Review Article

Neutrino Yukawa Textures within Type-I Seesaw

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We review neutrino Yukawa textures with zeros within the framework of the type-I seesaw with three heavy right chiral neutrinos and in the basis where the latter and the charged leptons are mass diagonal. An assumed nonvanishing mass of every ultralight neutrino and the observed nondecoupling of any neutrino generation allow a maximum of four zeros in the Yukawa coupling matrix in family space. We show that the requirement of an exact $\mu\tau$ symmetry, coupled with the observational constraints, reduces seventy-two allowed such textures to only four corresponding to just two different forms of the light neutrino mass matrix: one with an inverted and the other with a normal mass ordering. The masses and Majorana phases of ultralight neutrinos are predicted within definite ranges with laboratory and cosmological observational inputs. Within the same framework, we also study Yukawa textures with a fewer number of zeros, but with exact $\mu\tau$ symmetry. We further formulate the detailed scheme of the explicit breaking of $\mu\tau$ symmetry in terms of three small parameters for allowed four zero textures. The observed sizable mixing between the first and third generations of neutrinos is shown to follow for a suitable choice of these symmetry breaking parameters.

1. Introduction

The impressive experimental progress from neutrino oscillation studies [1–5] and the sharpening [6, 7] of the cosmological upper bound on the neutrino mass sum have underscored two fundamental but distinct puzzles. (1) Why are the observed neutrinos so ultralight, that is, with masses in the sub-eV range? (2) Why is the three neutrino mixing pattern of two large and one small (but measurable) angles so different from the sequentially small CKM mixing angles of quarks? There is a widespread feeling that the former is due to some kind of a seesaw mechanism [8–14] yielding ultralight Majorana neutrinos. It is our contention that the latter has to do with zeros in neutrino Yukawa textures plus a broken $\mu\tau$ symmetry. Let us start with the simplest scheme of three weakly interacting flavored ultralight neutrinos discarding any possible light sterile ones mixing with them. We hold that there should be a fundamental principle behind a massless particle, as with gauge invariance and the photon. Since no such principle is identifiable with any single neutrino, we take each to have a nonzero mass. Though there are other types of

proposed seesaw mechanisms, such as type-II [11, 12], type-III [13], and inverse seesaw [14, 15], in a minimalist approach we stick to the original type-I with three heavy right chiral electroweak singlet neutrinos denoted by the column vector N_R .

We next turn to the issue of texture zeros. By a texture we mean a configuration of a Yukawa coupling matrix with some vanishing elements. Texture zeros have a long history in the quark sector where four zero Yukawa textures [18–20] have had distinguished success in fitting the known quark masses and CKM parameters. The problem is simpler there since the Dirac quark mass matrix of a given charge, which is the corresponding Yukawa coupling matrix times the Higgs VEV, contains all information about physical quark masses. In the case of seesaw induced ultralight Majorana neutrinos, the elements of the Dirac mass matrix M_D do not carry all information about physical neutrino masses. The latter are contained in the elements of the complex symmetric Majorana neutrino mass matrix M_ν , which is related to M_D through the standard seesaw formula. There have been initial as well as continuing efforts [21–25] to assume the vanishing of certain elements in M_ν . But, we

strongly feel that an occurrence of zeros must be linked to some fundamental symmetry [26–28] or suppression mechanism [29] inherent in the Lagrangian itself. It seems more natural then to postulate the occurrence of such zeros in some elements of the neutrino Yukawa coupling matrix (equivalently M_D) which appears in the Lagrangian [16, 17, 30–36]. There are ways [37–39] to ensure the stability of such zeros under quantum corrections in type-I seesaw models.

An important point in the context of texture zeros is that of Weak Basis dependence. Both M_D and M_ν change [40] under general (and different) unitary transformations of the left and right chiral fermion fields. In consequence, any Yukawa texture is basis dependent. It is further known that those fermions, which do not couple mutually in the Lagrangian, can be simultaneously put into a mass diagonal form by suitable basis transformations. Without loss of generality, we can therefore choose a Weak Basis in which the charged lepton fields l and the very heavy right chiral neutrino fields N_R are mass diagonal with real masses. The question arises as to how a flavor model, corresponding to a given set of texture zeros in such a basis, would be recognized in a different basis. It has been shown [40] that the vanishing of certain Weak Basis invariants would be a hallmark of those zeros. This is also related to the linkage of CP violation at low energies, probed in short or long baseline experiments, and at high energies, as relevant to leptogenesis. Though that linkage is a major motivation for postulating Yukawa texture zeros [30–32], it is outside the scope of the present paper.

In this paper we focus on the role of texture zeros, occurring in M_D , in understanding the observed pattern of neutrino masses and mixing angles. More generally, we show how they affect key aspects of low energy neutrino phenomenology. Four is shown to be the maximum number of such zeros allowed within our framework [30]. We classify all possible four zero textures, seventy two in total [30]. Then we introduce $\mu\tau$ symmetry [16, 41–66] as an invariance under the interchange of flavors μ (2) and τ (3) in the neutrino sector which is motivated by an automatic prediction of vanishing (maximal) mixing between the first (second) and third generations of neutrinos. This symmetry reduces the preceding seventy two textures to four which lead to only two distinct forms of M_ν , whose phenomenological consequences are worked out [16, 17, 32]. Three zero textures with $\mu\tau$ symmetry are also shown to have similar consequences, while textures with a lesser number of zeros have little predictivity [33]. We then discuss the general explicit breaking of $\mu\tau$ symmetry in terms of three small parameters and show, within the lowest order of perturbation in those parameters, that the observed small mixing of first and third generations of neutrinos can be explained within our framework [33].

In Section 2 we set up our formalism. Section 3 contains the classification of all four zero textures and a discussion of $\mu\tau$ symmetry. Section 4 addresses the consequent phenomenological implications. In Section 5 we discuss the realization of other $\mu\tau$ symmetric texture zeros. Section 6 contains a general discussion of explicit $\mu\tau$ symmetry breaking and how that fits observation. Finally, in Section 7 we summarize our conclusions.

2. Framework and Formalism

The relevant mass terms in our starting Lagrangian are

$$\begin{aligned} -\mathcal{L}^m &= \overline{\nu}_L^0 M_D N_R^0 + \frac{1}{2} \overline{N}_L^{0C} M_R N_R^0 + \overline{l}_L^0 M_l l_R^0 + \text{h.c.} \\ &= \frac{1}{2} \begin{pmatrix} \overline{\nu}_L^0 & \overline{N}_L^{0C} \end{pmatrix} \begin{pmatrix} 0 & M_D \\ M_D^T & M_R \end{pmatrix} \begin{pmatrix} \nu_R^0 \\ N_R^0 \end{pmatrix} + \overline{l}_L^0 M_l l_R^0 + \text{h.c.}, \end{aligned} \quad (1)$$

where we have used the general definition of a conjugate fermion field $\psi^C = \gamma_0 C \psi^*$ (C being the charge conjugation matrix) and the identity

$$\overline{\psi}_L m \psi'_R = \overline{\psi}_L^C m^T \psi_R^C. \quad (2)$$

Here M_R , M_D , and M_l , respectively, denote the right chiral complex symmetric Majorana mass matrix, the neutrino Dirac mass, matrix and the charged lepton mass matrix in a three-dimensional family space. The superscripts “0” identifies the corresponding fields as flavor eigenstate ones. The complex symmetric 6×6 neutrino mass matrix in the second line of (1) is denoted by $\mathcal{M}$, that is,

$$\mathcal{M} = \begin{pmatrix} 0 & M_D \\ M_D^T & M_R \end{pmatrix}. \quad (3)$$

The energy scale of M_R is taken to be very high ($>10^9$ GeV), as compared with the electroweak scale $v \simeq 246$ GeV.

