Research Article

Hermitian (ϵ, δ) -Freudenthal-Kantor Triple Systems and Certain Applications of $*$ -Generalized Jordan Triple Systems to Field Theory

Noriaki Kamiya¹ and Matsuo Sato²

¹ Department of Mathematics, University of Aizu, Aizuwakamatsu 965-8580, Japan

² Department of Natural Science, Faculty of Education, Hirosaki University, Bunkyo-cho 1, Hirosaki, Aomori 036-8560, Japan

Correspondence should be addressed to Matsuo Sato; msato@cc.hirosaki-u.ac.jp

Received 15 August 2013; Revised 5 January 2014; Accepted 3 March 2014; Published 3 April 2014

Academic Editor: Anastasios Petkou

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We define Hermitian (ϵ, δ) -Freudenthal-Kantor triple systems and prove a structure theorem. We also give some examples of triple systems that are generalizations of the $u(N) \oplus u(M)$ and $sp(2N) \oplus u(1)$ Hermitian 3-algebras. We apply a $*$ -generalized Jordan triple system to a field theory and obtain a Chern-Simons gauge theory. We find that the novel Higgs mechanism works, where the Chern-Simons gauge theory reduces to a Yang-Mills theory in a certain limit.

1. Introduction

The structure theory of (ϵ, δ) -Freudenthal-Kantor triple systems of finite dimensions, which contain the concept of Jordan triple systems, has been studied in [1–11].

On the other hand, it is well known that symmetric bounded domains have one-to-one correspondence to Hermitian Jordan triple systems [12–15], for which a certain trace form is positive definite Hermitian. Moreover, structure theorems of Hermitian generalized Jordan triple systems of the second-order can be found in [16]. Hence, as a generalization of these concepts, it is interesting to investigate the structure theory of Hermitian (ϵ, δ) -Freudenthal-Kantor triple systems (HFKTSs). In this generalization, Hermitian generalized Jordan triple systems of the second-order or the so-called Kantor triple systems are included as Hermitian $(-1, 1)$ -Freudenthal-Kantor triple systems.

From the viewpoint of string theory, HFKTSs are generalizations (A different generalization, the so-called differential crossed module, was extensively investigated in [17–19].) of Hermitian 3-algebras [20–62], which have played crucial roles in M-theory. The field theories applied with

Hermitian 3-algebras are the Chern-Simons gauge theories that describe the effective actions of membranes in M-theory. In a certain limit, the novel Higgs mechanism works, where the Chern-Simons theories become the Yang-Mills theories that describe the effective actions of D-branes in string theory. Moreover, 3-algebra models of M-theory itself have been proposed and were studied in [63–67]. Therefore, it is interesting to apply HFKTSs to field theories. It is particularly important to study whether the novel Higgs mechanism works in a field theory with an HFKTS.

The Hermitian 3-algebras are special cases, where $K(a, b) = 0$ or, equivalently, $\langle abc \rangle = -\langle cba \rangle$, of Hermitian $(-1, -1)$ -Freudenthal-Kantor triple systems of the second order. Also, the Hermitian 3-algebras are classified into $u(N) \oplus u(M)$ and $sp(2N) \oplus u(1)$ Hermitian 3-algebras [34, 36, 37, 43, 67]. Therefore, it is natural to extend these triple systems to more general Hermitian $(-1, -1)$ -Freudenthal-Kantor triple systems or Hermitian generalized Jordan triple systems.

Here, we are concerned with algebras and triple systems that are finite or infinite over a complex number field, unless

otherwise specified. We refer to [68–72] for nonassociative algebras and to [73–75] for Lie superalgebras, for example.

2. Definitions and Preamble

A triple system U is a vector space over a field F of characteristic $\neq 2, 3$ with a trilinear map $U \otimes U \otimes U \rightarrow U$. In this paper, we are concerned with triple systems over the complex number $\mathbf{C}$, and we denote the trilinear product by $\langle xyz \rangle \in U$ for $x, y, z \in U$, assuming that $\langle xyz \rangle$ is $\mathbf{C}$ -linear on x and z and $\mathbf{C}$ -antilinear on y . Also we use the notations of two operations $L(x, y) \in \text{End } U$ and $K(x, y) \in \text{End } U$ with respect to the triple systems [1, 2], where $L(x, y)z = \langle xyz \rangle$ and $K(x, y)z = \langle xzy \rangle - \delta \langle yzx \rangle$ ($\delta = \pm 1$).

