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Research Article

Relativistic Bound States of Spinless Particle by the Cornell Potential Model in External Fields

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The two-dimensional (2D) relativistic bound states of a spinless particle placed in scalar $S(r)$ and vector $V(r)$ Cornell potentials (with $S(r) > V(r)$) are obtained under the influence of external magnetic and Aharonov-Bohm (AB) flux fields using the wave function ansatz method. The relativistic energy eigenvalues and wave functions are found for any arbitrary state with principal n and magnetic m quantum numbers. Further, we obtain the eigensolutions in any dimensional space D without external fields. We also find the relativistic and nonrelativistic bound states for Coulomb, harmonic oscillator, and Kratzer potentials.

1. Introduction

Relativistic wave equations such as Dirac and Klein-Gordon equations have been of much concern for theoretical physicists [1, 2]. The KG is almost employed in describing the spin-zero particle dynamics in relativistic quantum mechanics. The existence of strong fields or high speeds introduces relativistic phenomena that cannot be described using the Schrödinger equation. A discussion of the relativistic behavior of spin-zero particles needs understanding the single particle spectrum and consequently the exact solutions to the KG which are constructed using the four-vector potential A_ν ($\nu = 0, 1, 2, 3$) and the scalar potential $S(r)$. It can be said that the motion of a relativistic spin-zero particle in a potentials is described by the KG equation with the potential $S(r)$ and $V(r)$ [3].

Over the past few years, there has been an increasing interest in obtaining the exact solutions to the spinless KG particles exposed to different potential models [4–6]. Various techniques are used in solving such wave equations as the Nikiforov-Uvarov (NU) method [7, 8], the supersymmetric approach [9–11], the point canonical method [12, 13], the asymptotic iteration method [14, 15], the exact quantization rule [16, 17], the shifted $1/N$ expansion technique [18], parametrized version of NU method [19], perturbative treatment [20], and wave function ansatz method [21, 22].

One of the potentials that has received much attention in particle physics was the Cornell (Coulomb plus linear) potential. It has been used successfully in models describing binding states of heavy quarks [23–25]. The Cornell potential contains a short range part dominated by a Coulombic-term of quark and gluon interaction $\sim \alpha_s/r$ from perturbative quark chromodynamics (QCD) and the large distance quark confinement known from lattice QCD as a linear term $\sim r$ [26–31]. In some situations when the parameter b is small, it provides a particular case of perturbed Coulomb problem in atomic physics [32]. This potential has been used to study the strange, charmed, and beautiful baryon masses in the framework of variational approach [33].

The ground state energy ($n = 0$) of the KG equation with scalar-vector Killingbeck potentials [34] has been found using the ansatz method [35–37]. Furthermore, the ground state energy of the KG equation with scalar-vector Cornell potentials [38] has been calculated under the influence of the magnetic and Aharonov-Bohm fields.

Very recently, we have studied the scalar charged particle in scalar-vector (harmonic oscillator plus Cornell) potentials with and without external magnetic and Aharonov-Bohm flux fields [39] and obtained its relativistic energy eigenvalues and wave functions using the analytical exact iteration method [40, 41]. The behavior of a spinless relativistic particle moving under the Cornell potential with the Coulomb

singularity in the presence of static magnetic and AB flux fields has not been investigated yet for any principal n and magnetic m quantum numbers.

The aim of this work is to extend [38] to include the bound state solutions for any n and m quantum numbers. Thus we solve the radial KG equation in 2D space for unequal mixture of scalar-vector Cornell potentials with and without constant magnetic and AB flux fields for the first time. For the real relativistic bound state solutions, we should choose $S(r) \geq V(r)$ and $m_e < E$, where m_e is the mass of particle (electron) and E is its energy while the nonrelativistic solution can be obtained when $S(r) = V(r)$. We choose not to pursue the case $S(r) = -V(r)$ since its nonrelativistic limit is the trivial interaction-free mode. This does not diminish the importance of such problems. It only limits its contribution to the relativistic regime. This includes the general case $V(r) = \beta S(r)$, where $|\beta| \leq 1$, where β is a real number. On the other hand, the class of problems $S(r) = \beta V(r)$ with $|\beta| < 1$ are unbounded as we will see from our solution. Moreover, we present the exact energy eigenvalues and wave functions of the pure scalar Cornell potential in presence of magnetic field with various values of Larmor frequencies for any vibrational n and rotational m quantum numbers using the wave function ansatz and series method [40–44]. In addition, some special cases of KG solutions are obtained for the Coulomb and harmonic oscillator interactions in 3D space without the external fields.

The structure of this paper is as follows. In Section 2, we introduce the KG equation in 2D space for unequal mixture of scalar-vector Cornell potential under external magnetic and AB flux fields. We obtain the energy eigenvalues and wave functions for any n and m state. Section 3 is devoted for discussing some particular cases such as KG-scalar Cornell problem with and without external fields, KG-Coulomb, and KG-harmonic oscillator problems without external fields in any dimensional space. Our concluding remarks are given in Section 3.

