparison between the results of the arbitrary function $\mathfrak{F}(r)$, other constants, and the associated energy.

Spacetime	Arbitrary function $B(r)$	The constants c_1 and c_2	Energy	Total conserved charge	Regularized conserved charge
Schwarzschild	$\mathfrak{F}(r) = 0$ or	$c_1 = 0$ $\mathfrak{F}(r) = \text{const.}$	$\tilde{E} = M$	$\mathcal{Q}(\xi, \vartheta) = M\zeta^0$	$\tilde{\mathcal{Q}}(\xi, \vartheta) = M\zeta^0$
	$\mathfrak{F}(r) = 0$ or $\mathfrak{F}(r) = \text{const.}$	$c_1 = M$	$\tilde{E} = \frac{M}{3}$	$\mathcal{Q}(\xi, \vartheta) = \frac{5M}{3}\zeta^0$	$\tilde{\mathcal{Q}}(\xi, \vartheta) = M\zeta^0$
	$\mathfrak{F}(r) = \int \frac{1 - \sqrt{1 - (2c_1/r)}}{r} dr$	$c_1 \neq 0$	$\tilde{E} = M$	$\mathcal{Q}(\xi, \vartheta) = M\zeta^0$	$\tilde{\mathcal{Q}}(\xi, \vartheta) = M\zeta^0$
Reissner-Nordström	$\mathfrak{F}(r) = 0$ or $\mathfrak{F}(r) = \text{const.}$	$c_1 = 0$ and $c_2 = 0$	$\tilde{E} = M - \frac{q^2}{2R}$	$\mathcal{Q}(\xi, \vartheta) \approx \left(M - \frac{q^2 + M^2}{2r}\right)\zeta^0 + O\left(\frac{1}{r^2}\right)$	$\tilde{\mathcal{Q}}(\xi, \vartheta) \approx \left(M - \frac{q^2 + M^2}{2r}\right)\zeta^0 + O\left(\frac{1}{r^2}\right)$
	$\mathfrak{F}(r) = 0$ or $\mathfrak{F}(r) = \text{const.}$	$c_1 = M$ and $c_2 = q$	$\tilde{E} = \frac{1}{3}\left(M - \frac{q^2}{2R}\right)$	$\mathcal{Q}(\xi, \vartheta) \approx \left(\frac{5M}{3} - \frac{5q^2 + M^2}{6r}\right)\zeta^0 + O\left(\frac{1}{r^2}\right)$	$\tilde{\mathcal{Q}}(\xi, \vartheta) \approx \left(M - \frac{q^2 + M^2}{2r}\right)\zeta^0 + O\left(\frac{1}{r^2}\right)$
	$\mathfrak{F}(r) = \int \frac{1 - \sqrt{1 - (2c_1/r) + (c_2^2/r^2)}}{r} dr$	$c_1 \neq 0$ and $c_2 \neq 0$	$\tilde{E} = M - \frac{q^2}{2R}$	$\mathcal{Q}(\xi, \vartheta) \approx \left(M - \frac{q^2 + M^2}{2r}\right)\zeta^0 + O\left(\frac{1}{r^2}\right)$	$\tilde{\mathcal{Q}}(\xi, \vartheta) \approx \left(M - \frac{q^2 + M^2}{2r}\right)\zeta^0 + O\left(\frac{1}{r^2}\right)$

TABLE 2: Comparison between the results of energy using different translational momentum.

Translational momentum	Energy of Schwarzschild and Reissner-Nordström spacetimes	Equation
Møller within the framework of tetrad spacetime	Energy of Schwarzschild spacetime $E = 2M - R^2\mathcal{B}'(\mathcal{R})$	(46) in [94]
Equation (9) with trivial Weitzenböck connection, that is, $\Gamma_\alpha^\beta = 0$, $q = c_2 = 0$	Energy of Schwarzschild spacetime $E = \frac{R}{3} \left\{ 2 + \beta \left(1 - \frac{3}{\alpha_1} \right) \right\}$	(24)
Equation (9) with nontrivial Weitzenböck connection, that is, $\Gamma_\alpha^\beta \neq 0$, $q = c_1 = 0$	Energy of Schwarzschild spacetime $E \approx M + O\left(\frac{1}{R}\right)$	(33)
Møller within the framework of tetrad spacetime	Energy of Reissner-Nordström spacetime $E = 2M - \frac{q^2}{R} - R^2\mathfrak{F}'(\mathcal{R})$	(38) in [93]
Equation (9) with trivial Weitzenböck connection, that is, $\Gamma_\alpha^\beta = 0$	Energy of Reissner-Nordström spacetime $E = \frac{R}{3} \left\{ 2 + \beta \left(1 - \frac{3}{\alpha_1} \right) \right\}$	(24)
Equation (9) with nontrivial Weitzenböck connection, that is, $\Gamma_\alpha^\beta \neq 0$	Energy of Reissner-Nordström spacetime $E \approx M - \frac{q^2}{2R} + O\left(\frac{1}{R^2}\right)$	(33)