Definition 1. A triple system U is said to be a $\ast$ -(ϵ, δ)-Freudenthal-Kantor triple system if the following relations (0)–(iv) are satisfied:

- (0) U is a Banach space,
- (i) $[L(x, y), L(z, w)] = L(\langle xyz \rangle, w) + \epsilon L(z, \langle yxw \rangle)$, where $[A, B] = AB - BA$, $A, B \in \text{End } U$, and $\epsilon = \pm 1$,
- (ii) $K(\langle xyz \rangle, w) + K(z, \langle xyw \rangle) + \delta K(x, K(z, w)y) = 0$,
- (iii) $\langle xyz \rangle$ is a $\mathbf{C}$ -linear operator on x and z and a $\mathbf{C}$ -antilinear operator on y ,
- (iv) $\langle xyz \rangle$ is continuous with respect to a norm, $\|\cdot\|$; that is, there exists $K > 0$ such that

$$\|\langle xxx \rangle\| \leq K\|x\|^3 \quad \forall x \in U. \quad (1)$$

Furthermore, a $\ast$ -(ϵ, δ)-Freudenthal-Kantor triple system is said to be Hermitian if it satisfies the following condition.

- (v) All operators $L(x, y)$ are positive Hermitian operators with a Hermitian metric

$$(x, y) = \text{tr } L(x, y); \quad (2)$$

that is, $(L(x, y)z, w) = (z, L^*(x, y)w)$ and $(x, y) = \overline{(y, x)}$.

This definition is a generalization of the concept of known (ϵ, δ) -Freudenthal-Kantor triple systems (see [1–10]) to that of Hermitian triple systems. Note that there are many simple Lie algebras (the case of $\delta = 1$) and simple Lie superalgebras (the case of $\delta = -1$) constructed from (ϵ, δ) -Freudenthal-Kantor triple systems [4, 7, 8].

Let U be a $\ast$ -(ϵ, δ)-Freudenthal-Kantor triple system. Then we may define the notations of tripotent and bitripotent as follows.

Definition 2. It is said to be a tripotent of U if

$$\langle ccc \rangle = c, \quad c \in U. \quad (3)$$

Definition 3. It is said to be a strong bitripotent of U if a pair (c_1, c_2) of tripotents satisfies the relations,

$$\begin{aligned} \langle c_1 c_1 c_2 \rangle &= -\frac{1}{2} c_2, \\ \langle c_2 c_2 c_1 \rangle &= -\frac{1}{2} c_1, \end{aligned} \quad (4)$$

and other products are zero.

Definition 4. It is said to be a bitripotent of U if a pair (c_1, c_2) of tripotents satisfies the relations

$$\begin{aligned} \langle c_1 c_1 c_2 \rangle &= \alpha c_2, & \langle c_2 c_2 c_1 \rangle &= \alpha c_1, \\ \langle c_1 c_2 c_1 \rangle &= \beta c_2, & \langle c_2 c_1 c_2 \rangle &= \beta c_1, \\ \langle c_2 c_1 c_1 \rangle &= \gamma c_2, & \langle c_1 c_2 c_2 \rangle &= \gamma c_1, \end{aligned} \quad (5)$$

and other products are zero, where

$$\alpha^2 + \beta^2 + \gamma^2 \neq 0, \quad \alpha, \beta, \gamma \in \mathbf{R} \text{ (real number)}. \quad (6)$$

From now on, we will consider a finite-dimensional triple system equipped with a tripotent, unless otherwise specified.

Following [16], theorems concerning tripotents (Theorems 5 and 6) are proved for Hermitian generalized Jordan triple systems. Actually, the proofs of [16] are valid in Hermitian $(-1, \delta)$ -Freudenthal-Kantor triple systems because they are independent of δ . Here, we show the theorems below for completeness.

Theorem 5. Let U be a Hermitian $(-1, \delta)$ -Freudenthal-Kantor triple system. If $W \subset U$ is flat (i.e., $L(x, y) = L(y, x)$ for all $x, y \in W$), then we have the decomposition

$$W = \mathbf{R}e_1 \oplus \cdots \oplus \mathbf{R}e_n, \quad (7)$$

where e_i are tripotents or bitripotents.

