Research Article

A Topologically Charged Black Hole in the $f(\mathcal{R})$ Braneworld

Alexis Larrañaga,¹ Andres Rengifo,² and Luis Cabarique²

¹ National Astronomical Observatory, National University of Colombia, Bogotá 11001000, Colombia

² Department of Physics, National University of Colombia, Bogotá 11001000, Colombia

Correspondence should be addressed to Alexis Larrañaga; ealarranaga@unal.edu.co

Received 2 March 2013; Accepted 12 May 2013

Academic Editor: Elias C. Vagenas

Copyright © 2013 Alexis Larrañaga et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We find a new general black hole solution in the braneworld scenario, considering a modified 5-dimensional $f(\mathcal{R})$ action in the bulk. We study the horizon structure and find the possibility of two, one, or no horizon depending on the value of the topological parameter β . On the thermodynamics side, we show that the value of the topological parameter determines the black hole temperature to have a divergent behaviour at small scales or to present a maximum value before cooling down towards a zero temperature remnant.

1. Introduction

The braneworld scenario describes our 4-dimensional world as a brane that is embedded in a higher dimensional bulk and that supports all gauge fields excluding the gravitational field that lives in the whole spacetime. There are many braneworld models in the cosmological context as well as descriptions of local self-gravitating objects. One of the most successful models is that proposed by Randall and Sundrum [1, 2], where the 5-dimensional bulk has the geometry of an AdS space. The confinement of ordinary matter and gauge fields to the brane is achieved by the application of the Israel junction conditions. Even more, Shiromizu et al. [3] showed how to systematically project the Einstein field equations, written in the high dimensional bulk, onto the brane using the Gauss-Codazzi equations and the Israel junction conditions and assuming $\mathbb{Z}_2$ symmetry.

On the other hand, theories of gravity, in which the Einstein-Hilbert action is replaced with a generic function of the Ricci scalar, $f(R)$, the Einstein tensor, $f(G)$, or both, $f(R, G)$, have been studied recently as natural scenarios that unify and explain both, the inflationary paradigm and the dark energy problem [4]. However, all these models were considered in four dimensions. Therefore, it is interesting to consider both, the braneworld model and the modified gravity, as a unified scenario that we hope could be able to describe some

of the cosmological puzzles and also give some new lights on the behaviour of black holes.

The study of a Randall-Sundrum type model with $f(\mathcal{R})$ as the action in bulk ($\mathcal{R}$ is the Ricci scalar in the bulk) has been presented by Borzou et al. [5]. They derive the field equations on the brane by using the Shiromizu-Maeda-Sasaki methods and the result is a set of field equations that differ from the standard Einstein field equations in 4D by additional terms like $\pi_{\mu\nu}$, which depend on the energy-momentum tensor on the brane, the projection of the Weyl tensor, $E_{\mu\nu}$, and a quantity $Q_{\mu\nu}$ which originates in the geometry of the bulk space, specifically in the function $f(\mathcal{R})$. It is interesting that, for a conformally flat bulk, the quantity $Q_{\mu\nu}$ is conserved and could therefore be identified with a new kind of matter.

Although black hole solutions on the brane are interesting because they have considerably richer physical aspects than black holes in general relativity and have been studied extensively in recent years [6–14], the obtention of such types of solutions in $f(\mathcal{R})$ braneworlds has been lacking. Therefore, the main purpose of this paper is to solve the gravitational field equations on the brane presented in [5] to obtain a topological braneworld black hole, which will be the generalisation of the one presented in [15] to include the effects of the modified action $f(\mathcal{R})$. In the last section of this work, we also present the temperature associated with the event horizon of the braneworld black hole and, from its characteristics, we

conclude that our solution presents the interesting case of an extremal configuration for which the surface gravity vanishes and so does its temperature. As is well known, this kind of configurations present in general relativity [16], and alternative gravitational theories (see e.g., [17, 18]) and its properties have been studied extensively [19–21]. However, the possible interpretation of extremal configurations as elementary string excitations gives them a special interest in string theory [22].