2. 2D Relativistic Bound State Solutions in External Fields

The Klein-Gordon atom for the spinless particle with mass m_e and charge $-e$ moving in external electromagnetic field and AB flux field given by potentials $V(r)$, $S(r)$, and $\vec{A}$ reads [45, 46]

$$\left[c^2\left(\vec{p}+\frac{e}{c}\vec{A}\right)^2-(E-V(r))^2 +\left(m_ec^2+S(r)\right)^2\right]\psi(r,\phi)=0. \tag{1}$$

We choose [47–52]

$$\vec{A}=\frac{1}{2}\vec{B}\times\vec{r}+\frac{\Phi_{\mathrm{AB}}}{2\pi r}\hat{\phi}, \tag{2}$$

and the scalar-vector potentials in the form of the Cornell (Coulomb plus linear) potentials describing quark-antiquark interactions [23, 24, 28–31] are as follows:

$$V(r)=-\frac{a_v}{r}+b_vr,\qquad S(r)=-\frac{a_s}{r}+b_sr, \tag{3}$$

which are extensively used in particle physics [30, 31]. Moreover, the vector potential in the symmetric gauge is defined by $\vec{A}=\vec{A}_1+\vec{A}_2$ such that $\vec{\nabla}\times\vec{A}_1=\vec{B}$ and $\vec{\nabla}\times\vec{A}_2=0$, where the applied magnetic field $\vec{B}=(0,0,B)$ is perpendicular to the plane of transversal motion of the particle and $\vec{A}_2$ describes the additional AB flux field Φ_{AB} created by a solenoid in cylindrical coordinates [50–52]. The wave function in (1) is defined by

$$\psi(r,\phi)=\frac{1}{\sqrt{2\pi}}e^{im\phi}\frac{R(r)}{\sqrt{r}},\quad m=0,\pm1,\pm2,\ldots, \tag{4}$$

where m is the eigenvalue of angular momentum. Now in the solution of the KG equation the relationship between the attractive scalar and repulsive vector potentials is given by $V(r)=\beta S(r)$, where $|\beta|\leq 1$ is arbitrary constant that refers to the existence of energy eigenvalues and hence the KG equation could be reduced to the Schrödinger-type second order differential equation as

$$\left[c^2\left(\vec{p}+\frac{e}{c}\vec{A}\right)^2+2\left(EV(r)+m_ec^2S(r)\right) +S^2(r)-V^2(r)+m_e^2c^4-E^2\right]\psi(r,\phi)=0. \tag{5}$$

In case when $S(r)=V(r)$, the above equation reduces to the Schrödinger equation. Now we will treat the energy solutions of the two cases in (5) as follows.

2.1. The Energy Solution. The energy states require that $S(r)=V(r)$ (i.e., $\beta=1$ case) which, in the nonrelativistic limit, corresponds to the solution of the wave equation:

$$\left\{\frac{1}{2\mu}\left[-i\hbar\vec{\nabla}+\frac{e}{c}\left(\frac{Br}{2}+\frac{\Phi_{\mathrm{AB}}}{2\pi r}\right)\hat{\phi}\right]^2+2V_K(r)-E\right\} \times\psi(r,\phi)=0, \tag{6}$$

where $\psi(r,\phi)$ stands for nonrelativistic wave function. Thus, the choice of the case $S(r)=V(r)$, $\beta=1$ is not mostly to do with finding exact analytical solution but simply to enable one to obtain a non-relativistic limit with a potential function $2V(r)$ and not $V(r)$. Accordingly, it would be natural to scale the potential term in (6) so that in the non-relativistic limit the interaction potential becomes $V(r)$ not $2V(r)$. With the scalar $S(r)$ and vector $V(r)$ Cornell potentials: given in (3), we can rewrite (5) as

$$\left\{c^2\left[-i\hbar\vec{\nabla}+\frac{e}{c}\left(\frac{Br}{2}+\frac{\Phi_{\mathrm{AB}}}{2\pi r}\right)\hat{\phi}\right]^2 -\frac{w_1}{r^2}-\frac{w_2}{r}+w_3+w_4r+w_5r^2\right\}\psi(r,\phi)=0, \tag{7}$$

with

$$w_1 = a_v^2 - a_s^2, \qquad w_2 = 2\left(Ea_v + m_e c^2 a_s\right),$$
$$w_3 = m_e^2 c^4 - E^2 - 2\left(a_s b_s - a_v b_v\right), \tag{8}$$
$$w_4 = 2\left(Eb_v + m_e c^2 b_s\right), \qquad w_5 = b_s^2 - b_v^2,$$
$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}. \tag{9}$$

By inserting (4) and (9) into (7), we can obtain [40, 41, 53]

$$\frac{d^2R(r)}{dr^2} + \left(\frac{C_1}{r^2} + \frac{C_2}{r} - C_3 - C_4 r - C_5 r^2\right)R(r) = 0, \tag{10}$$

with

$$C_1 = -m'^2 + \frac{1}{4} + \frac{a_v^2 - a_s^2}{\hbar^2 c^2}, \qquad C_2 = \frac{w_2}{\hbar^2 c^2},$$
$$C_3 = \frac{2m_e\omega_L m'}{\hbar} + \frac{w_3}{\hbar^2 c^2}, \tag{11a}$$
$$C_4 = \frac{w_4}{\hbar^2 c^2}, \qquad C_5 = \frac{w_5}{\hbar^2 c^2} + \left(\frac{m_e\omega_L}{\hbar}\right)^2, \tag{11b}$$

and the asymptotic behaviors $R(0) = 0$ and $R(\infty) \to 0$. Notice the energy depending on the strength of the magnetic field characterized by the Larmor frequency, ω_L,

$$\omega_L = \frac{\Omega}{2}, \quad \Omega = \frac{|e|B}{m_e c}, \tag{12}$$

and the new eigenvalue of angular momentum (magnetic quantum number), m',

$$m' = m + \xi, \quad \xi = \frac{\Phi_{AB}}{\Phi_0}, \; m' = 1, 2, \ldots, \tag{13}$$

where $\Phi_0 = hc/e$ refers to the flux quantum and ξ is an integer. The frequency Ω is called the cyclotron frequency of rotation [54] corresponding to the classical motion of a charged particle in a uniform magnetic field and $\Omega/2$ is the Larmor frequency in units of Hz [54].