Proof. Let $S \subset \text{End}_{\mathbf{R}}(W)$ be the $\mathbf{R}$ -linear span of all $L(x, x)|_W$, where $x \in W$.

From

$$2L(x, y)z = L(x + y, x + y)z - L(x, x)z - L(y, y)z, \quad (8)$$

it follows that $L(x, y)$ belongs to S for all $x, y \in W$. Choose a scalar product, (x, y) , for example, $(x, y) := \text{trace } L(x, y)$, satisfying $L(x, y)^* = L(y, x)$.

By taking the real part of (x, y) and restricting W , we obtain a Euclidean scalar product on W such that S consists of self-adjoint transformations. Furthermore, S is commutative since $[L(x, x), L(y, y)] = L(\langle xxy \rangle, y) - L(y, \langle xxy \rangle) = 0$.

By a standard calculation, S may be simultaneously diagonalized. Therefore, there exists a basis $e_1, \dots, e_n$ of W such that $fe_i \in \mathbf{R}e_i$ for all $f \in S$. Hence, from $L(e_i, e_i) \in S$, we have

$$\langle e_i e_i e_i \rangle = \lambda e_i, \quad \lambda \in \mathbf{R}. \quad (9)$$

Furthermore, after sign changes, we may assume that $\lambda > 0$.

Indeed, if $\lambda < 0$, we write $e'_i = -e_i$, then $\lambda e_i = -\lambda e'_i$ such that $-\lambda$ is positive.

Replacing e_i by $(\sqrt{\lambda})^{-1} e_i$, we may assume that e_i are tripotents.

On the other hand, if $i \neq j$, we have

$$\begin{aligned} \langle e_i e_i e_j \rangle &= L(e_i, e_i) e_j \in Se_j \\ &\subseteq Re_j. \end{aligned} \quad (10)$$

These imply that e_i and e_j are tripotents or bitripotents. This completes the proof. $\square$

We define odd powers of x inductively as follows:

$$\begin{aligned} x^{(3)} &:= \langle xxx \rangle, \\ x^{(2n+1)} &:= \langle xx^{(2n-1)}x \rangle. \end{aligned} \quad (11)$$

By using this theorem, we have the following.

Theorem 6. *Let U be a Hermitian $(-1, \delta)$ -Freudenthal-Kantor triple system. Then every $x \in U$ can be written uniquely as*

$$x = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n, \quad (12)$$

where e_i are tripotents or bitripotents and λ_i satisfy

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n. \quad (13)$$

Proof. Applying Theorem 5 to the subspace W spanned by the powers of x , from definition (i), we have $0 = [L(x, x), L(x, x)] = L(\langle xxx \rangle, x) - L(x, \langle xxx \rangle)$ for $x \in W$, and thus $L(x, \langle xxx \rangle) = L(\langle xxx \rangle, x)$. Hence, W is flat.

Without loss of generality, we can assume that $W = U$. This implies that any element $x \in U$ can be represented by $x = \mu_1 e_1 + \mu_2 e_2 + \cdots + \mu_n e_n$, with real numbers μ_i , after permutations, and we may assume that $0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$, where e_i are tripotents or bitripotents.

Furthermore, because e_i are tripotents or bitripotents, the powers of x are

$$x^{(2k+1)} = \mu_1^{(2k+1)} e_1 + \mu_2^{(2k+1)} e_2 + \cdots + \mu_n^{(2k+1)} e_n. \quad (14)$$

That is, we obtain

$$\begin{aligned} x^{(3)} &= \langle xxx \rangle = \mu_1^{(3)} e_1 + \mu_2^{(3)} e_2 + \cdots + \mu_n^{(3)} e_n, \\ x^{(5)} &= \langle x \langle xxx \rangle x \rangle = \mu_1^{(5)} e_1 + \mu_2^{(5)} e_2 + \cdots + \mu_n^{(5)} e_n, \\ &\vdots \\ x^{(2n-1)} &= \langle xx^{(2n-3)}x \rangle \\ &= \mu_1^{(2n-1)} e_1 + \mu_2^{(2n-1)} e_2 + \cdots + \mu_n^{(2n-1)} e_n, \end{aligned} \quad (15)$$

where $\mu_i^{(k)}$ ($i = 1, \dots, n$, $k = 3, 5, \dots, 2n-1$) are polynomials in $\mu_1, \dots, \mu_n$ with their coefficients in $\mathbf{R}$.