2. The Field Equations on the Brane

The starting point is the action for the bulk,

$$S = \int d^5x \sqrt{-g} f(\mathcal{R}) + S_M, \quad (1)$$

where g is the determinant of the bulk metric, $\mathcal{R}$ is the bulk Ricci scalar, and S_M is the matter action. This gives the 5-dimensional field equations:

$$\mathcal{G}_{AB} = \mathcal{R}_{AB} - \frac{1}{2} g_{AB} \mathcal{R} = \mathcal{T}_{AB}, \quad (2)$$

where

$\mathcal{T}_{AB}$

$$= \frac{1}{f'(\mathcal{R})} \left[\kappa_5^2 T_{AB} - \left(\frac{1}{2} \mathcal{R} f'(\mathcal{R}) - \frac{1}{2} f(\mathcal{R}) + f'(\mathcal{R}) \right) g_{AB} + \nabla_A \nabla_B f'(\mathcal{R}) \right], \quad (3)$$

and $f'(\mathcal{R}) = (df/d\mathcal{R})$. Assuming that the brane is located at $y = 0$, we write the energy-momentum tensor as

$$T_{AB} = -\Lambda_5 g_{AB} + S_{AB} \delta(y), \quad (4)$$

where Λ_5 is the cosmological constant in the bulk and

$$S_{\mu\nu} = -\lambda g_{\mu\nu} + \tau_{\mu\nu} \quad (5)$$

with λ being the brane tension and $\tau_{\mu\nu}$ the energy-momentum tensor on the brane. Contracting the 5-dimensional field equation and taking into account the decomposition of the energy-momentum tensor in (4) and (5), the 5-dimensional Ricci scalar can be written as

$$\mathcal{R} = -\frac{5\kappa_5^2 \Lambda_5}{f'(\mathcal{R})} + \frac{5}{2} \frac{f(\mathcal{R})}{f'(\mathcal{R})} - 4 \frac{\square f'(\mathcal{R})}{f'(\mathcal{R})}. \quad (6)$$

In order to obtain the gravitational field on the brane, the Shiromizu-Maeda-Sasaki method considers the Gauss-Codazzi equations of 5-dimensional gravity and the Israel junction conditions to obtain the effective field equations [3]:

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + 8\pi G \tau_{\mu\nu} + \kappa_5^2 \pi_{\mu\nu} + Q_{\mu\nu} - E_{\mu\nu}, \quad (7)$$

where $G_{\mu\nu} = R_{\mu\nu} - (1/2)g_{\mu\nu}R$, is the 4-dimensional Einstein tensor, and κ_5 is the 5-dimensional gravity coupling constant,

$$\kappa_5^4 = (8\pi G_5)^2 \quad (8)$$

with G_5 being the gravitational constant in five dimensions and Λ the 4-dimensional cosmological constant that is given in terms of the 5-dimensional cosmological constant Λ_5 and the brane tension λ by

$$\Lambda = \frac{\kappa_5^2}{2} \left(\Lambda_5 + \frac{\kappa_5^2}{6} \lambda^2 \right). \quad (9)$$

$\pi_{\mu\nu}$ is a quadratic tensor in the energy-momentum tensor given by

$$\pi_{\mu\nu} = \frac{1}{12} \tau \tau_{\mu\nu} - \frac{1}{4} \tau_{\mu\sigma} \tau_{\nu}^{\sigma} + \frac{1}{8} g_{\mu\nu} \left(\tau_{\alpha\beta} \tau^{\alpha\beta} - \frac{1}{3} \tau^2 \right), \quad (10)$$

with $\tau = \tau_{\sigma}^{\sigma}$, while $E_{\mu\nu}$ is the projection of the 5-dimensional bulk Weyl tensor C_{ABCD} on the brane and can be written as $E_{\mu\nu} = \delta_{\mu}^A \delta_{\nu}^B C_{ABCD} n^A n^B$ with n^A the unit normal to the brane. $E_{\mu\nu}$ represents the nonlocal bulk effect and its only known property is that it is traceless, $E_{\sigma}^{\sigma} = 0$.