Hence, the Schrödinger-type equation with unequal scalar-vector Cornell potential can be written as

$$\frac{d^2R(r)}{dr^2} + \frac{\left(E + m_e c^2\right)}{\hbar^2 c^2} \times \left[E - m_e c^2 - U_{\text{eff}}(r, \omega_L, \xi)\right]R(r) = 0, \tag{14}$$

with the effective potential

$$\begin{aligned} U_{\text{eff}}(r, \omega_L, \xi) &= \frac{1}{(E + m_e c^2)}\Bigg[-\frac{2\left(Ea_v + m_e c^2 a_s\right)}{r} \\ &\quad + 2\left(Eb_v + m_e c^2 b_s\right)r \\ &\quad + \left(m_e^2 c^2 \omega_L^2 + b_s^2 - b_v^2\right)r^2 \\ &\quad + \frac{\hbar^2 c^2\left(m'^2 - 1/4\right) + a_s^2 - a_v^2}{r^2} \\ &\quad + 2\hbar\omega_L m_e c^2 m'\Bigg]. \end{aligned} \tag{15}$$

It is interesting to look at the results obtained from (10) for a special case $C_2 = C_4 = 0$; that is, we have KG-pseudoharmonic problem:

$$\frac{d^2R(r)}{dr^2} + \left(\frac{C_1}{r^2} - C_5 r^2\right)R(r) = C_3 R(r), \tag{16}$$

which corresponds to the differential equation of a harmonic oscillator with a centrifugal term. It is similar form to the one given by (7) in [55] with the equivalence $C_1 \to -l'(l'+1)$, $C_5 \to \alpha^2$, and $C_3 \to -\lambda_n$ (α, l, and λ are the parameters used in [55]). So by putting the parameter values,

$$\begin{aligned} l' &= \pm\sqrt{m'^2 + \frac{a_s^2 - a_v^2}{\hbar^2 c^2}} - \frac{1}{2}, \\ \alpha &= \frac{1}{\hbar}\sqrt{\frac{b_s^2 - b_v^2}{c^2} + \left(m_e\omega_L\right)^2}, \\ \lambda_n &= \frac{1}{\hbar^2 c^2}\left[m_e^2 c^4 - E^2 + d_s^2 - d_v^2 \right. \\ &\quad \left. + 2\left(a_v b_v - a_s b_s + Ed_v + m_e c^2 d_s\right)\right] \\ &\quad - \frac{2m_e\omega_L m'}{\hbar}, \end{aligned} \tag{17}$$

into (14) of [54], we obtain the energy equation as

$$\begin{aligned} E^2 - m_e^2 c^4 + 2\left(a_s b_s - a_v b_v\right) &= 2m_e c^2 \hbar\omega_L m' \\ &\quad + 2\hbar c^2\left(2n + 1 + \sqrt{m'^2 + \frac{a_s^2 - a_v^2}{\hbar^2 c^2}}\right) \\ &\quad \times \sqrt{\frac{b_s^2 - b_v^2}{c^2} + \left(m_e\omega_L\right)^2}, \quad a_s > a_v, \; b_s > b_v. \end{aligned} \tag{18}$$

Now we find a solution to (10) by making the following choice of the wave function [40, 41]:

$$R_{nm}(r) = \exp\left(\frac{1}{2}pr^2 + qr\right)\sum_{n=0}^{\infty} a_n r^{n+\delta}, \tag{19}$$

where p and q are parameters whose values are to be determined in terms of the potential parameters. Substituting (19) into (10), we obtain the following series [40]:

$$\sum_{n=0}^{\infty} a_n S_n r^{n+\delta-2} + \sum_{n=1}^{\infty} a_{n-1} T_{n-1} r^{n+\delta-2} + \sum_{n=2}^{\infty} a_{n-2} W_{n-2} r^{n+\delta-2} = 0, \tag{20}$$

with

$$\begin{aligned} S_n &= (n+\delta)(n+\delta-1) + C_1, \\ T_{n-1} &= 2q(n+\delta-1) + C_2, \\ W_{n-2} &= q^2 + 2p\left(n+\delta-\frac{3}{2}\right) - C_3, \end{aligned} \tag{21}$$

$$\begin{aligned} p &= \pm\sqrt{\frac{1}{\hbar^2 c^2}\left(b_s^2 - b_v^2\right) + \left(\frac{m_e \omega_L}{\hbar}\right)^2}, \\ pq &= \frac{1}{\hbar^2 c^2}\left(E b_v + m_e c^2 b_s\right), \quad b_s > b_v, \end{aligned} \tag{22}$$

where the two parameters p and q are determined in order that the radial wave functions $R_{nm}(r)$ must be finite everywhere and vanish at $r = 0$ and as $r \to \infty$. In the absence of the magnetic field we require that $a_s > a_v$.