These expressions may also be written as

$$\begin{pmatrix} \mu_1 & \cdots & \mu_n \\ \mu_1^{(3)} & \cdots & \mu_n^{(3)} \\ \vdots & & \vdots \\ \mu_1^{(2n-1)} & \cdots & \mu_n^{(2n-1)} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} = \begin{pmatrix} x \\ x^{(3)} \\ \vdots \\ x^{(2n-1)} \end{pmatrix}. \quad (16)$$

This completes the proof. $\square$

For the HFKTS U , we can define a norm $\| \cdot \|$ as follows:

$$\|x\| = \max |\lambda_i|, \quad \text{if } x = \sum \lambda_i e_i \in U, \quad (17)$$

where e_i are tripotents or bitripotents. Note that

$$\|x\| \geq 0, \quad \|x + y\| \leq \|x\| + \|y\|. \quad (18)$$

3. Peirce Decomposition

In this section, we briefly consider the Peirce decomposition of a $*(-1, -1)$ -Freudenthal-Kantor triple system equipped with the tripotent $\langle ccc \rangle = c$.

We recall conditions (i) and (ii) in Section 2, which are equivalent to the following conditions (i)' and (ii)':

$$[L(x, y), L(z, w)] = L(\langle xyz \rangle, w) + \varepsilon L(z, \langle yxw \rangle), \quad (i)'$$

$$K(K(x, y)z, w) - L(w, z)K(x, y) + \varepsilon K(x, y)L(z, w) = 0. \quad (ii)'$$

From condition (i)' with $\varepsilon = -1$, we have

$$L(c, c)R(c, c) = R(c, c)L(c, c), \quad (i)''$$

where we define $R(x, y)z = \langle zxy \rangle$. From condition (ii)' with $\varepsilon = -1$, we have

$$K(K(x, y)z, w) - L(w, z)K(x, y) - K(x, y)L(z, w) = 0. \quad (19)$$

We write $x = z = w = c$ (if c is a tripotent element, $\langle ccc \rangle = c$), and then we obtain

$$K(K(c, y)c, c) - L(c, c)K(c, y) - K(c, y)L(c, c) = 0. \quad (20)$$

When $\delta = -1$, we obtain

$$L(c, c)R(c, c) + R(c, c)^2 - L(c, c) - R(c, c) = 0. \quad (ii)''$$

From conditions (i)'' and (ii)'', we have

$$\begin{aligned} (R(c, c) - Id)(R(c, c) + L(c, c)) &= 0, \\ L(c, c)R(c, c) &= R(c, c)L(c, c). \end{aligned} \quad (21)$$

Summarizing the results, we have the following.

Theorem 7. *Let U be a $*(-1, -1)$ -Freudenthal-Kantor triple system. Then, we have the following decomposition with respect to a tripotent c (i.e., $\langle ccc \rangle = c$):*

$$U = U_1(c) \oplus U_{1/2}(c) \oplus U_0(c), \quad (22)$$

where

$$\begin{aligned}
 U_1(c) &= \{x \mid (L(c, c) + R(c, c))x = 0, \\
 &\quad (R(c, c) - Id)x \neq 0\}, \\
 U_{1/2}(c) &= \{x \mid (L(c, c) + R(c, c))x \neq 0, \\
 &\quad (R(c, c) - Id)x = 0\}, \\
 U_0(c) &= \{x \mid (L(c, c) + R(c, c))x = 0, \\
 &\quad (R(c, c) - Id)x = 0\}.
 \end{aligned} \tag{23}$$

4. Examples

Hermitian 3-algebras are classified into $u(N) \oplus u(M)$ and $sp(2N) \oplus u(1)$ Hermitian 3-algebras [34, 36, 37, 43, 67]. In this section, we extend these 3-algebras to Hermitian $(-1, -1)$ -Freudenthal-Kantor triple systems and $*$ -generalized Jordan triple systems.