The complete diagonalization of $\mathcal{M}$ leads to

$$\mathcal{V}^\dagger \mathcal{M} \mathcal{V}^* = \text{diag}(d, D), \quad (4)$$

where $\mathcal{V}$ is a 6×6 unitary matrix

$$\mathcal{V} = \begin{pmatrix} K & G \\ S & T \end{pmatrix}, \quad (5)$$

with 3×3 blocks K , G , S , and T . In (4) d and D are three dimensional diagonal mass matrices, each with ultralight and heavy real positive entries, respectively:

$$\begin{aligned} d &= \text{diag}(m_1, m_2, m_3), \quad m_{1,2,3} < 1 \text{ eV}, \\ D &= \text{diag}(M_1, M_2, M_3), \quad M_{1,2,3} > 10^9 \text{ GeV}. \end{aligned} \quad (6)$$

Charged current interactions can then be written in terms of the semiweak coupling strength g as well as the respective ultralight neutrino and heavy neutrino fields ν_i and N_i :

$$\mathcal{L}^{cc} = -\frac{g}{\sqrt{2}} \left(\overline{l}_{iL} \gamma_\mu K_{ij} \nu_{jL} + \overline{l}_{iL} \gamma_\mu G_{ij} N_{jL} \right) W_\mu^\dagger. \quad (7)$$

In an excellent approximation, the ultralight neutrino masses and mixing angles can now be obtained from

$$M_\nu \simeq -M_D M_R^{-1} M_D^T, \quad (8)$$

$$U^\dagger M_\nu U^* = d. \quad (9)$$

Equation (8) is the well-known seesaw formula. We also choose to define the matrix $h_\nu = M_\nu M_\nu^\dagger$ and have

$$U^\dagger h_\nu U = d^2. \quad (10)$$

In (9), U is the Pontecorvo, Maki, Nakagawa, Sakata (PMNS) matrix admitting the standard parametrization:

$$U_{\text{PMNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_D} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_D} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_D} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_D} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_D} & c_{23}c_{13} \end{pmatrix} \times \begin{pmatrix} e^{i\alpha_{M_1}/2} & 0 & 0 \\ 0 & e^{i\alpha_{M_2}/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (11)$$

with $c_{ij} = \cos \theta_{ij}$, $s_{ij} = \sin \theta_{ij}$, and $\delta_D(\alpha_{M_1}, \alpha_{M_2})$ being the yet unknown Dirac (Majorana) phase(s). We note for the sake of completeness that the unitary transformation between the column of mass eigenstate of left chiral neutrino fields ν_L and the corresponding flavor eigenstate ν_L^0 is

$$\nu_L^0 = U \nu_L. \quad (12)$$

The additional approximate relations to keep in mind are those between U and the submatrices K , G of $\mathcal{V}$ and M_D of $\mathcal{M}$:

$$K_{ij} = U_{ij} + O\left(\frac{v^2}{M_R^2}\right), \quad (13)$$

$$M_k G_{jk} = (M_D)_{jk} + O\left(\frac{v^3}{M_R^2}\right), \quad k \text{ not summed.}$$

Needless to add, we always neglect terms of order v^2/M_R^2 .

As mentioned earlier, without loss of generality, we can choose the Weak Basis in which M_l and M_R are

$$M_l = \text{diag}(m_e, m_\mu, m_\tau), \quad (14)$$

$$M_R = \text{diag}(M_1, M_2, M_3),$$

with real positive entries. All CP-violating phases, stemming from $\mathcal{M}$, are contained in the Dirac mass matrix M_D in this Weak Basis. As a consequence of (6) and (8), (9) M_D can be written in the Casas-Ibarra form [67]:

$$M_D = iU \sqrt{d} R \sqrt{M_R}, \quad (15)$$

where R is a complex orthogonal matrix: $R^T R = R R^T = I$. An important comment on M_D , following from (8), is that, our condition of no massless neutrino, that is $\det M_\nu \neq 0$, implies that $\det M_D \neq 0$. This means that textures of M_D with one vanishing row or column or with a quartet of zeros (i.e., zeros in ij , lk , ik , and lj elements with $i \neq l$ and $k \neq j$ and $l = 1, 2$, or 3) are inadmissible since they make $\det M_D$ vanish. Furthermore, in our Weak Basis, for any nonzero entry in M_D with all other elements in its row or column

being zero, M_ν from (8) develops a block diagonal form that is incompatible with the observed simultaneous mixing of three neutrinos. The same logic holds for any block diagonal texture of M_D . Indeed, if any row in a texture of M_D is orthogonal, element by element, to both the others, one neutrino family decouples and therefore makes such a texture inadmissible. These arguments have been shown [30] to be sufficient to rule out all textures in M_D with more than four zeros. Four is then the maximum permitted number of zeros in a neutrino Yukawa texture.

3. Classification of Four Zero Textures and the Role of $\mu\tau$ Symmetry

In this section we provide the classification of all possible four zero neutrino Yukawa textures and forms of the surviving textures, since these details were not given in [30, 68]. There are ${}^9C_4 = 126$ possible four zero neutrino Yukawa textures which can be classified into four classes [68]. In making this classification, we rule out the orthogonality between any two rows or columns by some artificial cancellation; orthogonality is to be ensured in terms of a vanishing product, element by element. We can now enumerate four cases.

- (i) $\det M_D \neq 0$ and one family of neutrinos decouples: 9 textures. For each texture of M_D here, one row is orthogonal to the other rows. It follows that, in the neutrino mass matrix M_ν in our chosen basis with a diagonal M_R , one neutrino family always decouples. So, though all neutrinos are massive here, these textures are to be discarded.
- (ii) $\det M_D = 0$ and one family of neutrinos decouples: 18 textures. Here each texture has a vanishing row and there are six such textures for every such row. Such a row generates a vanishing mass eigenvalue and the corresponding family decouples. Hence this class is also excluded.
- (iii) $\det M_D = 0$ and no family decouples: 27 textures [68]. Each of 18 textures in this class has a vanishing column and each of the remaining 9 has a quartet of zeros, leading to a vanishing $\det M_\nu$. So, this class is rejected.
- (iv) $\det M_D \neq 0$ and no family decouples: 72 textures. These remaining textures are allowed by the criteria we have set up.

The retained textures are subdivided into two categories A and B. We wish to elaborate on this categorization [30]. Let us consistently use the complex parameters a_k , b_k , and c_k for elements in M_D belonging to the k th column and the first, second, and third rows, respectively. The two categories then are as follows.

Category A. Here every texture has two mutually orthogonal rows (i, j say, with $i \neq j$) and the corresponding derived M_ν has $(M_\nu)_{ij} = 0$. Thus there are 54 such textures divided into three sub-categories, each containing 18 textures: (A1) those with orthogonal rows 1 and 2 which generate

TABLE 1: Four zero Yukawa textures of M_D in Category A with subcategories A1, A2, and A3.

Category A1 (orthogonality between rows 1 and 2)					
$\begin{pmatrix} 0 & a_2 & 0 \\ b_1 & 0 & b_3 \\ 0 & c_2 & c_3 \end{pmatrix}$	$\begin{pmatrix} 0 & a_2 & 0 \\ b_1 & 0 & b_3 \\ c_1 & c_2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & a_3 \\ b_1 & b_2 & 0 \\ 0 & c_2 & c_3 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & a_3 \\ b_1 & b_2 & 0 \\ c_1 & 0 & c_3 \end{pmatrix}$	$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & b_2 & b_3 \\ c_1 & 0 & c_3 \end{pmatrix}$	$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & b_2 & b_3 \\ c_1 & c_2 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & a_2 & 0 \\ 0 & 0 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$	$\begin{pmatrix} 0 & a_2 & 0 \\ b_1 & 0 & 0 \\ c_1 & c_2 & c_3 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & a_3 \\ b_1 & 0 & 0 \\ c_1 & c_2 & c_3 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & a_3 \\ 0 & b_2 & 0 \\ c_1 & c_2 & c_3 \end{pmatrix}$	$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & 0 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$	$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & b_2 & 0 \\ c_1 & c_2 & c_3 \end{pmatrix}$
$\begin{pmatrix} a_1 & 0 & a_3 \\ 0 & b_2 & 0 \\ 0 & c_2 & c_3 \end{pmatrix}$	$\begin{pmatrix} a_1 & 0 & a_3 \\ 0 & b_2 & 0 \\ c_1 & c_2 & 0 \end{pmatrix}$	$\begin{pmatrix} a_1 & a_2 & 0 \\ 0 & 0 & b_3 \\ 0 & c_2 & c_3 \end{pmatrix}$	$\begin{pmatrix} a_1 & a_2 & 0 \\ 0 & 0 & b_3 \\ c_1 & 0 & c_3 \end{pmatrix}$	$\begin{pmatrix} 0 & a_2 & a_3 \\ b_1 & 0 & 0 \\ c_1 & 0 & c_3 \end{pmatrix}$	$\begin{pmatrix} 0 & a_2 & a_3 \\ b_1 & 0 & 0 \\ c_1 & c_2 & 0 \end{pmatrix}$
Category A2 (orthogonality between rows 2 and 3)					
$\begin{pmatrix} 0 & a_2 & a_3 \\ 0 & b_2 & 0 \\ c_1 & 0 & c_3 \end{pmatrix}$	$\begin{pmatrix} a_1 & a_2 & 0 \\ 0 & b_2 & 0 \\ c_1 & 0 & c_3 \end{pmatrix}$	$\begin{pmatrix} 0 & a_2 & a_3 \\ 0 & 0 & b_2 \\ c_1 & c_2 & 0 \end{pmatrix}$	$\begin{pmatrix} a_1 & 0 & a_3 \\ 0 & 0 & b_3 \\ c_1 & c_2 & 0 \end{pmatrix}$	$\begin{pmatrix} a_1 & 0 & a_3 \\ b_1 & 0 & 0 \\ 0 & c_2 & c_3 \end{pmatrix}$	$\begin{pmatrix} a_1 & a_2 & 0 \\ b_1 & 0 & 0 \\ 0 & c_2 & c_3 \end{pmatrix}$
$\begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}$	$\begin{pmatrix} a_1 & a_2 & a_1 \\ 0 & b_2 & 0 \\ c_1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & b_2 \\ 0 & c_2 & 0 \end{pmatrix}$	$\begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & b_3 \\ c_1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & 0 & 0 \\ 0 & 0 & c_3 \end{pmatrix}$	$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & 0 & 0 \\ 0 & c_2 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & a_2 & a_3 \\ b_1 & 0 & b_3 \\ 0 & c_2 & 0 \end{pmatrix}$	$\begin{pmatrix} a_1 & a_2 & 0 \\ b_1 & 0 & b_3 \\ 0 & c_2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & a_2 & a_3 \\ b_1 & b_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}$	$\begin{pmatrix} a_1 & 0 & a_3 \\ b_1 & b_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}$	$\begin{pmatrix} a_1 & 0 & a_3 \\ 0 & b_2 & b_3 \\ c_1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a_1 & a_2 & 0 \\ 0 & b_2 & b_3 \\ c_1 & 0 & 0 \end{pmatrix}$
Category A3 (orthogonality between rows 1 and 3)					
$\begin{pmatrix} 0 & a_2 & 0 \\ 0 & b_2 & b_3 \\ c_1 & 0 & c_3 \end{pmatrix}$	$\begin{pmatrix} 0 & a_2 & 0 \\ b_1 & b_2 & 0 \\ c_1 & 0 & c_3 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & a_3 \\ 0 & b_2 & b_3 \\ c_1 & c_2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & a_3 \\ b_1 & 0 & b_3 \\ c_1 & c_2 & 0 \end{pmatrix}$	$\begin{pmatrix} a_1 & 0 & 0 \\ b_1 & b_2 & 0 \\ 0 & c_2 & c_3 \end{pmatrix}$	$\begin{pmatrix} a_1 & 0 & 0 \\ b_1 & 0 & b_3 \\ 0 & c_2 & c_3 \end{pmatrix}$
$\begin{pmatrix} 0 & a_2 & 0 \\ b_1 & b_2 & b_3 \\ 0 & 0 & c_3 \end{pmatrix}$	$\begin{pmatrix} 0 & a_2 & 0 \\ b_1 & b_2 & b_3 \\ c_1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & c_2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a_1 & 0 & 0 \\ b_1 & b_2 & b_3 \\ 0 & c_2 & 0 \end{pmatrix}$	$\begin{pmatrix} a_1 & 0 & 0 \\ b_1 & b_2 & b_3 \\ 0 & 0 & c_3 \end{pmatrix}$
$\begin{pmatrix} a_1 & 0 & a_3 \\ 0 & b_2 & b_3 \\ 0 & c_2 & 0 \end{pmatrix}$	$\begin{pmatrix} a_1 & 0 & a_3 \\ b_1 & b_2 & 0 \\ 0 & c_2 & 0 \end{pmatrix}$	$\begin{pmatrix} a_1 & a_2 & 0 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{pmatrix}$	$\begin{pmatrix} a_1 & a_2 & 0 \\ b_1 & 0 & b_3 \\ 0 & 0 & c_3 \end{pmatrix}$	$\begin{pmatrix} 0 & a_2 & a_3 \\ b_1 & b_2 & 0 \\ c_1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & a_2 & a_3 \\ b_1 & 0 & b_3 \\ c_1 & 0 & 0 \end{pmatrix}$