Finally, the term $Q_{\mu\nu}$ encompasses the $f(\mathcal{R})$ effects because it completely depends on this function and its derivatives. It is given by

$$Q_{\mu\nu} = \left[\mathcal{F}(\mathcal{R}) g_{\mu\nu} + \frac{2}{3} \frac{\nabla_A \nabla_B f'(\mathcal{R})}{f'(\mathcal{R})} \left(\delta_{\mu}^A \delta_{\nu}^B + n^A n^B g_{\mu\nu} \right) \right]_{y=0}, \quad (11)$$

where

$$\begin{aligned} \mathcal{F}(\mathcal{R}) = & -\frac{4}{15} \frac{\square f'(\mathcal{R})}{f'(\mathcal{R})} - \frac{1}{10} \mathcal{R} \left(\frac{3}{2} + f'(\mathcal{R}) \right) \\ & + \frac{1}{4} f(\mathcal{R}) - \frac{2}{5} \square f'(\mathcal{R}). \end{aligned} \quad (12)$$

For simplicity, in order to obtain the black hole solution, we will consider the following traced field equation:

$$R = 4\Lambda - 8\pi G \tau - \frac{\kappa_5^4}{4} \left(\tau_{\alpha\beta} \tau^{\alpha\beta} - \frac{1}{3} \tau^2 \right) - Q \quad (13)$$

with $Q = Q_{\sigma}^{\sigma}$.

3. Black Hole on the Brane

We will propose a solution on the brane with the following form:

$$ds^2 = -A(r) dt^2 + \frac{dr^2}{A(r)} + r^2 d\Omega_k^2, \quad (14)$$

where $d\Omega_k^2$ is the line element of a two-dimensional hypersurface with constant curvature. For $k = 1$, the topology of the event horizon is a two-sphere, for $k = 0$, the event horizon corresponds to a torus, and for $k = -1$, it corresponds to a hypersurface with constant negative curvature. The corresponding line elements can be written as

$$d\Omega_k^2 = \begin{cases} d\theta^2 + \sin^2 \theta d\varphi^2 & \text{if } k = 1 \\ d\theta^2 + \theta^2 d\varphi^2 & \text{if } k = 0 \\ d\theta^2 + \sinh^2 \theta d\varphi^2 & \text{if } k = -1. \end{cases} \quad (15)$$

We are interested primarily on the $k = 1$ solution, and we impose the condition $g_{rr} = -g_{tt}^{-1}$ because we want the induced metric to be close to Schwarzschild's solution. By considering a constant Ricci curvature scalar $\mathcal{R}$ and solving the field equations (13) in vacuum ($\tau_{\mu\nu} = 0$), we obtain the line element coefficient as follows:

$$A(r) = k - \frac{2M}{r} + \frac{\beta}{r^2} + \frac{\Lambda_{\text{eff}}}{3} r^2, \quad (16)$$

where M and β are integration constants, and

$$\Lambda_{\text{eff}} = \Lambda - \frac{Q}{4} \quad (17)$$

plays the role of an effective cosmological constant on the brane depending on the tension of the brane and the function $f(\mathcal{R})$. The constant β can be interpreted as a five-dimensional mass parameter [7, 15], but it is more useful to think β as a tidal charge associated with the bulk Weyl tensor, and therefore, it can take positive as well as negative values. Indeed, the projected Weyl tensor transmits the tidal charge stresses from the bulk to the brane. This fact can be seen by inserting solution (16) into field equations (7) to obtain the following components:

$$E_t^t = E_r^r = -E_\theta^\theta = -E_\varphi^\varphi = -\frac{\beta}{r^4}. \quad (18)$$

It is clear that the traceless nature of the Weyl tensor is obeyed, and it is interesting to note that the horizon topology of the braneworld black hole does not affect the bulk geometry, that is, the bulk Weyl tensor is independent of the constant curvature k .

In the special case $k = 1$ and $Q = 4\Lambda$ (i.e., $\Lambda_{\text{eff}} = 0$), our solution reduces to the uncharged braneworld black hole solution found in [23]. For $Q = 0$, our solution reduces to the topologically charged black hole solution in the braneworld presented in [15]. Now, let us consider two particular cases for the function $f(\mathcal{R})$.