To obtain the recurrence relation which can connect various expansion coefficients a_n, we make identical powers of r in (20), that is, equate the coefficients of $r^{n+\delta-2}$ to zero. Thus, the relations become

$$\begin{aligned} &a_0\left[\delta(\delta-1) + C_1\right] = 0 \Longrightarrow \delta^2 + C_1 = \delta, \quad a_0 \neq 0, \\ &\delta = \pm\sqrt{m'^2 + \frac{1}{\hbar^2 c^2}\left(a_s^2 - a_v^2\right)} + \frac{1}{2}, \quad \delta \neq 0 \Longrightarrow a_s > a_v \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } m' = 0, \\ &a_1 = -\frac{(2q\delta + C_2)}{2\delta} a_0, \quad \delta \neq 0 \\ &a_2 = -\frac{\left[p(2\delta+1) + q^2 - C_3\right] a_0 + \left[2q(\delta+1) + C_2\right] a_1}{2(2\delta+1)}, \\ &a_3 = -\frac{(2pq - C_4) a_0 + \left[p(2\delta+3) + q^2 - C_3\right] a_1}{6(\delta+1)} \\ &\qquad + \frac{\left[2q(\delta+2) + C_2\right] a_2}{6(\delta+1)}, \\ &\vdots \end{aligned}$$

$$\begin{aligned} a_n = &-\frac{\left(p^2 - C_5\right) a_{n-4} + (2pq - C_4) a_{n-3}}{n(2\delta+n-1)} \\ &+ \frac{\left[p(2\delta+2n-3) + q^2 - C_3\right] a_{n-2}}{n(2\delta+n-1)} \\ &+ \frac{\left[2q(\delta+n-1) + C_2\right] a_{n-1}}{n(2\delta+n-1)}, \end{aligned} \tag{23}$$

where $n = 0, 1, 2, \ldots$ and $a_0 \neq 0$. Here, the positive sign of parameter

$$\delta = \sqrt{m'^2 + \frac{1}{\hbar^2 c^2}\left(a_s^2 - a_v^2\right)} + \frac{1}{2} \tag{24}$$

has been selected. The power series for large values of δ or m' is convergent. For convenience, we take the ratio of two successive terms, that is, a_{n+1}/a_n, which becomes

$$\begin{aligned} \frac{a_1}{a_0} &= -\left(q + \frac{C_2}{2\delta}\right) \longrightarrow -q \quad \text{when } \delta \longrightarrow \infty, \\ \frac{a_2}{a_0} &= \frac{q^2\left(\delta + (C_2/2q)\right)\left[\delta + (C_2/2q) + 1\right]}{\delta(2\delta+1)} \\ &\quad - \frac{\left[p(2\delta+1) + q^2 - C_3\right]}{2(2\delta+1)} \\ &\longrightarrow \frac{q^2 - p}{2} \quad \text{when } \delta \longrightarrow \infty. \end{aligned} \tag{25}$$

It is apparent from the above relations that the power series converges when $\delta \to \infty$. Hence, the series must be truncated (bounded) for $n = n_{\max}$. At this value of n, we obtain the following equations:

$$p = -\sqrt{\frac{1}{\hbar^2 c^2}\left(b_s^2 - b_v^2\right) + \left(\frac{m_e \omega_L}{\hbar}\right)^2}, \quad p \neq 0, \tag{26}$$

where the negative sign of the coefficient p has been chosen in (22). Also the parameter q in (22) is physically taken as

$$q = -\frac{\left(E b_v + m_e c^2 b_s\right)}{\hbar^2 c^2 \sqrt{\left(1/\hbar^2 c^2\right)\left(b_s^2 - b_v^2\right) + \left(m_e \omega_L/\hbar\right)^2}}, \tag{27}$$

and the following relationship between the potential parameters must be achieved:

$$\begin{aligned} &\frac{E b_v + m_e c^2 b_s}{\sqrt{\left(1/\hbar^2 c^2\right)\left(b_s^2 - b_v^2\right) + \left(m_e \omega_L/\hbar\right)^2}} \\ &\quad = \frac{E a_v + m_e c^2 a_s}{\left(n + \sqrt{m'^2 + \left(1/\hbar^2 c^2\right)\left(a_s^2 - a_v^2\right)} - (1/2)\right)}. \end{aligned} \tag{28}$$

The energy equation can be obtained via (23) as

$$\begin{aligned} E^2 - m_e^2c^4 + 2\left(a_sb_s - a_vb_v\right) &= 2m_ec^2\hbar\omega_L m' \\ &+ 2\hbar^2c^2\sqrt{\frac{\left(b_s^2 - b_v^2\right)}{\hbar^2c^2} + \left(\frac{m_e\omega_L}{\hbar}\right)^2} \\ &\times\left(\sqrt{m'^2 + \frac{1}{\hbar^2c^2}\left(a_s^2 - a_v^2\right)} + n - 1\right) \\ &- \frac{\left(Ea_v + m_ec^2a_s\right)^2}{\hbar^2c^2\left(n + \sqrt{m'^2 + \left(1/\hbar^2c^2\right)\left(a_s^2 - a_v^2\right)} - (1/2)\right)^2}, \\ & n = 0, 1, 2, \ldots. \end{aligned} \tag{29}$$

We may find solutions to the above transcendental equation in terms of m, n, ξ, and ω_L. The above energy equation is correct for $S > V$ and not only for $S = V$ (i.e, $\beta = 1$). On the other hand, the solution for the case $S = -V$ can be obtained very simply from the above energy equation by making some transformations: $E \to -E$ and $V(r) \to -V(r)$, that is, $a_v \to -a_v$, $a_s \to -a_s$, $b_v \to -b_v$, and $b_s \to -b_s$ [39].