$(M_\nu)_{12} = (M_\nu)_{21} = 0$; (A2) those with orthogonal rows 2 and 3 which generate $(M_\nu)_{23} = (M_\nu)_{32} = 0$; (A3) those with orthogonal rows 1 and 3 which generate $(M_\nu)_{13} = (M_\nu)_{31} = 0$. The explicit form of each of the 54 textures in Category A within the three sub-categories is shown in Table 1.

Category B. There are 18 textures in this category. Each has two orthogonal columns, while no pair of rows is orthogonal. Invariably, then, it turns out that one row (i , say) has two zeroes and the other two rows (say $k, l \neq i$) have one zero each. It is now a consequence of (8) that, in the derived neutrino mass-matrix M_ν , we have the relation

$$\det \text{cofactor} [(M_\nu)_{kl}] = 0. \quad (16)$$

Once again, one can make three subcategories with six entries each. B1 has two zeros in the first row and one zero in each of the other two rows. B2 has two zeros in the second row and one zero in each of the other two rows. B3 has two zeros in the third row and one zero in each of the other two rows. All 18 textures of Category B are shown in Table 2 within the three subcategories.

We now raise the question of $\mu\tau$ symmetry which we had explained in the Introduction. This symmetry is evidently invalid for the charged lepton mass terms. However, for elements in the Dirac mass matrix M_D of neutrinos, it immediately implies the relations

$$\begin{aligned} (M_D)_{12} &= (M_D)_{13}, & (M_D)_{21} &= (M_D)_{31}, \\ (M_D)_{23} &= (M_D)_{32}, & (M_D)_{22} &= (M_D)_{33}. \end{aligned} \quad (17)$$

Moreover, for the masses of the very heavy right-chiral neutrinos, we have

$$(M_R)_{22} = (M_R)_{33}, \quad (18)$$

a result which is transparent as $M_2 = M_3$ in our chosen basis. On account of (8) and (17) as well as (18), one is immediately led to the following relations among elements of the complex symmetric ultralight neutrino Majorana mass matrix M_ν :

$$(M_\nu)_{12} = (M_\nu)_{13}, \quad (M_\nu)_{22} = (M_\nu)_{33}. \quad (19)$$

We take these as statements of a custodial $\mu\tau$ symmetry in the ultralight neutrino sector. One can now invert (9) and

TABLE 2: Four zero Yukawa textures of M_D in Category B with subcategories B1, B2, and B3.

Category B1			
First row with two zeros, a pair of orthogonal columns, non-orthogonal rows			
$\begin{pmatrix} 0 & a_2 & 0 \\ 0 & b_1 & b_3 \\ c_1 & c_2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & a_2 & 0 \\ b_1 & b_2 & 0 \\ 0 & c_2 & c_3 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & a_3 \\ 0 & b_2 & b_3 \\ c_1 & 0 & c_3 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & a_3 \\ b_1 & 0 & b_3 \\ 0 & c_2 & c_3 \end{pmatrix}$
$\begin{pmatrix} a_1 & 0 & 0 \\ b_1 & 0 & b_3 \\ c_1 & c_2 & 0 \end{pmatrix}$	$\begin{pmatrix} a_1 & 0 & 0 \\ b_1 & b_2 & 0 \\ c_1 & 0 & c_3 \end{pmatrix}$		
Category B2			
Second row with two zeros, a pair of orthogonal columns, non-orthogonal rows			
$\begin{pmatrix} 0 & a_2 & a_3 \\ 0 & b_2 & 0 \\ c_1 & c_2 & 0 \end{pmatrix}$	$\begin{pmatrix} a_1 & a_2 & 0 \\ 0 & b_2 & 0 \\ 0 & c_2 & c_3 \end{pmatrix}$	$\begin{pmatrix} 0 & a_2 & a_3 \\ 0 & 0 & b_2 \\ c_1 & 0 & c_3 \end{pmatrix}$	$\begin{pmatrix} a_1 & 0 & a_3 \\ 0 & 0 & b_3 \\ 0 & c_2 & c_3 \end{pmatrix}$
$\begin{pmatrix} a_1 & 0 & a_3 \\ b_1 & 0 & 0 \\ c_1 & c_2 & 0 \end{pmatrix}$	$\begin{pmatrix} a_1 & a_2 & 0 \\ b_1 & 0 & 0 \\ c_1 & 0 & c_3 \end{pmatrix}$		
Category B3			
Third row with two zeros, a pair of orthogonal columns, non-orthogonal rows			
$\begin{pmatrix} a_1 & a_2 & 0 \\ 0 & b_2 & b_3 \\ 0 & c_2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & a_2 & a_3 \\ b_1 & b_2 & 0 \\ 0 & c_2 & 0 \end{pmatrix}$	$\begin{pmatrix} a_1 & 0 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{pmatrix}$	$\begin{pmatrix} 0 & a_2 & a_3 \\ b_1 & 0 & b_3 \\ 0 & 0 & c_3 \end{pmatrix}$
$\begin{pmatrix} a_1 & 0 & a_3 \\ b_1 & b_2 & 0 \\ c_1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a_1 & a_2 & 0 \\ b_1 & 0 & b_3 \\ c_1 & 0 & 0 \end{pmatrix}$		

explore the consequences of (19) in the parametrization of (11). An immediate consequence is the fixing of the two mixing angles pertaining to the third flavor: $\theta_{23} = \pi/4$, $\theta_{13} = 0$. Since the measured former angle is compatible with 45° within errors and the latter has been found to be small ($\approx 9^\circ$), the occurrence of at least a broken $\mu\tau$ symmetry in nature is a reasonable supposition that we adhere to. An interesting footnote to this discussion is the issue of tribimaximal mixing [69, 70] which subsumes $\mu\tau$ symmetry but posits the additional relation

$$(M_\nu)_{11} + (M_\nu)_{13} = (M_\nu)_{22} + (M_\nu)_{23}, \quad (20)$$

leading to a fixation of the remaining mixing angle $\theta_{12} = \sin^{-1}(1/\sqrt{3}) \approx 35.26^\circ$. However, we will not make use of (20).