3.1. $f(\mathcal{R}) = \mathcal{R}^n$ Solution. Considering constant curvature and the particular function $f(\mathcal{R}) = \mathcal{R}^n$, (6) and (11) give the components of $Q_{\mu\nu}$ as

$$Q_\mu^\nu = \left[\frac{\kappa_5^2 \Lambda_5}{2} - \frac{3}{20} \left(\frac{10\kappa_5^2 \Lambda_5}{5-2n} \right)^{1/n} \right] \delta_\mu^\nu, \quad (19)$$

and therefore, its trace is simply

$$Q = 2\kappa_5^2 \Lambda_5 - \frac{3}{5} \left(\frac{10\kappa_5^2 \Lambda_5}{5-2n} \right)^{1/n}. \quad (20)$$

The effective cosmological constant on the brane induced by the tension of the brane and the function $f(\mathcal{R})$ is

$$\Lambda_{\text{eff}} = \frac{\kappa_5^4}{12} \lambda^2 + \frac{3}{20} \left(\frac{10\kappa_5^2 \Lambda_5}{5-2n} \right)^{1/n}. \quad (21)$$

Note that the particular case $n = 1$ gives the usual result $\Lambda_{\text{eff}} = \Lambda$, and the black hole reduces to Schwarzschild's metric with cosmological constant and tidal charge.

3.2. $f(\mathcal{R}) = \mathcal{R} + (\mu^4/\mathcal{R})$ Solution. The function $f(\mathcal{R}) = \mathcal{R} + (\mu^4/\mathcal{R})$ is a good model to explain the positive acceleration of the expanding universe. In this model, a large value of $\mathcal{R}$ gives $f(\mathcal{R}) \approx \mathcal{R}$, so we may expect for those values of the 5-dimensional Ricci scalar a negligible modification of the usual solution. However, for small values of $\mathcal{R}$, gravity is modified.

From (6) and (11), we obtain two possible values for components of $Q_{\mu\nu}$ as follows:

$$Q_\mu^\nu = \left[-\frac{21\mu^4}{20 \left(5\Lambda_5 \kappa_5^2 \pm \sqrt{21\mu^4 + 25\Lambda_5^2 \kappa_5^4} \right)} \right] \delta_\mu^\nu, \quad (22)$$

and therefore, its trace is

$$Q = -\frac{21\mu^4}{5 \left(5\Lambda_5 \kappa_5^2 \pm \sqrt{21\mu^4 + 25\Lambda_5^2 \kappa_5^4} \right)}. \quad (23)$$

The effective cosmological constant on the brane takes the values as follows:

$$\Lambda_{\text{eff}} = \Lambda + \frac{21\mu^4}{20 \left(5\Lambda_5 \kappa_5^2 \pm \sqrt{21\mu^4 + 25\Lambda_5^2 \kappa_5^4} \right)}. \quad (24)$$

The result $\Lambda_{\text{eff}} = \Lambda$ is obtained when $\mu = 0$ (i.e., when the modification in $f(\mathcal{R})$ is turned off).

4. Thermodynamics

Now we will consider only the case in which $k = 1$, that is, when the surface of the event horizon is a 2-sphere. The largest root of the depressed quartic equation $A(r_+) = 0$, or

$$r_+^4 + \frac{3}{\Lambda_{\text{eff}}} r_+^2 - \frac{6}{\Lambda_{\text{eff}}} M r_+ + \frac{3\beta}{\Lambda_{\text{eff}}} = 0, \quad (25)$$

corresponds to the event horizon radius $r = r_+$. The nature of the roots of this equation is commented in the appendix. However, we will present here a graphical analysis of the characteristic behaviour of this polynomial. In Figure 1, we plot function $A(r)$ for a fixed value of parameter M and different values of Λ and β . The position of the horizons can be seen by looking at the intersections with the r -axis. In Figure 1(a), we set $\Lambda = 0$, while Figure 1(b) has $\Lambda > 0$. Note that in both cases, values of $\beta \leq 0$ give one horizon ($\beta = 0$ corresponds to Schwarzschild's black hole), while values of $\beta > 0$ present the possibility of two horizons (when β is less than certain critical value β_c) and the existence of an extremal configuration ($\beta = \beta_c$), resembling the behaviour of Reissner-Nordström or Reissner-Nordström-AdS solutions, respectively. Values of β above the extremal value β_c do not show horizons, and therefore, we conclude that they do not correspond to black holes. In Figure 1(c) we consider the $\Lambda < 0$ case, and the behaviour shows no black hole solutions.