Example 1. Let $D_{N,M}^*$ be the set of all $N \times M$ matrices with the operation

$$\langle xyz \rangle = x\bar{y}^T z - z\bar{y}^T x + z\bar{x}^T \bar{y}, \tag{24}$$

where x^T and $\bar{x}$ denote the transpose and conjugation of x , respectively.

Then $D_{N,M}^*$ is a Hermitian $(-1, -1)$ -Freudenthal-Kantor triple system. In fact, it satisfies conditions (0), (i), (ii), (iii), (iv), and (v) in Section 2. This is an extension of the $u(N) \oplus u(M)$ Hermitian 3-algebra (the metric of the Hermitian 3-algebra is defined as $(x, y) = \text{tr}(xy^\dagger)$, which is different from our Definition 1(v)) $\langle xyz \rangle = x\bar{y}^T z - z\bar{y}^T x$, which is a basis for the effective action of multiple membranes in M-theory.

One of the tripotents is given by

$$c = \begin{pmatrix} Id & 0 \\ 0 & 0 \end{pmatrix}, \tag{25}$$

where Id is the $n \times n$ identity matrix ($n \leq N, M$). Because any element can be decomposed as

$$x = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(A - A^T) & B \\ 0 & D \end{pmatrix} + \begin{pmatrix} \frac{1}{2}(A + A^T) & 0 \\ C & 0 \end{pmatrix}, \tag{26}$$

the Peirce decomposition is given by

$$\begin{aligned}
 U_1(c) &= \left\{ \begin{pmatrix} \frac{1}{2}(A - A^T) & B \\ 0 & D \end{pmatrix} \right\}, \\
 U_{1/2}(c) &= \left\{ \begin{pmatrix} \frac{1}{2}(A + A^T) & 0 \\ C & 0 \end{pmatrix} \right\}, \\
 U_0(c) &= 0.
 \end{aligned} \tag{27}$$

As in Theorem 5, we can expand any element as $x = \sum(\lambda_{ij}E_{ij} + \mu_{ij}\sqrt{-1}E_{ij})$, where E_{ij} denotes that element (i, j) is

1 and other elements are zero, and E_{ij} and $\sqrt{-1}E_{ij}$ are tripotents, that is, $\langle E_{ij}E_{ij}E_{ij} \rangle = E_{ij}$ and $\langle \sqrt{-1}E_{ij}\sqrt{-1}E_{ij}\sqrt{-1}E_{ij} \rangle = \sqrt{-1}E_{ij}$.

Example 2. Let $S_{n,2k}^*$ be the set of all $n \times 2k$ matrices with the operation

$$\langle XYZ \rangle = X(\bar{Y})^T Z - Z(\bar{Y})^T X + Z\phi(X)^T \phi(\bar{Y}), \tag{28}$$

where $X = (x_1, x_2)$, x_i is an $n \times k$ matrix and $\phi(X) = (x_2, -x_1)$.

Then $S_{n,2k}^*$ is a Hermitian $(-1, -1)$ -Freudenthal-Kantor triple system. In fact, it satisfies conditions (0), (i), (ii), (iii), (iv), and (v) in Section 2. This triple system with $k = 1$ reduces to the $sp(2N) \oplus u(1)$ Hermitian 3-algebra.

In this example, we can show that $e = (E_{ij}, E_{ij})$, where $i \neq l$ are tripotents and any element can be expanded by using part of them.

Example 3. Let $D_{N,M}^*$ be the set of all $N \times M$ matrices with the operation

$$\langle xyz \rangle = x\bar{y}^T z - z\bar{y}^T x + z\bar{x}^T \bar{y} - \bar{y}x^T z. \tag{29}$$

Then $D_{N,M}^*$ is a $*$ -generalized Jordan triple system. In fact, it satisfies conditions (0), (i) with $\epsilon = -1$, (iii), and (iv) in Section 2 but does not satisfy (ii) and (v). This is an extension of the triple system in Example 1.

5. Application to a Field Theory

In this section, we apply a $*$ -generalized Jordan triple system to a field theory.