2.2. The Wave Function Solution. We can find the wave function in (4) with the help of (19), (24)–(28) as

$$\begin{aligned} \psi_{n,m}(r,\phi) = C_{n,m}\frac{1}{\sqrt{2\pi}}e^{im\phi}r^{\sqrt{(m+\xi)^2+(1/\hbar^2c^2)(a_s^2-a_v^2)}} \\ \times e^{-(1/2)\sqrt{(1/\hbar^2c^2)(b_s^2-b_v^2)+(m_e\omega_L/\hbar)^2}r^2} \\ \times e^{-((Ea_v+m_ec^2a_s)/\hbar^2c^2(n+\sqrt{m'^2+(1/\hbar^2c^2)(a_s^2-a_v^2)}-(1/2)))r} \\ \times \sum_{n=0}^{n_{\max}} a_nr^n, \end{aligned} \tag{30}$$

with

$$\begin{aligned} a_1 &= -a_0 \\ &\times\left[-\frac{Eb_v + m_ec^2b_s}{\hbar^2c^2\sqrt{\left(1/\hbar^2c^2\right)\left(b_s^2 - b_v^2\right) + \left(m_e\omega_L/\hbar\right)^2}}\right. \\ &\left.+ \frac{Ea_v + m_ec^2a_s}{\hbar^2c^2\left(\sqrt{m'^2 + \left(1/\hbar^2c^2\right)\left(a_s^2 - a_v^2\right)} + (1/2)\right)}\right], \end{aligned} \tag{31}$$

$$\begin{aligned} a_2 &= -a_0\frac{1}{2\hbar^2c^2\left(2\sqrt{m'^2 + \left(1/\hbar^2c^2\right)\left(a_s^2 - a_v^2\right)} + (3/2)\right)} \\ &\times\left\{\frac{\left(Ea_v + m_ec^2a_s\right)^2}{\hbar^2c^2\left(n + \sqrt{m'^2 + \left(1/\hbar^2c^2\right)\left(a_s^2 - a_v^2\right)} - (1/2)\right)^2}\right. \\ &- \frac{2\hbar^2c^2\left(\sqrt{m'^2 + \left(1/\hbar^2c^2\right)\left(a_s^2 - a_v^2\right)} + 1\right)}{\sqrt{\left(1/\hbar^2c^2\right)\left(b_s^2 - b_v^2\right) + \left(m_e\omega_L/\hbar\right)^2}} \\ &+ E^2 - m_e^2c^4 + 2\left(a_sb_s - a_vb_v\right) \\ &- 2m_ec^2\hbar\omega_L m' - \frac{2\left(Ea_v + m_ec^2a_s\right)^2}{\hbar^2c^2} \\ &\times\left[1 - \frac{\sqrt{m'^2 + \left(1/\hbar^2c^2\right)\left(a_s^2 - a_v^2\right)} + (1/2)}{\left(n + \sqrt{m'^2 + \left(1/\hbar^2c^2\right)\left(a_s^2 - a_v^2\right)} - (1/2)\right)}\right] \\ &\times\left[1 - \frac{\sqrt{m'^2 + \left(1/\hbar^2c^2\right)\left(a_s^2 - a_v^2\right)} + (3/2)}{\left(n + \sqrt{m'^2 + \left(1/\hbar^2c^2\right)\left(a_s^2 - a_v^2\right)} - (1/2)\right)}\right]\Bigg\}, \end{aligned} \tag{32}$$

where $C_{n,m}$ is the normalization constant.

Notice that the present model has been solved in 2D space with an external uniform magnetic field since it is perpendicular to the plane where the vector and scalar Cornell potentials have the dimensions of $r = \rho$ and ϕ; that is, $V = V(\rho, \phi)$ [52]. However, without the magnetic field, the model can be solved in any desired dimensional space by considering the change $m \to l + (D-2)/2$ [56, 57], where D refers to a spatial dimensional space and also the existence of interdimensional degeneracy.

3. Some Special Cases

In this section, we will obtain the energy levels with and without external magnetic field and AB flux field and the energy levels in three dimensional (3D) space. Two special cases of much interest are considered, namely, the Coulomb and harmonic oscillator.