FIGURE 1: $A(r)$ as function of r for different values of Λ and β . The intersections with the r -axis give inner and outer horizons. In (a), we set $\Lambda = 0$, and (b) has $\Lambda > 0$. In both cases, values of $\beta \leq 0$ give one horizon, while $\beta > 0$ give two horizons or an extremal configuration ($\beta = \beta_c$), resembling the behaviour of Reissner-Nordström solution. Above this extremal value, there is no black hole. In (c), we consider the $\Lambda < 0$ case, and the behaviour shows no black hole solution.

The extremal black hole is described by the following conditions:

$$\begin{aligned}
 A(r_e) &= 1 - \frac{2M}{r_e} + \frac{\beta}{r_e^2} + \frac{\Lambda_{\text{eff}}}{3} r_e^2 = 0, \\
 \partial_r A(r_e) &= \frac{2}{r_e} \left[\frac{M}{r_e} - \frac{\beta}{r_e^2} + \frac{\Lambda_{\text{eff}}}{3} r_e^2 \right] = 0,
 \end{aligned} \tag{26}$$

from which we obtain the value of the critical parameter β_c as

$$\beta_c = \frac{9}{8} M^2$$

$$\begin{aligned}
 & -\frac{1}{8} \left[\frac{1}{\Lambda_{\text{eff}}} \left[\left(6M + 2\sqrt{\frac{9\Lambda_{\text{eff}}M^2 + 2}{\Lambda_{\text{eff}}}} \right) \Lambda_{\text{eff}}^2 \right]^{1/3} \right. \\
 & \quad \left. - \frac{2}{\left[\left(6M + 2\sqrt{(9\Lambda_{\text{eff}}M^2 + 2)/\Lambda_{\text{eff}}} \right) \Lambda_{\text{eff}}^2 \right]^{1/3}} \right. \\
 & \quad \left. - 3M \right]^2.
 \end{aligned} \tag{27}$$

FIGURE 2: Temperature of the black hole as function of the radius of the event horizon r_+ for different values of Λ and β . In (a), we set $\Lambda = 0$. For $\beta = 0$, we obtain Schwarzschild's temperature and a similar behaviour for $\beta < 0$. Values of $\beta > 0$ give a temperature that goes to zero for an extremal black hole remnant configuration. In (b), we set $\Lambda > 0$. The value of $\beta = 0$ gives Schwarzschild-AdS temperature and similar behaviour for $\beta < 0$. Values of $\beta > 0$ give again a function that goes to the zero temperature extremal black hole. In both cases, the value $\beta = \beta_c$.

The temperature of the black hole is defined in terms of the surface gravity at r_+ by

$$T = \frac{\kappa_+}{2\pi} = \frac{1}{4\pi} \partial_r A(r)|_{r=r_+} = \frac{1}{2\pi r_+} \left[\frac{M}{r_+} - \frac{\beta}{r_+^2} + \frac{\Lambda_{\text{eff}}}{3} r_+^2 \right]. \quad (28)$$

In Figure 2, we can see the plot of the temperature of the black hole as function of the radius of the event horizon r_+ for different values of Λ and β . In Figure 2(a), we set $\Lambda = 0$, and, is clear that $\beta = 0$ gives Schwarzschild's Hawking temperature. The behaviour for $\beta < 0$ reproduces Schwarzschild's divergence for $r_+ \rightarrow 0$. However, it is very interesting to note that values of $\beta > 0$ give a new behaviour: for large values of r_+ , the temperatures coincide with Schwarzschild's, but for small radii, our solution admits a maximum value T_{max} . From this point, the function goes to the zero temperature of the extremal black hole remnant configuration. In Figure 2(b), we set $\Lambda > 0$. This time, the value of $\beta = 0$ gives Schwarzschild-AdS temperature, and the behaviour for $\beta < 0$ is similar to this function. Again, values of $\beta > 0$ give a function that goes to the zero temperature of the extremal black hole. We do not consider the $\Lambda < 0$ case because there is no black hole solution and therefore no interesting temperature.