An immediate consequence of the imposition of $\mu\tau$ symmetry, via (17), is the drastic reduction of the seventy two allowed four zero textures of M_D to only four [16]. This is seen just by inspection. The allowed $\mu\tau$ symmetric textures are the following, each involving only three complex parameters.

Category A. One has

$$M_{DA1} = \begin{pmatrix} a_1 & a_2 & a_2 \\ 0 & 0 & b_1 \\ 0 & b_1 & 0 \end{pmatrix}, \quad M_{DA2} = \begin{pmatrix} a_1 & a_2 & a_2 \\ 0 & b_1 & 0 \\ 0 & 0 & b_1 \end{pmatrix}. \quad (21)$$

Category B. One has

$$M_{DB1} = \begin{pmatrix} a_1 & 0 & 0 \\ b_1 & 0 & b_2 \\ b_1 & b_2 & 0 \end{pmatrix}, \quad M_{DB2} = \begin{pmatrix} a_1 & 0 & 0 \\ b_1 & b_2 & 0 \\ b_1 & 0 & b_2 \end{pmatrix}. \quad (22)$$

It may be noted that, in either category, any texture can be obtained from the other by the interchange of rows 2 and 3 or columns 2 and 3. Because of $\mu\tau$ symmetry, this means that the physical content of the two textures in each category is the same. Indeed, by use of (8), we obtain the same M_ν for either

of them. Thus we have just two allowed ultralight neutrino Majorana mass matrices

$$M_{\nu A} = - \begin{pmatrix} \frac{a_1^2}{M_1} + \frac{2a_2^2}{M_2} & \frac{a_2 b_1}{M_2} & \frac{a_2 b_1}{M_2} \\ \frac{a_2 b_1}{M_2} & \frac{b_1^2}{M_2} & 0 \\ \frac{a_2 b_1}{M_2} & 0 & \frac{b_1^2}{M_2} \end{pmatrix}, \quad (23)$$

$$M_{\nu B} = - \begin{pmatrix} \frac{a_1^2}{M_1} & \frac{a_1 b_1}{M_1} & \frac{a_1 b_1}{M_1} \\ \frac{a_1 b_1}{M_1} & \frac{b_1^2}{M_1} + \frac{b_2^2}{M_2} & \frac{b_1^2}{M_1} \\ \frac{a_1 b_1}{M_1} & \frac{b_1^2}{M_1} & \frac{b_1^2}{M_1} + \frac{b_2^2}{M_2} \end{pmatrix}, \quad (24)$$

for Categories A and B, respectively.

4. Phenomenology with $\mu\tau$ Symmetric Four Zero Yukawa Textures

Given $\mu\tau$ symmetry, one automatically obtains that $\theta_{23} = \pi/4$ and $\theta_{13} = 0$. The current 3σ limits on these are $36.86^\circ < \theta_{23} < 55.55^\circ$ and $7.49^\circ < \theta_{13} < 10.46^\circ$ [1]. We shall later consider a small breaking of $\mu\tau$ symmetry. But, for the moment, let us assume the latter to be the exact. The other mass and mixing parameters in the ultralight neutrino sector are kept free. Their experimentally allowed 3σ ranges to be used to constrain the nonzero elements of $M_{\nu A}$ and $M_{\nu B}$ in Table 3. We define $\Delta_{ij}^2 = m_i^2 - m_j^2$, where i, j ($= 1, 2, 3$) refer to the mass eigenstate neutrinos. It will now be convenient to reparametrize the elements of $M_{\nu A}$ and $M_{\nu B}$ in (23) and (24), respectively, in the way given in Table 3. Here k_1, k_2, l_1 , and l_2 are real and positive quantities while α, α', β , and β' are phases. However, the phases α' and β' can be absorbed in

TABLE 3: Reparametrized quantities and relevant functions in Category A and Category B.

Category A	Category B
$M_{\nu A}$	$M_{\nu B}$
$m_A = \frac{-b_1^2}{M_2}$	$m_B = \frac{-b_2^2}{M_2}$
$k_1 e^{i(\alpha+\alpha')} = \frac{a_1}{b_1} \sqrt{\frac{M_2}{M_1}}$	$l_1 e^{i\beta'} = \frac{a_1}{b_2} \sqrt{\frac{M_2}{M_1}}$
$k_2 e^{i\alpha'} = \frac{a_2}{b_1}$	$l_2 e^{i\beta} = \frac{b_1}{b_2} \sqrt{\frac{M_2}{M_1}}$
$\alpha = \arg \frac{a_1}{a_2}$	$\beta = \arg \frac{b_1}{b_2}$
$X_1^A = 2\sqrt{2}k_2[(1+2k_2^2)^2 + k_1^4 + 2k_1^2(1+2k_2^2)\cos 2\alpha]^{1/2}$	$X_1^B = 2\sqrt{2}l_1l_2[(l_1^2+2l_2^2)^2 + 1 + 2(l_1^2+2l_2^2)\cos 2\beta]^{1/2}$
$X_2^A = 1 - k_1^4 - 4k_2^4 - 4k_1^2k_2^2\cos 2\alpha$	$X_2^B = 1 + 4l_2^2\cos 2\beta + 4l_1^4 - l_1^4$
$X_3^A = 1 - 4k_2^4 - k_1^4 - 4k_1^2k_2^2\cos 2\alpha - 4k_2^2$	$X_3^B = 1 - (l_1^2+2l_2^2)^2 - 4l_2^2\cos 2\beta$
$X_4^A = k_1^4 + 4k_2^4 + 4k_1^2k_2^2\cos 2\alpha$	$X_4^B = l_1^4$

TABLE 4: Input experimental values [1].

Quantity	Experimental 3σ range
Δ_{21}^2	$7.12 \times 10^{-5} \text{ eV}^2 < \Delta_{21}^2 < 8.20 \times 10^{-5} \text{ eV}^2$
$\Delta_{32}^2 < 0$	$-2.76 \times 10^{-3} \text{ eV}^2 < \Delta_{32}^2 < -2.22 \times 10^{-3} \text{ eV}^2$
$\Delta_{32}^2 > 0$	$2.18 \times 10^{-3} \text{ eV}^2 < \Delta_{32}^2 < 2.70 \times 10^{-3} \text{ eV}^2$
θ_{12}	$31.30^\circ < \theta_{12} < 37.46^\circ$

the definition of the first family neutrino field ν_e for $M_{\nu A}$ and $M_{\nu B}$ respectively and therefore are not physical. Moreover, the overall phase in $m_{A,B}$ can also be absorbed by a further redefinition of all flavor eigenstate neutrino fields. So we can treat $m_{A,B}$ as real for further discussions. In addition, we have defined in Table 3 sets of derived real quantities $X_{1-4}^{A,B}$ which will be related to various observables.

With the reparametrization given in Table 3, $M_{\nu A}$ and $M_{\nu B}$ assume the simple forms

$$M_{\nu A} = m_A \begin{pmatrix} k_1^2 e^{2i\alpha} + 2k_2^2 & k_2 & k_2 \\ k_2 & 1 & 0 \\ k_2 & 0 & 1 \end{pmatrix},$$

$$M_{\nu B} = m_B \begin{pmatrix} l_1^2 & l_1 l_2 e^{i\beta} & l_1 l_2 e^{i\beta} \\ l_1 l_2 e^{i\beta} & l_2^2 e^{2i\beta} + 1 & l_2^2 e^{2i\beta} \\ l_1 l_2 e^{i\beta} & l_2^2 e^{2i\beta} & l_2^2 e^{2i\beta} + 1 \end{pmatrix}. \quad (25)$$

These lead us, through the diagonalization of the matrix h_ν of (10), to the relations

$$\Delta_{21}^2 = m^2 X, \quad \Delta_{32}^2 = \frac{m^2}{2} (X_3 - X), \quad \tan 2\theta_{12} = \frac{X_1}{X_2}, \quad (26)$$

where $m = m_{A,B}$ for Categories A, B and

$$X = \sqrt{X_1^2 + X_2^2}. \quad (27)$$

One can further make use of (9) to calculate [17] the ultralight masses $m_{1,2,3}$ in terms Δ_{21}^2 and also the Majorana phases $\alpha_{M_1}, \alpha_{M_2}$, compare (11), in terms of m_1, m_2 , and m_3 . The former are given by

$$m_{1,2} = \left| \Delta_{21}^2 \left(\frac{2 - X_3 \mp X}{2X} \right) \right|^{1/2}, \quad m_3 = \left| \frac{\Delta_{21}^2}{X} \right|^{1/2} \quad (28)$$

and the latter by

$$\cos(\alpha_{M_1} - \arg \cdot Z) = \frac{|Z|^2 m_3^2 + m_1^2 \sin^4 \theta_{12} - m_2^2 \cos^4 \theta_{12}}{2m_1 m_3 \sin^2 \theta_{12} |Z|},$$

$$\cos(\alpha_{M_2} - \arg \cdot Z) = \frac{|Z|^2 m_3^2 + m_2^2 \cos^4 \theta_{12} - m_1^2 \sin^4 \theta_{12}}{2m_2 m_3 \cos^2 \theta_{12} |Z|}. \quad (29)$$

Here $Z = [(\mathcal{M}_\nu)_{22} + (\mathcal{M}_\nu)_{23}][(\mathcal{M}_\nu)_{22} - (\mathcal{M}_\nu)_{23}]^{-1}$. The last quantity of physical interest that we calculate in this section is the effective mass $m_{\beta\beta} = |(\mathcal{M}_\nu)_{11}|$ appearing in the transition amplitude for the yet unobserved neutrinoless nuclear double beta decay. That is given by

$$m_{\beta\beta} = |\Delta_{21}^2 X_4 X^{-1}|^{1/2}, \quad (30)$$

with X_4 as given in Table 3.