Firstly, let us consider the present system without external fields; that is, $\omega_L = \xi = 0$. We are solving the Schrödinger-type equation with the effective potential:

$$\begin{aligned} U_{\text{eff}}(r) = \frac{1}{\left(E + m_ec^2\right)} \\ \times\left[-\frac{2\left(Ea_v + m_ec^2a_s\right)}{r} + 2\left(Eb_v + m_ec^2b_s\right)r\right. \\ \left.+ \left(b_s^2 - b_v^2\right)r^2 + \frac{\hbar^2c^2\left(m^2 - 1/4\right) + a_s^2 - a_v^2}{r^2}\right]. \end{aligned} \tag{33}$$

Hence, the energy equation and wave functions reduce to

$$E_{nm} = \pm \left[m_e^2 c^4 - 2\gamma_1 + 2\hbar c \gamma_2 \left(\sqrt{m^2 + \frac{\gamma_3}{\hbar^2 c^2}} + n - 1 \right) - \frac{\left(Ea_v + m_e c^2 a_s\right)^2}{\hbar^2 c^2 \left(n + \sqrt{m^2 + (\gamma_3/\hbar^2 c^2)} - (1/2) \right)^2} \right]^{1/2}$$

$$\gamma_1 = a_s b_s - a_v b_v, \qquad \gamma_2 = \sqrt{b_s^2 - b_v^2},$$

$$\gamma_3 = a_s^2 - a_v^2, \qquad b_s > b_v, \qquad a_s > a_v, \tag{34}$$

$$\begin{aligned} \psi_{n,m}(r,\phi) &= C_{n,m} \frac{1}{\sqrt{2\pi}} e^{im\phi} r^{\sqrt{m^2 + (\gamma_3/\hbar^2 c^2)}} \\ &\times e^{-(\gamma_2/2\hbar c) r^2 - (Ea_v + m_e c^2 a_s)/\hbar^2 c^2 (n + \sqrt{m^2 + (\gamma_3/\hbar^2 c^2)} - (1/2)) r} \\ &\times \sum_{n=0}^{n_{\max}} a_n r^n, \end{aligned} \tag{35}$$

where

$$a_1 = -a_0 \left[-\frac{Eb_v + m_e c^2 b_s}{\hbar c \sqrt{b_s^2 - b_v^2}} + \frac{Ea_v + m_e c^2 a_s}{\hbar^2 c^2 \left(\sqrt{m^2 + (1/\hbar^2 c^2)(a_s^2 - a_v^2)} + (1/2) \right)} \right], \tag{36}$$

$$\begin{aligned} a_2 = {} & -a_0 \frac{1}{2\hbar^2 c^2 \left(2\sqrt{m^2 + (1/\hbar^2 c^2)(a_s^2 - a_v^2)} + (3/2) \right)} \\ & \times \left\{ E^2 - m_e^2 c^4 + \left(\left(\left(Ea_v + m_e c^2 a_s \right)^2 \right) \times \left(\hbar^2 c^2 \left(n + \sqrt{m^2 + \frac{1}{\hbar^2 c^2}(a_s^2 - a_v^2)} - \frac{1}{2} \right)^2 \right)^{-1} \right) \right. \\ & - \frac{2\hbar^2 c^2 \left(\sqrt{m^2 + (1/\hbar^2 c^2)(a_s^2 - a_v^2)} + 1 \right)}{\sqrt{(1/\hbar^2 c^2)(b_s^2 - b_v^2)}} \\ & + 2(a_s b_s - a_v b_v) - \frac{2\left(Ea_v + m_e c^2 a_s\right)^2}{\hbar^2 c^2} \\ & \times \left[1 - \left(\left(\sqrt{m^2 + \frac{1}{\hbar^2 c^2}(a_s^2 - a_v^2)} + \frac{1}{2} \right) \times \left(\left(n + \sqrt{m^2 + \frac{1}{\hbar^2 c^2}(a_s^2 - a_v^2)} - \frac{1}{2} \right) \right)^{-1} \right) \right] \\ & \times \left[1 - \left(\left(\sqrt{m^2 + \frac{1}{\hbar^2 c^2}(a_s^2 - a_v^2)} + \frac{3}{2} \right) \times \left(\left(n + \sqrt{m^2 + \frac{1}{\hbar^2 c^2}(a_s^2 - a_v^2)} - \frac{1}{2} \right) \right)^{-1} \right) \right] \Bigg\}. \end{aligned} \tag{37}$$

It is clear that, for the case $S(r) \geq V(r)$, there exist bound state solutions for a relativistic spin-zero particle. For $S(r) = V(r)$, the KG equation reduces to a Schrödinger-like equation and thus the bound state solutions are easy to obtain using well-known methods developed in nonrelativistic quantum mechanics [45].

If the vector potential is taken zero, that is, $V(r) = 0$, $a_v = b_v = 0$, the effective potential is

$$U_{\text{eff}}(r) = \frac{1}{(E + m_e c^2)} \times \left[-\frac{2m_e c^2 a_s}{r} + 2m_e c^2 b_s r + b_s^2 r^2 + \frac{\hbar^2 c^2 \left(m^2 - 1/4\right) + a_s^2}{r^2} \right]. \tag{38}$$

Therefore, the energy can be found as

$$E_{nm} = \pm \left[m_e^2 c^4 - 2a_s b_s - \frac{m_e^2 c^4 a_s^2}{\hbar^2 c^2 \left(n + \sqrt{m^2 + (a_s^2/\hbar^2 c^2)} - (1/2) \right)^2} + 2\hbar c b_s \left(\sqrt{m^2 + \frac{a_s^2}{\hbar^2 c^2}} + n - 1 \right) \right]^{1/2} \tag{39}$$

and wave function

$$\begin{aligned} \psi_{n,m}(r,\phi) &= C_{n,m} \frac{1}{\sqrt{2\pi}} e^{im\phi} r^{\sqrt{m^2 + (a_s^2/\hbar^2 c^2)}} \\ &\times e^{-(b_s/2\hbar c) r^2 - (m_e c^2 a_s/\hbar^2 c^2 (n + \sqrt{m^2 + (a_s^2/\hbar^2 c^2)} - (1/2))) r} \\ &\times \sum_{n=0}^{n_{\max}} a_n r^n. \end{aligned} \tag{40}$$