5. Conclusions

We have obtained a general black hole solution in the brane-world scenario modified by a bulk action in the form $f(\mathcal{R})$. Due to the complexity of the five-dimensional equations, we have solved the effective field equations for the induced

metric on the brane derived by the Shiromizu-Maeda-Sasaki method to obtain a topologically charged solution that generalises the metric presented in [15] to include the effects of the function $f(\mathcal{R})$. Therefore, we have not studied fully the effect of the braneworld black hole on the bulk geometry.

In the last section, we also present the temperature associated with the event horizon of the braneworld black hole. The presented analysis of the solution shows that the geometry admits two, one, or no horizon depending on the value of the topological parameter β with respect to two threshold masses $\beta = 0$ and $\beta = \beta_c$. From the thermodynamical analysis, the possibility of a degenerate horizon gives a temperature that, instead of a divergent behaviour at short scales, admits both a minimum and a maximum before cooling down towards a zero temperature remnant configuration.

Finally, it is important to stress that the quantity $Q_{\mu\nu}$ which originates from the extra geometry terms, is conserved for a conformally flat bulk, and therefore, it can be identified with a new kind of matter [5]. Since this quantity enters in function (16) through the parameter β , our solution could be used to explain the galaxy rotation curves by considering it as the metric describing the central black hole.

Appendix

The nature of the roots of the depressed quartic polynomial

$$f(x) = x^4 + a_3 x^3 + a_1 x + a_0 \quad (\text{A.1})$$

with a_0, a_1 , and a_3 being real can be determined by its discriminant Δ [24]. Denoting by α_j the roots of $f(x) = 0$, we

define the discriminant as the product of the squares of the differences of the roots:

$$\Delta = \prod_i \prod_{j>i} (\alpha_i - \alpha_j)^2. \quad (\text{A.2})$$

The criteria regarding the nature of the roots are as follows.

Case I ($a_3 < 0$)

(i) $a_0 > a_3^2/4$;

$\Delta < 0$: two roots real and distinct,
 $\Delta = 0$: two roots real and equal,
 $\Delta > 0$: no real roots.

(ii) $a_0 = a_3^2/4$;

$\Delta < 0$: two roots real and distinct,
 $\Delta = 0$: two pairs of equal real roots.

(iii) $-a_3^2/12 < a_0 < a_3^2/4$;

$\Delta < 0$: two roots real and distinct,
 $\Delta = 0$: all roots real, two equal,
 $\Delta > 0$: all roots real and distinct.

(iv) $a_0 = -a_3^2/12$;

$\Delta < 0$: two roots real and distinct,
 $\Delta = 0$: all roots real, three equal.

(v) $a_0 < -a_3^2/12$;

$\Delta < 0$: two roots real and distinct.

Case II ($a_3 = 0$)

(i) $a_0 > 0$;

$\Delta < 0$: two roots real and distinct,
 $\Delta = 0$: two roots real and equal,
 $\Delta > 0$: no real roots.

(ii) $a_0 = 0$;

$\Delta < 0$: two roots real and distinct,
 $\Delta = 0$: four equal real roots.

(iii) $a_0 < 0$;

$\Delta < 0$: two roots real and distinct.

Case III ($a_3 > 0$)

(i) $a_0 > 0$;

$\Delta < 0$: two roots real and distinct,
 $\Delta = 0$: two roots real and equal,

$\Delta > 0$: no real roots.

(ii) $a_0 = 0$;

$\Delta < 0$: two roots real and distinct,
 $\Delta = 0$: two roots real and equal.

(iii) $a_0 < 0$;

$\Delta < 0$: two roots real and distinct.

In particular, for the horizon equation (25), the coefficients of the quartic equation in terms of the parameters M, β , and $\Lambda_{\text{eff}} \neq 0$ are

$$\begin{aligned} a_3 &= \frac{3}{\Lambda_{\text{eff}}}, \\ a_1 &= -\frac{6M}{\Lambda_{\text{eff}}}, \\ a_0 &= \frac{3\beta}{\Lambda_{\text{eff}}}, \end{aligned} \quad (\text{A.3})$$

and following the condition for the parameters, the criteria for the nature of the possible roots become as follows.

Case I ($\Lambda_{\text{eff}} < 0$) No black holes; therefore, it is not considered.