Feeding the experimental 3σ ranges from Table 4, we find that only the inverted mass ordering $\Delta_{32}^2 < 0$ is allowed in Category A while only the normal mass ordering $\Delta_{32}^2 > 0$ is permitted for Category B. Moreover, in the corresponding

FIGURE 1: Variation of k_1 and k_2 in Category A and of l_1 and l_2 in Category B with $\mu\tau$ symmetry over the 3σ allowed ranges of Δ_{21}^2 , $|\Delta_{32}^2|$, and θ_{12} [16].

$k_1 - k_2/l_1 - l_2$ parameter plane [16], very constrained domains are allowed, as shown in Figure 1. The phases α, β are also severely restricted in magnitude, specifically $89.0^\circ \leq |\alpha| < 90^\circ$ and $87.0^\circ \leq |\beta| < 90^\circ$. These allow just a very limited region in the $X_3 - X$ plane, leading to 3σ lower and upper bounds on the neutrino mass sum $m_1 + m_2 + m_3$, namely, $[0.156, 0.5] \text{ eV}/[0.074, 0.132] \text{ eV}$ for an inverted/normal mass ordering [17]. It may be recalled that there is already a lower bound of 0.05 eV on the said sum from atmospheric neutrino data. Furthermore the general consensus [7] on the least model-dependent cosmological upper bound on it is 0.5 eV.

Turning to the individual neutrino masses m_1/eV , m_2/eV , and m_3/eV , respectively, we obtain by use of (28) the respective 3σ intervals $[0.0452, 0.1682]$, $[0.0457, 0.1684]$, $[0.077, 0.1632]$ for Category A and $[0.0110, 0.0335]$, $[0.0144, 0.0345]$, and $[0.0485, 0.0638]$ for Category B. However, there are correlated constraints among these masses. These are shown in the left-most panel of Figure 2. Given these allowed intervals and correlated constraints, it is not possible right now to distinguish between the hierarchical and quasi-degenerate possibilities. But a future reduction of these ranges and domains could pin this down. We next come to the Majorana phases $\alpha_{M_1}, \alpha_{M_2}$. One can ab initio restrict them to the interval $-\pi$ to π and utilize (29) as well as the expressions for Z, m_1, m_2, m_3 and $\tan 2\theta_{12}$ in terms of the basic parameters $(k_1, k_2, \alpha)/(l_1, l_2$ and $\beta)$, depending on the category. The further application of the phenomenologically acceptable ranges of these parameters, as given above, leads to the allowed 3σ intervals $-98.0^\circ \leq \alpha_{M_1} \leq 20.0^\circ$, $9.2^\circ \leq \alpha_{M_2} \leq 36.4^\circ$ for Category A and $-88.6^\circ \leq \alpha_{M_1} \leq 7.97^\circ$, $90.7^\circ \leq \alpha_{M_2} \leq 128.8^\circ$ for Category B. Allowed values of α_{M_1} and α_{M_2} are shown in the middle panel of Figure 2.

Another quantity to be considered in this section is the double β -decay effective mass $m_{\beta\beta}$, compare (30). The currently accepted upper bound [71] on it is $m_{\beta\beta} < 0.35 \text{ eV}$. In comparison, our allowed values $m_{\beta\beta}$ versus the neutrino mass sum $\sum_i m_i$ are shown in the rightmost panels of of Figure 2. More absolutely, we can say that $0.038 \leq m_{\beta\beta}/\text{eV} \leq 0.161$ for Category A and $0.003 \leq m_{\beta\beta}/\text{eV} \leq 0.0186$ for Category B. The region near the upper bound in Category A may be accessible in forthcoming experiments.

An interesting question pertains to the consequences of the effect of $\mu\tau$ symmetry on couplings between the heavy right chiral and the ultralight left chiral neutrinos. The corresponding neutral gauge boson induced interactions are down by factor $O(v^2/M_R^2)$. On the other hand, the Higgs boson induced interactions affect leptogenesis modes and have been discussed in detail in [32]. Since leptogenesis is outside the scope of the present paper, we do not go into those discussions here.

5. Realization of Other Texture Zeros with $\mu\tau$ Symmetry

Though four is the maximum number of allowed neutrino Yukawa texture zeros, we examine other textures with a lesser number of zeros for completeness [33]. Let us work in the same Weak Basis of real diagonal M_l and M_R . We wish to study only those textures that are compatible with $\mu\tau$ symmetry which we believe to be approximately valid in the real world. The $\mu\tau$ symmetric forms of M_D and M_R now are

$$M_D = \begin{pmatrix} a & b & b \\ c & d & e \\ c & e & d \end{pmatrix}, \quad M_R = \text{diag}(M_1, M_2, M_2), \quad (31)$$

with a, b, c, d , and e as complex numbers.

FIGURE 2: Allowed $m_{2,3}$ versus m_1 (left), α_{M_2} versus α_{M_1} (middle), and $m_{\beta\beta}$ versus Σm_i (right) for Category A (top) and Category B (bottom) [17].

5.1. Three Zero Textures. We first identify possible three zero textures which are compatible with (31). Apart from a , the other four complex parameters in M_D come in pairs. So, for any texture with an odd number of zeros, a must vanish. For three zero textures the remaining two zeros can be arranged in ${}^4C_1 = 4$ ways. So, the four allowed three zero $\mu\tau$ symmetric textures of M_D are

$$\begin{pmatrix} 0 & b & b \\ c & 0 & e \\ c & e & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & b & b \\ c & d & 0 \\ c & 0 & d \end{pmatrix}, \quad (32)$$

$$\begin{pmatrix} 0 & 0 & 0 \\ c & d & e \\ c & e & d \end{pmatrix}, \quad \begin{pmatrix} 0 & b & b \\ 0 & d & e \\ 0 & e & d \end{pmatrix}.$$

The last two textures have one vanishing row and one vanishing column, respectively. These can be discarded with our requirement of no massless neutrino, that is, $\det M_\nu \neq 0$, leaving only the first two textures as acceptable. In general, there can be ${}^9C_3 = 84$ three zero textures. The conditions of (1) $\mu\tau$ symmetry, (2) the non-zero value of $\det M_\nu$, and (3) the nondecoupling of any neutrino generation reduce this number to only two.

The first two textures of (32) have only three complex parameters each and we can just use b , c , and d for both allowed textures of M_D :

$$\begin{pmatrix} 0 & b & b \\ c & 0 & d \\ c & d & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & b & b \\ c & d & 0 \\ c & 0 & d \end{pmatrix}. \quad (33)$$

Using the seesaw formula, we obtain an identical form of M_ν for both textures in (33), namely,

$$M_\nu = - \begin{pmatrix} \frac{2b^2}{M_2} & \frac{bd}{M_2} & \frac{bd}{M_2} \\ \frac{bd}{M_2} & \frac{c^2}{M_1} + \frac{d^2}{M_2} & \frac{c^2}{M_1} \\ \frac{bd}{M_2} & \frac{c^2}{M_1} & \frac{c^2}{M_1} + \frac{d^2}{M_2} \end{pmatrix}. \quad (34)$$