Secondly, the energy equation and wave functions reduce to 3D space if we replace $m \rightarrow l + 1/2$ as

$$E_{nl} = \pm \left[m_e^2 c^4 - 2a_s b_s - \frac{m_e^2 c^4 a_s^2}{\hbar^2 c^2 \left(n + \sqrt{(l + (1/2))^2 + (a_s^2/\hbar^2 c^2)} - (1/2) \right)^2} + 2\hbar c b_s \left(\sqrt{\left(l + \frac{1}{2}\right)^2 + \frac{a_s^2}{\hbar^2 c^2}} + n - 1 \right) \right]^{1/2} \tag{41}$$

$$\psi_{n,l}(r,\phi) = C_{n,m} \frac{1}{\sqrt{2\pi}} e^{im\phi} r^{\sqrt{(l+1/2)^2 + (a_s^2/\hbar^2 c^2)}} \times e^{-(b_s/2\hbar c) r^2 - (m_e c^2 a_s/\hbar^2 c^2 (n + \sqrt{(l+1/2)^2 + (a_s^2/\hbar^2 c^2)} - (1/2)))r} \times \sum_{n=0}^{n_{\max}} a_n r^n. \tag{42}$$

Thirdly, if the potential parameter $b_s = 0$, the Cornell potential reduces to the relativistic Coulomb potential with $n = n_r + 1$. The effective potential becomes

$$U_{\text{eff}}(r) = \frac{1}{(E + m_e c^2)} \left[-\frac{2m_e c^2 a_s}{r} + \frac{\hbar^2 c^2 l(l+1) + a_s^2}{r^2} \right], \tag{43}$$

which consists of the Coulomb plus new centrifugal term potentials. Therefore the energy levels and wave function become

$$E_{nl} = \pm m_e c^2 \left[1 - \frac{a_s^2}{\hbar^2 c^2 (n_r + l + 1)^2} \right]^{1/2}, \tag{44}$$

where

$$\left(n_r + \sqrt{(l + 1/2)^2 + \frac{a_s^2}{\hbar^2 c^2}} + \frac{1}{2} \right) \longrightarrow (n_r + l + 1), \tag{45}$$

is being obtained from (28). The wave function is

$$\psi_{n,l}(r,\phi) = C_{n,m} \frac{1}{\sqrt{2\pi}} e^{im\phi} r^{l+1/2} \times e^{-(m_e c^2 a_s/\hbar^2 c^2 (n_r + l + 1))r} \sum_{n=0}^{n_{\max}} a_n r^n. \tag{46}$$

Thus, there exist solutions both for positive $E = +m_e c^2 [1 - (a_s^2/(\hbar^2 c^2 (n_r + l + 1)^2))]^{1/2}$ as well as for negative $E = -m_e c^2 [1 - (a_s^2/(\hbar^2 c^2 (n_r + l + 1)^2))]^{1/2}$ energies, respectively. The solutions yielding negative energy are physically connected with antiparticles. Since antiparticles can indeed be observed in nature, we have already obtained an indication of the value of extending the nonrelativistic theory. In the nonrelativistic limit the difference of total energy E of the particle and the rest mass $m_e c^2$ is small. Therefore, we define $E' = E - m_e c^2$ and remark that the kinetic energy nonrelativistic which means $E' \ll m_e c^2$ [45]. In the nonrelativistic limit we obtain

$$E'_{nl} = -\frac{m_e a_s^2}{2\hbar^2 (n_r + l + 1)^2}. \tag{47}$$

Fourthly, if the potential parameter $b_s = 0$, the Cornell potential reduces to the relativistic Kratzer potential $V(r) = -D_e((2r_e/r) - (r_e^2/r^2))$ with $n = n_r + 1$. The effective potential is

$$U_{\text{eff}}(r) = \frac{1}{(E + m_e c^2)} \left[-\frac{2m_e c^2 a_s}{r} + \frac{\hbar^2 c^2 l(l+1) + d_e}{r^2} \right], \quad a_s = 2D_e r_e, \quad d_s = \left(E + m_e c^2\right) D_e r_e^2. \tag{48}$$

The energy formula is as follows:

$$E_{nl} = \pm m_e c^2 \left[1 - \left(\left(4r_e^2 D_e^2\right) \times \left(\hbar^2 c^2 \times \left(n_r + \sqrt{\left(l + \frac{1}{2}\right)^2 + \frac{2r_e D_e^2 \left(E + m_e c^2\right)}{\hbar^2 c^2}} + \frac{1}{2} \right)^2 \right)^{-1} \right) \right]^{1/2}, \tag{49}$$

and, the first inspection to the solution in [58], we can obtain the nonrelativistic energy as

$$E'_{nl} = -\frac{8m_e r_e^2 D_e^2}{\hbar^2 \left(1 + 2n_r + \sqrt{(2l+1)^2 + \left(8m_e r_e D_e^2/\hbar^2\right)} \right)^2}, \tag{50}$$

which is identical to (14) of [58].