Case II ($1/\Lambda_{\text{eff}} \rightarrow 0$)

(i) $\beta > 0$;

$\Delta < 0$: two roots real and distinct,
 $\Delta = 0$: two roots real and equal,
 $\Delta > 0$: no real roots.

(ii) $\beta = 0$;

$\Delta < 0$: two roots real and distinct,
 $\Delta = 0$: four equal real roots.

(iii) $\beta < 0$;

$\Delta < 0$: two roots real and distinct.

Case III ($\Lambda_{\text{eff}} > 0$)

(i) $\beta > 0$;

$\Delta < 0$: two roots real and distinct,
 $\Delta = 0$: two roots real and equal,
 $\Delta > 0$: no real roots.

(ii) $\beta = 0$;

$\Delta < 0$: two roots real and distinct,
 $\Delta = 0$: two roots real and equal.

(iii) $\beta < 0$;

$\Delta < 0$: two roots real and distinct.

In order to solve the depressed polynomial (25), we may note that it can be factorized by solving the following resolvent cubic [25]:

$$z^3 - \frac{3}{\Lambda_{\text{eff}}} z^2 - \frac{12\beta}{\Lambda_{\text{eff}}} z + \frac{36}{\Lambda_{\text{eff}}^2} (\beta - M^2) = 0. \quad (\text{A.4})$$

The nature of the roots of this equation depends on the discriminant as follows:

$$\delta = \frac{1}{\Lambda_{\text{eff}}^4} \left[12\beta - 18M^2 - \frac{1}{\Lambda_{\text{eff}}} \right]^2 - \frac{1}{\Lambda_{\text{eff}}^3} \left[4\beta + \frac{1}{\Lambda_{\text{eff}}} \right]^3. \quad (\text{A.5})$$

If $\delta > 0$, there is one real root and two conjugates imaginary roots.

If $\delta = 0$, there are three real roots of which at least two are equal.

If $\delta < 0$, there are three real and unequal roots.

In any case, we can obtain one real root $z = z_1$ of the resolvent cubic from which we build the four roots of the original quartic equation as

$$\begin{aligned} r_1 &= \frac{1}{2} (R + D), \\ r_2 &= \frac{1}{2} (R - D), \\ r_3 &= -\frac{1}{2} (R - E), \\ r_4 &= -\frac{1}{2} (R + E), \end{aligned} \quad (\text{A.6})$$

where

$$\begin{aligned} R &= \sqrt{z_1 - \frac{3}{\Lambda_{\text{eff}}}}, \\ D &= \begin{cases} \sqrt{\frac{12M}{\Lambda_{\text{eff}} R} - z_1 - \frac{3}{\Lambda_{\text{eff}}}} & \text{if } R \neq 0, \\ \sqrt{-\frac{6}{\Lambda_{\text{eff}}} + 2\sqrt{z_1^2 - \frac{12\beta}{\Lambda_{\text{eff}}}}} & \text{if } R = 0, \end{cases} \\ E &= \begin{cases} \sqrt{-\frac{12M}{\Lambda_{\text{eff}} R} - z_1 - \frac{3}{\Lambda_{\text{eff}}}} & \text{if } R \neq 0, \\ \sqrt{-\frac{6}{\Lambda_{\text{eff}}} - 2\sqrt{z_1^2 - \frac{12\beta}{\Lambda_{\text{eff}}}}} & \text{if } R = 0. \end{cases} \end{aligned} \quad (\text{A.7})$$

Acknowledgment

This work was supported by the Universidad Nacional de Colombia, Hermes Project Code 17318.