Equation (34) can be written in the following form under a further reparametrization:

$$M_\nu = m_0 \begin{pmatrix} 2k_1^2 e^{i2\alpha_1} & k_1 e^{i\alpha_1} & k_1 e^{i\alpha_1} \\ k_1 e^{i\alpha_1} & k_2^2 e^{i\alpha_2} + 1 & k_2^2 e^{i\alpha_2} \\ k_1 e^{i\alpha_1} & k_2^2 e^{i\alpha_2} & k_2^2 e^{i\alpha_2} + 1 \end{pmatrix}, \quad (35)$$

with $m_0 = -d^2/M_2$, $k_1 e^{i\alpha_1} = b/d$, and $k_2 e^{i\alpha_2} = c/d\sqrt{M_2/M_1}$. From M_ν , we can remove the phase α_1 and any phase θ_m in m_0 by rotating M_ν with the diagonal phase matrix $\text{diag}(e^{-i\alpha_1}, 1, 1)e^{-i\theta_m/2}$. Thus, M_ν has three real parameters, namely, m_0 , k_1 and k_2 , and only one phase α_2 . The interesting point to be noted is that the number of independent parameters in M_ν for $\mu\tau$ symmetric three zero Yukawa textures is the same as that for $\mu\tau$ symmetric four zero textures. We then have the same phenomenological

expressions as in (26)–(30), but now with changed definitions of $X_{1,2,3,4}$, namely,

$$\begin{aligned} X_1 &= 2\sqrt{2}k_1\sqrt{(1+2k_1^2)^2+4k_2^4+4k_2^2(1+2k_1^2)\cos 2\alpha_2}, \\ X_2 &= 4k_2^4+1+4k_2^2\cos 2\alpha_2-4k_1^4, \\ X_3 &= 1-4k_1^4-4k_1^2-4k_2^4-4k_2^2\cos 2\alpha_2, \\ X_4 &= 4k_1^2. \end{aligned} \quad (36)$$

5.2. Two Zero Textures. Again, looking at the $\mu\tau$ symmetric form of M_D in (31), we can conclude that, for any even zero textures, $a \neq 0$. Two zeros can be fitted to each of the remaining four pairs of parameters in four ways. The four possible two zero textures of M_D are

$$\begin{aligned} \begin{pmatrix} a & b & b \\ c & 0 & d \\ c & d & 0 \end{pmatrix}, \quad \begin{pmatrix} a & b & b \\ c & d & 0 \\ c & 0 & d \end{pmatrix}, \\ \begin{pmatrix} a & 0 & 0 \\ c & b & d \\ c & d & b \end{pmatrix}, \quad \begin{pmatrix} a & b & b \\ 0 & c & d \\ 0 & d & c \end{pmatrix}, \end{aligned} \quad (37)$$

and are all allowed. So, the number of allowed $\mu\tau$ symmetric two zero textures is the same as that of similar four zero textures. We need only the four parameters a, b, c and d to write down all four textures. The latter lead to three allowed forms of M_ν ; the first two such textures yield one form and the remaining two lead to two forms of M_ν . These are respectively given by

$$M_\nu = - \begin{pmatrix} \frac{a^2}{M_1} + \frac{2b^2}{M_2} & \frac{ac}{M_1} + \frac{bd}{M_2} & \frac{ac}{M_1} + \frac{bd}{M_2} \\ \frac{ac}{M_1} + \frac{bd}{M_2} & \frac{c^2}{M_1} + \frac{d^2}{M_2} & \frac{c^2}{M_1} \\ \frac{ac}{M_1} + \frac{bd}{M_2} & \frac{c^2}{M_1} & \frac{c^2}{M_1} + \frac{d^2}{M_2} \end{pmatrix}, \quad (38)$$

$$M_\nu = - \begin{pmatrix} \frac{a^2}{M_1} & \frac{ac}{M_1} & \frac{ac}{M_1} \\ \frac{ac}{M_1} & \frac{c^2}{M_1} + \frac{d^2}{M_2} + \frac{b^2}{M_2} & \frac{c^2}{M_1} + \frac{2bd}{M_2} \\ \frac{ac}{M_1} & \frac{c^2}{M_1} + \frac{2bd}{M_2} & \frac{c^2}{M_1} + \frac{d^2}{M_2} + \frac{b^2}{M_2} \end{pmatrix}, \quad (39)$$

$$M_\nu = - \begin{pmatrix} \frac{a^2}{M_1} + \frac{2b^2}{M_2} & \frac{bc}{M_2} + \frac{bd}{M_2} & \frac{bc}{M_2} + \frac{bd}{M_2} \\ \frac{bc}{M_2} + \frac{bd}{M_2} & \frac{c^2}{M_2} + \frac{d^2}{M_2} & \frac{2cd}{M_2} \\ \frac{bc}{M_2} + \frac{bd}{M_2} & \frac{2cd}{M_2} & \frac{c^2}{M_2} + \frac{d^2}{M_2} \end{pmatrix}. \quad (40)$$

Compared to four and three zero Yukawa textures, there are more independent parameters here. Apart from the overall mass scale, there will be three moduli and two irremovable phases. It is easier to fit the neutrino data with such a larger number of parameters and we do not discuss it any further.

5.3. One Zero. One zero textures represent the most trivial case among the $\mu\tau$ symmetric neutrino Yukawa texture zeros. This is since, as an odd zero texture, it must have $a = 0$ in (31). The single allowed texture of M_D is

$$\begin{pmatrix} 0 & b & b \\ c & d & e \\ c & e & d \end{pmatrix}, \quad (41)$$

and yields the following form of M_ν :

$$M_\nu = - \begin{pmatrix} \frac{2b^2}{M_2} & \frac{be}{M_2} + \frac{bd}{M_2} & \frac{be}{M_2} + \frac{bd}{M_2} \\ \frac{be}{M_2} + \frac{bd}{M_2} & \frac{c^2}{M_1} + \frac{d^2}{M_2} + \frac{e^2}{M_2} & \frac{c^2}{M_1} + \frac{2ed}{M_2} \\ \frac{be}{M_2} + \frac{bd}{M_2} & \frac{c^2}{M_1} + \frac{2ed}{M_2} & \frac{c^2}{M_1} + \frac{d^2}{M_2} + \frac{e^2}{M_2} \end{pmatrix}. \quad (42)$$

Like the two zero textures, this allowed one zero texture to have six parameters: one overall real mass scale, three moduli, and two phases. These can easily fit the extant neutrino data.

6. The Breaking of $\mu\tau$ Symmetry

As mentioned in previous sections, the results $\theta_{13} = 0$ and $\theta_{23} = \pi/4$ are consequences of the custodial $\mu\tau$ symmetry in M_ν . But present neutrino data from T2K, DOUBLE CHOOZ, RENO, and DAYA BAY experiments rule out $\theta_{13} = 0$ by 5σ . So, the breaking of $\mu\tau$ symmetry is an inevitable need in order to generate a nonzero value of θ_{13} . In addition, a departure from $\theta_{23} = \pi/4$ would arise from the same breaking. Another interesting consequence of a nonzero θ_{13} would be the observability of a CKM-type of CP violation in the lepton sector. Our previous expressions for Δ_{21}^2 , Δ_{32}^2 and $\tan 2\theta_{12}$ will be modified if $\mu\tau$ symmetry is broken.

This symmetry can be broken explicitly or spontaneously or dynamically as with the Renormalization Group evolution of Lagrangian parameters. Spontaneous breakdown generally requires the presence of extra scalars and needs a model with them. We do not go into a discussion of such models here. On the other hand, RG effects on neutrino masses can be incorporated with the methodology presented in [72, 73] in terms of the τ lepton mass arising through the running of the Yukawa coupling strength from the GUT scale Λ to the weak scale λ . The effect is characterized by the parameter Δ_τ which has the 1-loop expression

$$\Delta_\tau \simeq \frac{m_\tau^2}{8\pi^2 v^2} (\tan^2 \beta + 1) \ln \left(\frac{\Lambda}{\lambda} \right), \quad (43)$$

where $\tan \beta$ is the ratio of the VEVs of the up-type and down-type neutral Higgs fields in the MSSM and v^2 is twice the sum

of their squares. Even for a large $\tan \beta$, ($\tan \beta \simeq 60$) Δ_τ is $O(10^{-3})$ and cannot generate a θ_{13} of the order of 9° .

We therefore turn to an explicit breaking of $\mu\tau$ symmetry in M_D . That can be realized in the following way.

Category A. One has

$$M_{DA1} = \begin{pmatrix} a_1 & a_2 & a_2(1 - \epsilon_1 e^{i\theta}) \\ 0 & 0 & b_1(1 - \epsilon_2 e^{i\phi}) \\ 0 & b_1 & 0 \end{pmatrix}, \quad (44)$$

$$M_{DA2} = \begin{pmatrix} a_1 & a_2 & a_2(1 - \epsilon_1 e^{i\theta}) \\ 0 & b_1(1 - \epsilon_2 e^{i\phi}) & 0 \\ 0 & 0 & b_1 \end{pmatrix}.$$

Category B. One has

$$M_{DB1} = \begin{pmatrix} a_1 & 0 & 0 \\ b_1(1 - \epsilon_1 e^{i\theta}) & 0 & b_2(1 - \epsilon_2 e^{i\phi}) \\ b_1 & b_2 & 0 \end{pmatrix}, \quad (45)$$

$$M_{DB2} = \begin{pmatrix} a_1 & 0 & 0 \\ b_1(1 - \epsilon_1 e^{i\theta}) & b_2(1 - \epsilon_2 e^{i\phi}) & 0 \\ b_1 & 0 & b_2 \end{pmatrix}.$$

Furthermore,

$$M_R = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_2(1 - \epsilon_3) \end{pmatrix}, \quad (46)$$

where $\epsilon_1 e^{i\theta}$, $\epsilon_2 e^{i\phi}$ are complex $\mu\tau$ symmetry breaking parameters ($\epsilon_{1,2}$ real) and ϵ_3 is a real $\mu\tau$ symmetry breaking parameter in M_R . For these modified M_D and M_R , we have the following M_ν 's.