Fifthly, if potential parameter $a_s = 0$, the Cornell potential reduces to the relativistic harmonic oscillator potential with $n = 2(n_r + 1)$. The effective potential is

$$U_{\text{eff}}(r) = \frac{1}{(E + m_e c^2)} \left[2m_e c^2 b_s r + b_s^2 r^2 + \frac{\hbar^2 c^2 l(l+1)}{r^2} \right], \tag{51}$$

which consists of the linear plus harmonic plus centrifugal term potentials. The energy levels can be found as

$$E_{nl} = m_e c^2 \left[1 + \frac{2\hbar c b_s}{m_e^2 c^4} \left(2n_r + l + \frac{3}{2} \right) \right]^{1/2} \tag{52}$$

and wave function

$$\psi_{n,l}(r,\phi) = C_{n,m}\frac{1}{\sqrt{2\pi}}e^{im\phi}r^{l+(1/2)}e^{-(b_s/2\hbar c)r^2}\sum_{n=0}^{n_{\max}}a_n r^n, \quad (53)$$

where

$$a_1 = \frac{m_e c}{\hbar}a_0,$$
$$a_2 = -a_0\frac{1}{2\hbar^2c^2(2l+(5/2))}\left\{E^2 - m_e^2c^4 - \frac{2\hbar^3c^3(l+3/2)}{b_s}\right\}. \quad (54)$$

As before, in the nonrelativistic limit we can also obtain

$$E' = \frac{\hbar b_s}{m_e}\left(2n_r + l + \frac{3}{2}\right). \quad (55)$$

Finally, we can take the vector potential of any mixture, say, for example, $a_v = 0.5a_s$ and $b_v = 0.5b_s$, then energy equation reads as

$$E_{nm} = \pm\left[m_e^2c^4 - 1.5a_sb_s + 2\hbar c\sqrt{0.75b_s^2}\left(\sqrt{m^2 + \frac{0.75a_s^2}{\hbar^2c^2}} + n - 1\right) - \frac{0.25a_s^2\left(E + 2m_ec^2\right)^2}{\hbar^2c^2\left(n + \sqrt{m^2 + (0.75a_s^2/\hbar^2c^2)} - (1/2)\right)^2}\right]^{1/2} \quad (56)$$

and the wave function ≤

$$\psi_{n,m}(r,\phi) = C_n\frac{1}{\sqrt{2\pi}}e^{m\phi}r^{\sqrt{m^2+(0.75a_s^2/\hbar^2c^2)}} \times e^{-(\sqrt{0.75b_s^2}/2\hbar c)r^2-((0.5E+m_ec^2)a_s/\hbar^2c^2(n+\sqrt{m^2+(0.75a_s^2/\hbar^2c^2)}-(1/2)))r} \times \sum_{n=0}^{n_{\max}}a_nr^n. \quad (57)$$

4. Concluding Remarks

To sum up, in this paper, we used the wave function ansatz method to study the bound state solutions of the KG equation in 2D space for unequal mixture of the Cornell potential with and without external magnetic and AB flux fields of arbitrary Larmor frequency $\omega_L \neq 0$ and AB strength $\xi \neq 0$. We obtained the energy eigenvalues and wave functions for any arbitrary principal (vibrational) and magnetic (rotational) quantum numbers n and m. We applied an appropriate transformation: $m \to l+(D-1)/2$ to make our solution valid for any dimension D without magnetic and AB flux fields. The present results show that the problems of relativistic quantum mechanics can be also solved exactly as in the non-relativistic ones.

It is noticed that the solution of KG with pure scalar Cornell potential provides the general bound state solution for the well-known Cornell plus harmonic oscillator solution for a spinless particle. The bound state solution exists for the relativistic spin-0 particle only when $S(r) > V(r)$. This is obvious from (29) where the KG possesses bound state spectrum when $|a_v| \leq a_s$ and $|b_v| \leq b_s$ or when the scalar potential is larger than the vector potential there are very few exactly solvable KG equations [59]. The relativistic bound state solutions can also be obtained in this work for any mixture with $V(r) = \beta S(r)$, where $|\beta| \leq 1$. The significance of this bound on $|\beta|$ is the existence of energy eigenvalues (see [60–64] and references therein). The choice of the case $\beta = 1$ is simply to reduce our solution to the nonrelativistic (Schrodinger) energy states of the single particle. However, the solution of KG equation for the case $\beta = -1$ corresponds to the energy states of the antiparticle (antisymmetric case) with some simple transformations in (29); that is, $E \to -E$ and $V(r) \to -V(r)$ [39]. The well-known bound state solutions for some special cases including the Coulomb and harmonic oscillator potentials can be recovered without the magnetic and AB flux fields. The results show that the splitting is not constant and mainly dependent on the strength of the external magnetic field and AB flux field. We have seen that the effective potential function and corresponding energy levels are raised in energy when magnetic and AB flux field strengths increase.

The method is effective in solving the singular Cornell potential and produces the desired results. To test the accuracy of our results, specific choices of the potential parameters recover the results of the exactly solvable Coulomb, harmonic oscillator, and Kratzer potentials.

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