References

- [1] L. Randall and R. Sundrum, "Large mass hierarchy from a small extra dimension," *Physical Review Letters*, vol. 83, no. 17, pp. 3370–3373, 1999.
- [2] L. Randall and R. Sundrum, "An alternative to compactification," *Physical Review Letters*, vol. 83, no. 23, pp. 4690–4693, 1999.
- [3] T. Shiromizu, K. Maeda, and M. Sasaki, "The Einstein equation on the 3-brane world," *Physical Review D*, vol. 62, no. 2, Article ID 024012, 6 pages, 2000.
- [4] S. Nojiri and S. D. Odintsov, "Introduction to modified gravity and gravitational alternative for dark energy," *International Journal of Geometric Methods in Modern Physics*, vol. 4, no. 1, pp. 115–145, 2007.
- [5] A. Borzou, H. R. Sepangi, S. Shahidi, and R. Yousefi, "Brane $f(\mathcal{R})$ gravity," *Europhysics Letters*, vol. 88, no. 2, Article ID 29001, 2009.
- [6] A. Chamblin, S. W. Hawking, and H. S. Reall, "Brane-world black holes," *Physical Review D*, vol. 61, no. 6, Article ID 065007, pp. 1–6, 2000.
- [7] A. Chamblin, H. S. Reall, H. Shinkai, and T. Shiromizu, "Charged brane-world black holes," *Physical Review D*, vol. 63, no. 6, Article ID 064015, 2001.
- [8] S. Nojiri, O. Obregon, S. D. Odintsov, and S. Ogushi, "Dilatonic brane-world black holes, gravity localization, and Newton's constant," *Physical Review D*, vol. 62, no. 6, Article ID 064017, 10 pages, 2000.
- [9] P. Kanti and K. Tamvakis, "Quest for localized 4D black holes in brane worlds," *Physical Review D*, vol. 65, no. 8, Article ID 084010, 12 pages, 2002.
- [10] R. Casadio, A. Fabbri, and L. Mazzacurati, "New black holes in the brane world?" *Physical Review D*, vol. 65, no. 8, Article ID 084040, 5 pages, 2002.
- [11] C. Germani and R. Maartens, "Stars in the braneworld," *Physical Review D*, vol. 64, no. 12, Article ID 124010, 6 pages, 2001.
- [12] R. Casadio and L. Mazzacurati, "Bulk shape of brane-world black holes," *Modern Physics Letters A*, vol. 18, no. 9, pp. 651–660, 2003.
- [13] R. Casadio, "On brane-world black holes and short scale physics," *Annals of Physics*, vol. 307, no. 2, pp. 195–208, 2003.
- [14] R. Casadio and B. Harms, "Black hole evaporation and large extra dimensions," *Physics Letters B*, vol. 487, no. 3–4, pp. 209–214, 2000.
- [15] A. Sheykhi and B. Wang, "On topological charged braneworld black holes," *Modern Physics Letters A*, vol. 24, no. 31, pp. 2531–2538, 2009.
- [16] B. Wang, R. Su, and P. Yu, "Geometry of the extreme Kerr black hole," *Physics Letters D*, vol. 58, no. 12, Article ID 124026, 4 pages, 1998.
- [17] L. Modesto and P. Nicolini, "Charged rotating noncommutative black holes," *Physics Letters D*, vol. 82, no. 10, Article ID 104035, 2010.
- [18] W. Kim, E. J. Son, and M. Yoon, "Thermodynamic similarity between the noncommutative Schwarzschild black hole and the Reissner-Nordström black hole," *Journal of High Energy Physics A*, vol. 2008, article 42, 2008.
- [19] L. Vanzo, "Radiation of extreme black holes," *Physical Review D*, vol. 55, no. 4, pp. 2192–2201, 1997.
- [20] A. Gosh and P. Miltra, "Entropy for extremal Reissner-Nordström black holes," *Physics Letters B*, vol. 357, no. 3, pp. 295–299, 1995.

- [21] R. Balbinot, A. Fabbri, S. Farese, and R. Parentani, “Hawking radiation from extremal and nonextremal black holes,” *Physical Review D*, vol. 76, no. 12, Article ID 124010, 8 pages, 2007.
- [22] A. Sen, “Extremal black holes and elementary string states,” *Modern Physics Letters A*, vol. 10, no. 28, pp. 2081–2094, 1995.
- [23] N. Dadhich, R. Maartens, P. Papadopoulos, and V. Rezanian, “Quark mixings in $SU(6) \times SU(2)_R$ and suppression of V_{ub} ,” *Physics Letters B*, vol. 487, no. 1-2, pp. 104–109, 2000.
- [24] E. L. Rees, “Graphical discussion of the roots of a quartic equation,” *The American Mathematical Monthly*, vol. 29, no. 2, pp. 51–55, 1922.
- [25] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables*, Dover Publications, New York, NY, USA, 1965.