Category A. One has

$$M_{\nu A1} = m_A \left[\begin{pmatrix} k_1^2 e^{2i\alpha} + 2k_2^2 & k_2 & k_2 \\ k_2 & 1 & 0 \\ k_2 & 0 & 1 \end{pmatrix} - \epsilon_1 e^{i\theta} \begin{pmatrix} 2k_2^2 & k_2 & 0 \\ k_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \epsilon_2 e^{i\phi} \begin{pmatrix} 0 & k_2 & 0 \\ k_2 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \epsilon_3 \begin{pmatrix} -k_2^2 & -k_2 & 0 \\ -k_2 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right],$$

$$M_{\nu A2} = m_A \left[\begin{pmatrix} k_1^2 e^{2i\alpha} + 2k_2^2 & k_2 & k_2 \\ k_2 & 1 & 0 \\ k_2 & 0 & 1 \end{pmatrix} - \epsilon_1 e^{i\theta} \begin{pmatrix} 2k_2^2 & 0 & k_2 \\ 0 & 0 & 0 \\ k_2 & 0 & 0 \end{pmatrix} - \epsilon_2 e^{i\phi} \begin{pmatrix} 0 & k_2 & 0 \\ k_2 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \epsilon_3 \begin{pmatrix} -k_2^2 & 0 & -k_2 \\ 0 & 0 & 0 \\ -k_2 & 0 & -1 \end{pmatrix} \right]. \quad (47)$$

Category B. One has

$$M_{\nu B1} = m_B \left[\begin{pmatrix} l_1^2 & l_1 l_2 e^{i\beta} & l_1 l_2 e^{i\beta} \\ l_1 l_2 e^{i\beta} & l_2^2 e^{2i\beta} + 1 & l_2^2 e^{2i\beta} \\ l_1 l_2 e^{i\beta} & l_2^2 e^{2i\beta} & l_2^2 e^{2i\beta} + 1 \end{pmatrix} - \epsilon_1 e^{i\theta} \begin{pmatrix} 0 & l_1 l_2 e^{i\beta} & 0 \\ l_1 l_2 e^{i\beta} & 2l_2^2 e^{2i\beta} & l_2^2 e^{2i\beta} \\ 0 & l_2^2 e^{2i\beta} & 0 \end{pmatrix} - \epsilon_2 e^{i\phi} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \epsilon_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right],$$

$$M_{\nu B2} = m_B \left[\begin{pmatrix} l_1^2 & l_1 l_2 e^{i\beta} & l_1 l_2 e^{i\beta} \\ l_1 l_2 e^{i\beta} & l_2^2 e^{2i\beta} + 1 & l_2^2 e^{2i\beta} \\ l_1 l_2 e^{i\beta} & l_2^2 e^{2i\beta} & l_2^2 e^{2i\beta} + 1 \end{pmatrix} - \epsilon_1 e^{i\theta} \begin{pmatrix} 0 & l_1 l_2 e^{i\beta} & 0 \\ l_1 l_2 e^{i\beta} & 2l_2^2 e^{2i\beta} & l_2^2 e^{2i\beta} \\ 0 & l_2^2 e^{2i\beta} & 0 \end{pmatrix} - \epsilon_2 e^{i\phi} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \epsilon_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right]. \quad (48)$$

The detailed diagonalization and expressions for mass differences and mixing angles are given in the appendix.

A nonzero θ_{13} arises after $\mu\tau$ symmetry breaking. The value $\theta_{13} \simeq 9^\circ$ is possible for 3σ variations of $(\Delta_{21}^2)^{\epsilon_{1,2,3}}$, $(\Delta_{32}^2)^{\epsilon_{1,2,3}}$, $\theta_{12}^{\epsilon_{1,2,3}}$, $\theta_{23}^{\epsilon_{1,2,3}}$, compare the appendix. An appropriate choice of symmetry breaking parameters, that is, $\epsilon_{1,2,3} \simeq 0.1$, and slightly shifted parameter spaces for k_1, k_2, α in Category A and l_1, l_2, β in Category B are needed. A nonzero CP violating effect can be effected through the Jarlskog invariant J_{CP} :

$$J_{CP} = \text{Im} \frac{H_{12} H_{23} H_{31}}{(\Delta_{21}^2)^{\epsilon_{1,2,3}} (\Delta_{32}^2)^{\epsilon_{1,2,3}} (\Delta_{31}^2)^{\epsilon_{1,2,3}}}, \quad (49)$$

where elements of H and mass squared differences $(\Delta_{21}^2)^{\epsilon_{1,2,3}}$, $(\Delta_{32}^2)^{\epsilon_{1,2,3}}$ are given in the appendix. Note that $(\Delta_{31}^2)^{\epsilon_{1,2,3}} = (\Delta_{21}^2)^{\epsilon_{1,2,3}} + (\Delta_{32}^2)^{\epsilon_{1,2,3}}$. A detailed treatment of explicitly broken $\mu\tau$ symmetric four zero and three zero textures is given in [33].

TABLE 5: Expressions for $P, Q, R, S, x_{1-6}, y_{1-6}$ and z_{1-6} for Categories A1, A2, B1 and B2 from (47) and (48).

Quantity	Category A1	Category A2	Category B1	Category B2
P	$k_1^2 e^{2i\alpha} + 2k_2^2$	$k_1^2 e^{2i\alpha} + 2k_2^2$	l_1^2	l_1^2
Q	k_2	k_2	$l_1 l_2 e^{i\beta}$	$l_1 l_2 e^{i\beta}$
R	1	1	$l_2^2 e^{2i\beta} + 1$	$l_2^2 e^{2i\beta} + 1$
S	0	0	$l_2^2 e^{2i\beta}$	$l_2^2 e^{2i\beta}$
x_1	$2k_2^2$	$2k_2^2$	0	0
x_2	k_2	0	$l_1 l_2 e^{i\beta}$	$l_1 l_2 e^{i\beta}$
x_3	0	k_2	0	0
x_4	0	0	$2l_2^2 e^{2i\beta}$	$2l_2^2 e^{2i\beta}$
x_5	0	0	$l_2^2 e^{2i\beta}$	$l_2^2 e^{2i\beta}$
x_6	0	0	0	0
y_1	0	0	0	0
y_2	k_2	k_2	0	0
y_3	0	0	0	0
y_4	2	2	2	2
y_5	0	0	0	0
y_6	0	0	0	0
z_1	$-k_2^2$	$-k_2^2$	0	0
z_2	$-k_2$	0	0	0
z_3	0	$-k_2$	0	0
z_4	-1	0	-1	0
z_5	0	0	0	0
z_6	0	-1	0	-1

7. Concluding Summary

We have reviewed neutrino Yukawa textures with zeros within the type-I seesaw with three heavy right chiral neutrinos and in the basis where the latter and the charged leptons are mass diagonal. The conditions of a nonvanishing mass of every ultralight neutrino and of the non-decoupling of any neutrino generation allow a maximum of four zeros in the neutrino Yukawa coupling matrix Y_ν . There are seventy two such textures. We show that the requirement of an exact $\mu\tau$ symmetry, coupled with observational constraints, reduces the *seventy two* allowed textures in such a Y_ν to *only four* corresponding to *just two* different forms of the light neutrino mass matrix $M_{\nu A}/M_{\nu B}$, resulting in an inverted/normal mass ordering. Apart from an overall mass scale, M_ν for every category has two real parameters and an irremovable phase. These parameters k_1, k_2 , and $|\alpha|$ for Category A and l_1, l_2 , and $|\beta|$ for Category B get highly restricted, given the 3σ ranges of measured neutrino mass squared differences and mixing angles. Neutrino masses and Majorana phases are predicted within definite ranges with 3σ laboratory and cosmological inputs. The predicted respective masses m_1/eV , m_2/eV , and m_3/eV are $[0.0452, 0.1682]$, $[0.0457, 0.1684]$, $[0.077, 0.1632]$ for Category A and $[0.0110, 0.0335]$, $[0.0144, 0.0345]$, and $[0.0485, 0.0638]$ for Category B. The corresponding intervals of the Majorana phases are $-98.0^\circ \leq \alpha_{M_1} \leq 20.0^\circ$, $9.2^\circ \leq \alpha_{M_2} \leq 36.4^\circ$ for Category A and $-88.6^\circ \leq \alpha_{M_1} \leq 7.97^\circ$, $90.7^\circ \leq \alpha_{M_2} \leq 128.8^\circ$ for Category B. In addition, we predict the range of the mass scale associated with $0\nu\beta\beta$ decay, most of which

is well below the reach of planned experiments. We have also studied Yukawa textures with a fewer number of zeros, but with exact $\mu\tau$ symmetry. Finally, we have formulated the detailed scheme of three-parameter explicit breaking of $\mu\tau$ symmetry for allowed four zero textures. A value of $\theta_{13} \approx 9^\circ$ can be arranged for a suitable choice of small values of these symmetry breaking parameters.