From (55), (44), and (45), we get the energy up to $O(1/r_+)$ in the form

$$E \approx M - \frac{q^2}{2r_+} + O\left(\frac{1}{r_+^2}\right), \quad (56)$$

which is consistent with (33) and (40).

Using (55) and (56) in (47), we get the entropy of tetrad (49) in the form

$$S \approx \pi r_+^2. \quad (57)$$

The Hawking temperature of tetrad (49) has the form [98]

$$T = \frac{(1 - (q^2/r_+^2))}{4\pi r_+}. \quad (58)$$

From (56), (57), and (58), we get

$$dM = TdS. \quad (59)$$

Equation (59) shows that the first law of thermodynamic is satisfied.

7. Discussion and Conclusion

Within the framework of MAG, the total Lagrangian is formed from a geometrical part plus a matter part; that is, $\mathcal{L}_{\text{total}} = V(g_{\alpha\beta}, \vartheta^\alpha, Q_{\alpha\beta}, T^\alpha, R_\alpha{}^\beta) + L_{\text{matter}}(g_{\alpha\beta}, \vartheta^\alpha, \Psi, D\Psi)$ with $Q_{\alpha\beta}$ is the nonmetricity, and Ψ is the matter fields. Baekler and Hehl [59] have obtained a stationary axially symmetric exact solution of the vacuum field equation for a specific gravitational Lagrangian. Their black hole solution embodies a Kerr-de Sitter metric and the post-Riemannian structures of torsion and nonmetricity and is characterized by mass, angular momentum, and shear charge which measure the violation of Lorentz invariance. When the cosmological constant vanishes, that is, $\lambda = 0$, the solution given by (57) derived in [59] will be identical with the third coframe defined by $\vartheta^R{}^\alpha = (\Lambda_1{}^\alpha{}_\gamma)(h^\gamma{}_i)_3 dx^i$.

To test the regularizing property of teleparallelism, we have used the superpotential 2-form to calculate energy of a spherically symmetric tetrad with arbitrary function. We have done such calculations when the Weitzenböck connection is taken into account and also when it is not taken into account. It has been shown that the energy depends on the arbitrary function in the trivial case and when it takes specific values, which make the local Lorentz transformations identical with the Kronecker delta; the total energy becomes physically acceptable [91, 94]. As Weitzenböck connection is taken into account and recalculation of energy is carried out, we have shown that it acts as a regularizing tool which separates inertia from the physical quantities.

In the context of TEGR, the total conserved charge of the spherically symmetric tetrad with arbitrary function is calculated. It has been shown how inertia spoils the total conserved charge along the time coordinate. Therefore, regularized expression to recalculate the total conserved charge is used, and a plausible result is obtained for energy. Finally, the energy of the tetrad field with arbitrary function is calculated using the Euclidean continuation method in the context of TEGR. It has been shown that energy depends on the arbitrary function. Therefore, a redefinition of the Riemannian connection is provided to recalculate energy. A satisfactory result which is consistent with the previous two methods is achieved. The thermodynamical quantities associated with the tetrad with arbitrary function are calculated, and satisfactory results are obtained [99]. *The main conclusion of the present work is that Weitzenböck connection which acts as a regularized term works correctly for the spherically symmetric case.* At present, it is not clear if this conclusion is correct for axially symmetric spacetimes. This needs a further study.

Blagojević [100] has assumed that spacetime behaves asymptotically as Minkowski space M_4 and has obtained the corresponding global Poincaré generators. The surface terms

related to the improved form of these generators appeared to represent the relevant conserved charges—energy, momentum, and angular momentum. Mielke and Wallner [101] discussed a Lagrangian form of the energy-momentum of some exact solutions in the teleparallel limit of PG theory.

We make a comparison between the possible values which reproduce the previous spacetimes of Schwarzschild and Reissner Nordström as shown in Table 1. Also, we give a comparison between the results of the total energy for taking into account Weitzenböck connection and for not taking into account Weitzenböck connection and the previous results of the total energy for the same spacetime with arbitrary function as shown in Table 2.

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