We start with

$$\begin{aligned}
 S &= \int d^3x \text{tr} \left(-D_\mu Z^A \overline{D^\mu Z^A}^T + L\epsilon^{\mu\nu\lambda} \right. \\
 &\quad \times \left(-A_{\mu\bar{b}c} \partial_\nu A_{\lambda\bar{d}a} \bar{T}^{T\bar{d}} \left[T^c, \bar{T}^{\bar{b}}, T^a \right] \right. \\
 &\quad \left. + \frac{2}{3} A_{\mu\bar{d}a} A_{\nu\bar{b}c} A_{\lambda\bar{f}e} \right. \\
 &\quad \left. \times \left[T^c, \bar{T}^{\bar{b}}, T^a \right] \overline{[T^f, \bar{T}^{\bar{e}}, T^d]} \right) \Bigg),
 \end{aligned} \tag{30}$$

where

$$D_\mu Z^A = \partial_\mu Z^A - A_{\mu\bar{b}a} \left[T^a, \bar{T}^{\bar{b}}, Z^A \right]. \tag{31}$$

Z^A and A_μ are matter and gauge fields, respectively. A runs from 1 to p , whereas μ runs from 0 to 2. $A_{\mu\bar{a}b}$ satisfies $\overline{A_{\mu\bar{a}b}} = -A_{\mu\bar{b}a}$. This action is invariant under the transformations generated by the operator $L(x, y)$. This action describes the bosonic parts of the effective actions of supermembranes in M-theory if appropriate potential terms of Z^A are added and a Lorentzian Lie 3-algebra or a Hermitian 3-algebra is applied. Here, we apply the $*$ -generalized Jordan triple

system $[x, \bar{y}, z] := \langle xyz \rangle = (x\bar{y}^T - \bar{y}x^T)z - z(\bar{y}^T x - x^T \bar{y})$ in Example 3 to this action.

The covariant derivative is explicitly written as

$$\mathbf{D}_\mu Z^A = \partial_\mu Z^A - iA_\mu^L Z^A + iZ^A A_\mu^R, \quad (32)$$

where $A_\mu^R := -iA_{\mu\bar{b}a}(\bar{T}^{T\bar{b}} T^a - T^{Ta} \bar{T}^{\bar{b}})$ and $A_\mu^L := -iA_{\mu\bar{b}a}(T^a \bar{T}^{T\bar{b}} - \bar{T}^{\bar{b}} T^{Ta})$ are real antisymmetric matrices, which generate the $o(N)$ and $o(M)$ Lie algebras, respectively. The action can be rewritten in a covariant form with respect to $o(N)$ and $o(M)$ and we obtain a Chern-Simons gauge theory:

$$\begin{aligned} S = \int d^3x \operatorname{tr} \bigg(& -(\partial_\mu Z^A - iA_\mu^L Z^A + iZ^A A_\mu^R) \\ & \times \overline{(\partial_\mu Z_A - iA_\mu^L Z_A + iZ_A A_\mu^R)}^T \\ & + L\epsilon^{\mu\nu\lambda} \left(\frac{1}{2} (A_\mu^L \partial_\nu A_\lambda^L - A_\mu^R \partial_\nu A_\lambda^R) \right. \\ & \left. + \frac{i}{3} (A_\mu^L A_\nu^L A_\lambda^L - A_\mu^R A_\nu^R A_\lambda^R) \right) \bigg). \end{aligned} \quad (33)$$

In this action, A_μ^L and A_μ^R transform as adjoint representations of $o(N)$ and $o(M)$, respectively, whereas Z^A transforms as a bifundamental representation of $o(N) \oplus o(M)$,

$$\begin{aligned} \delta A_\mu^R &= [i\Lambda^R, A_\mu^R], \\ \delta A_\mu^L &= [i\Lambda^L, A_\mu^L], \\ \delta Z^A &= i\Lambda^L Z^A - Z^A (i\Lambda^R), \end{aligned} \quad (34)$$

where gauge parameters Λ^R and Λ^L are defined in the same way as A_μ^R and A_μ^L , respectively.