Appendix

Expressions for Measurable Quantities

We can write the general form of a broken $\mu\tau$ symmetric M_ν in the following way [33]:

$$M_\nu = m \left[\begin{pmatrix} P & Q & Q \\ Q & R & S \\ Q & S & R \end{pmatrix} - \epsilon_1 e^{i\theta} \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{pmatrix} - \epsilon_2 e^{i\phi} \begin{pmatrix} y_1 & y_2 & y_3 \\ y_2 & y_4 & y_5 \\ y_3 & y_5 & y_6 \end{pmatrix} - \epsilon_3 \begin{pmatrix} z_1 & z_2 & z_3 \\ z_2 & z_4 & z_5 \\ z_3 & z_5 & z_6 \end{pmatrix} \right]. \quad (\text{A.1})$$

The explicit expressions of $P, Q, R, S, x_{1-6}, y_{1-6}$ and z_{1-6} for four forms of neutrino mass matrices after $\mu\tau$ symmetry breaking are given in Table 5.

We can now have

$$\begin{aligned}
 H &= M_\nu M_\nu^\dagger \\
 &= m^2 \left[\begin{pmatrix} |P|^2 + 2|Q|^2 & PQ^* + Q(R^* + S^*) & PQ^* + Q(R^* + S^*) \\ P^*Q + Q^*(R + S) & |Q|^2 + |R|^2 + |S|^2 & |Q|^2 + RS^* + R^*S \\ P^*Q + Q^*(R + S) & |Q|^2 + R^*S + RS^* & |Q|^2 + |R|^2 + |S|^2 \end{pmatrix} \right. \\
 &\quad \left. - \epsilon_1 \begin{pmatrix} u_1 & u_2^* & u_3^* \\ u_2 & u_4 & u_5^* \\ u_3 & u_5 & u_6 \end{pmatrix} - \epsilon_2 \begin{pmatrix} v_1 & v_2^* & v_3^* \\ v_2 & v_4 & v_5^* \\ v_3 & v_5 & v_6 \end{pmatrix} - \epsilon_3 \begin{pmatrix} w_1 & w_2^* & w_3^* \\ w_2 & w_4 & w_5^* \\ w_3 & w_5 & w_6 \end{pmatrix} \right]. \quad (A.2)
 \end{aligned}$$

Here

$$\begin{aligned}
 u_1 &= [P^* x_1 + Q^* (x_2 + x_3)] e^{i\theta} + [Px_1^* + Q(x_2^* + x_3^*)] e^{-i\theta}, \\
 u_2 &= [P^* x_2 + Q^* (x_4 + x_5)] e^{i\theta} + [Qx_1^* + Rx_2^* + Sx_3^*] e^{-i\theta}, \\
 u_3 &= [P^* x_3 + Q^* (x_5 + x_6)] e^{i\theta} + [Qx_1^* + Sx_2^* + Rx_3^*] e^{-i\theta}, \\
 u_4 &= [Q^* x_2 + R^* x_4 + S^* x_5] e^{i\theta} + [Qx_2^* + Rx_4^* + Sx_5^*] e^{-i\theta}, \\
 u_5 &= [Q^* x_3 + R^* x_5 + S^* x_6] e^{i\theta} + [Qx_2^* + Sx_4^* + Rx_5^*] e^{-i\theta}, \\
 u_6 &= [Q^* x_3 + S^* x_5 + R^* x_6] e^{i\theta} + [Qx_3^* + Sx_5^* + Rx_6^*] e^{-i\theta}. \quad (A.3)
 \end{aligned}$$

Note that v_i and w_i have similar functional forms as u_i . If we write u_i in the following way

$$u_i = f_i(x_1, x_2, \dots, x_6, \theta), \quad (A.4)$$

then v_i and w_i will be

$$\begin{aligned}
 v_i &= f_i(y_1, y_2, \dots, y_6, \phi), \\
 w_i &= f_i(z_1, z_2, \dots, z_6, 0). \quad (A.5)
 \end{aligned}$$

The diagonalization of H yields $\text{diag} \cdot (m_1^2, m_2^2, m_3^2)$ and also expressions for five relevant measurable quantities. The latter are: Δ_{21}^2 , Δ_{32}^2 , θ_{12} , θ_{23} and θ_{13} . We will associate

superscripts $\epsilon_{1,2,3}$ with all of these five quantities to distinguish them from their unperturbed expressions. The relevant functions for these physical quantities are given below

$$\begin{aligned}
 U_1 &= \frac{1}{2} \left[-2c_{12}^2 u_1 + \sqrt{2}c_{12}s_{12} \{ (u_2 + u_3) e^{-i\psi} + (u_2^* + u_3^*) e^{i\psi} \} \right. \\
 &\quad \left. - s_{12}^2 (u_4 + u_6 + u_5 + u_5^*) \right], \\
 U_2 &= \frac{1}{2} \left[-\sqrt{2}c_{12}^2 (u_2 + u_3) e^{-i\psi} + \sqrt{2}s_{12}^2 (u_2^* + u_3^*) e^{i\psi} \right. \\
 &\quad \left. + c_{12}s_{12} (u_4 + u_6 - 2u_1 + u_5 + u_5^*) \right], \\
 U_3 &= \frac{1}{2} \left[\sqrt{2}c_{12} (u_2 - u_3) e^{-i\psi} + s_{12} (u_6 - u_4 + u_5 - u_5^*) \right], \\
 U_4 &= \frac{1}{2} \left[-2s_{12}^2 u_1 - \sqrt{2}c_{12}s_{12} \{ (u_2 + u_3) e^{-i\psi} + (u_2^* + u_3^*) e^{i\psi} \} \right. \\
 &\quad \left. - c_{12}^2 (u_4 + u_6 + u_5 + u_5^*) \right], \\
 U_5 &= \frac{1}{2} \left[\sqrt{2}s_{12} (u_2 - u_3) e^{-i\psi} - c_{12} (u_6 - u_4 + u_5 - u_5^*) \right], \\
 U_6 &= \frac{1}{2} [u_5 + u_5^* - u_4 - u_6], \quad (A.6)
 \end{aligned}$$

where $\psi = \arg[P^*Q + Q^*(R + S)]$, $c_{12} = \cos \theta_{12}$, $s_{12} = \sin \theta_{12}$, θ_{12} being the unperturbed mixing angle in (26). There are also V_i and W_i , $i = 1-6$ which have similar functional forms as U_i . If we write U_i as

$$U_i = F_i(u_1, u_2, \dots, u_6), \quad (A.7)$$

then V_i and W_i will be

$$\begin{aligned}
 V_i &= F_i(v_1, v_2, \dots, v_6), \\
 W_i &= F_i(w_1, w_2, \dots, w_6). \quad (A.8)
 \end{aligned}$$

The final results with three $\mu\tau$ symmetry breaking parameters are

$$\begin{aligned}
 (\Delta_{21}^2)^{\epsilon_{1,2,3}} &= \Delta_{21}^2 + m^2 \{ (U_4 - U_1) \epsilon_1 + (V_4 - V_1) \epsilon_2 + (W_4 - W_1) \epsilon_3 \}, \\
 (\Delta_{32}^2)^{\epsilon_{1,2,3}} &= \Delta_{32}^2 + m^2 \{ (U_6 - U_4) \epsilon_1 + (V_6 - V_4) \epsilon_2 + (W_6 - W_4) \epsilon_3 \},
 \end{aligned}$$

$$\begin{aligned}
(\sin \theta_{12})^{\epsilon_{1,2,3}} &= \left| s_{12} + c_{12} m^2 \left\{ \frac{U_2^* \epsilon_1 + V_2^* \epsilon_2 + W_2^* \epsilon_3}{\Delta_{21}^2} \right\} \right|, \\
(\sin \theta_{23})^{\epsilon_{1,2,3}} &= \left| \frac{1}{\sqrt{2}} + \frac{s_{12} m^2}{\sqrt{2}} \left\{ \frac{U_3^* \epsilon_1 + V_3^* \epsilon_2 + W_3^* \epsilon_3}{\Delta_{21}^2 + \Delta_{32}^2} \right\} \right. \\
&\quad \left. - \frac{c_{12} m^2}{\sqrt{2}} \left\{ \frac{U_5^* \epsilon_1 + V_5^* \epsilon_2 + W_5^* \epsilon_3}{\Delta_{32}^2} \right\} \right|, \\
(\sin \theta_{13})^{\epsilon_{1,2,3}} &= \left| c_{12} m^2 \left\{ \frac{U_3^* \epsilon_1 + V_3^* \epsilon_2 + W_3^* \epsilon_3}{\Delta_{21}^2 + \Delta_{32}^2} \right\} \right. \\
&\quad \left. + s_{12} m^2 \left\{ \frac{U_5^* \epsilon_1 + V_5^* \epsilon_2 + W_5^* \epsilon_3}{\Delta_{32}^2} \right\} \right|.
\end{aligned} \tag{A.9}$$

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