Next, let us examine whether the novel Higgs mechanism works in this theory when $M = N$. By redefining the gauge fields as

$$\begin{aligned} A_\mu^L &= A_\mu + B_\mu, \\ A_\mu^R &= A_\mu - B_\mu, \end{aligned} \quad (35)$$

we can separate a nondynamical mode B_μ as

$$\begin{aligned} S = \int d^3x \operatorname{tr} \bigg(& -(D_\mu Z^A - i\{B_\mu, Z^A\}) \\ & \times \overline{(D_\mu Z^A - i\{B_\mu, Z^A\})}^T \\ & + L\epsilon^{\mu\nu\lambda} \left(B_\mu F_{\nu\lambda} + \frac{2i}{3} B_\mu B_\nu B_\lambda \right) \bigg), \end{aligned} \quad (36)$$

where

$$\begin{aligned} \{B_\mu, Z^A\} &= B_\mu Z^A + Z^A B_\mu, \\ D_\mu Z^A &= \partial_\mu Z^A - i[A_\mu, Z^A], \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]. \end{aligned} \quad (37)$$

We divide Z^A into two real matrices as

$$Z^A = iX^A + X^{p+A}, \quad (38)$$

and consider fluctuations around a background solution as $X^p = \nu I + \bar{X}^p$. If we rescale L and B_μ as

$$\begin{aligned} L &= \mathcal{O}(\nu), \\ B_\mu &= \mathcal{O}\left(\frac{1}{\nu}\right), \end{aligned} \quad (39)$$

we obtain

$$\begin{aligned} S = \int d^3x \operatorname{tr} \bigg(& -D_\mu Z^A \overline{(D^\mu Z_A)}^T - 4B_\mu D^\mu X^{2p} \nu \\ & - 4B_\mu B^\mu \nu^2 + L\epsilon^{\mu\nu\lambda} B_\mu F_{\nu\lambda} \bigg) + \mathcal{O}\left(\frac{1}{\nu}\right). \end{aligned} \quad (40)$$

By using the equation of motion of B_μ ,

$$B^\mu = \frac{L}{8\nu^2} \epsilon^{\mu\nu\lambda} F_{\nu\lambda} - \frac{1}{2\nu} D^\mu X^{2p} + \mathcal{O}\left(\frac{1}{\nu^2}\right), \quad (41)$$

the action reduces to

$$S \longrightarrow \int d^3x \operatorname{tr} \left(-g^2 F_{\mu\nu}^2 - (D_\mu X^i)^2 \right) \quad (42)$$

as $\nu \rightarrow \infty$, where $g = L/\nu$ and i run from 1 to $2p - 1$. Therefore, we conclude that the novel Higgs mechanism works in the Chern-Simons gauge theory with the $*$ -generalized Jordan triple system in Example 3 with $M = N$, and we obtain a Yang-Mills theory in this limit.

6. Concluding Remarks

For our triple systems, we emphasized in this paper that there exists a generalized concept of Hermitian 3-algebras, which have played crucial roles in M-theory. In particular, we find the novel Higgs mechanism also works in the generalized concept.

HKFTSs and $*$ -generalized Jordan triple systems $\langle xyz \rangle$ can be regarded as left and right actions on z by two Lie algebras, as one can see in the examples. For the novel Higgs mechanism, it is necessary that the two Lie algebras are the same because we need to define summation between the Lie algebras as in Section 5. Thus, the novel Higgs mechanism works only when Example 3 is applied among the examples in this paper.

Our principal physical motivation for generalizing the Hermitian 3-algebras to HKFTSs is to regularize the Nambu 3-algebra, which is defined by a Nambu bracket. The area-preserving diffeomorphism (APD) algebra defined by a Poisson bracket on a torus is regularized by the $u(N)$ algebra in the 't Hooft base. On the other hand, the APD algebra defined on a sphere is regularized by the $u(N)$ algebra in another base. That is, regularizations of infinite-dimensional algebras depend on the basis of the corresponding finite algebras. Although Hermitian 3-algebras ($u(N) \oplus u(M)$ and $sp(2N) \oplus u(1)$ Hermitian 3-algebras) are strong candidates for

the regularization of the Nambu 3-algebra because of their large symmetries and their relations to M2-branes, it is not clear how to choose a base for the regularization. To study this systematically, we generalized the Hermitian 3-algebras to HFKTs ($u(N) \oplus u(M)$ to Example 1 and $sp(2N) \oplus u(1)$ to Example 2), and we found a nilpotent basis for them. The next step is to generalize the Nambu 3-algebra to HFKTs and to find a corresponding nilpotent base. Then by restricting the HFKTs to the 3-algebras, we may prove that the Nambu 3-algebra is regularized by a Hermitian 3-algebra.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The work of Matsuo Sato is supported in part by Grant-in-Aid for Young Scientists (B) no. 25800122 from JSPS.

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