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Research Article

Noncommutative Phase Space Schrödinger Equation with Minimal Length

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We consider the Schrödinger equation under an external magnetic field in two-dimensional noncommutative phase space with an explicit minimal length relation. The eigenfunctions are reported in terms of the Jacobi polynomials, and the explicit form of energy eigenvalues is reported.

1. Introduction

Although the advent of Schrödinger equation dates back to many decades ago [1–3], the interest in the study of the equation has been renewed due to the noncommutative phase space commutations relations and the so-called minimal length (alternatively called generalized) uncertainty relation. The noncommutativity between space-time coordinates was first explained by Snyder [4] and provides us with motivating scenarios in string and M-theories [5, 6] as well as modern cosmology [7–9]. The application of the formulation is not limited to these fields and noncommutative theories are capable of explaining the IR/UV mixing and nonlocality [10], Lorentz violation [11], and new physics at very small scales [12, 13]. Extensive studies on the incorporation of ordinary quantum mechanics and classical mechanics with the noncommutative relations can be found in [14–16]. On the other hand, the modification of our ordinary uncertainty relation, which is inspired by the string theory, quantum gravity, and doubly special relativity [17–20], has become an appealing case of study in the recent years. In the minimal length formulation, the ordinary Heisenberg relation is modified by an ad hoc term and is written as $[\widehat{X},\widehat{P}] = i(1+\beta\widehat{P}^2)$, where β is a small parameter determined from a fundamental theory such as string theory [21, 22]. Such a commutation relation corresponds to the following uncertainty relation: $\Delta\widehat{X} \geq (1/2)((1/\Delta\widehat{P}) + \beta\Delta\widehat{P})$, which implies the existence of a minimal length $\Delta\widehat{X}_0 = \sqrt{\beta}$ [21, 22]. Recently, the universality of quantum gravity corrections was studied by Das and Vagenas. They indicated the existence of a minimum measurable length and the related generalized uncertainty principle (GUP) [23]. In another interesting paper, Ali et al. considered a GUP consistent with string theory, black hole physics, and doubly special relativity [24].

In the present work, we combine these topics within the framework of nonrelativistic Schrödinger equation. The organization of this work is as follows. In Section 2, we consider the Schrödinger equation in the presence of magnetic field in noncommutative phase space (NCPS). In Section 3, we study this problem with minimal length relations. In Section 4, we consider some special cases of the solutions to check the validity of the results.

2. The Two-Dimensional Schrödinger Equation in Noncommutative Phase Space

The free particle Schrödinger equation in two spatial dimensions ($\hbar = c = 1$) reads

$$\left(\left(\vec{P} - q\vec{A}\right)^2 - 2ME_{n,m}\right)\psi(x,y) = 0, \tag{1}$$

where M is mass of the particle. Using the operator relations

$$P_x = \frac{1}{i}\frac{d}{dx}, \qquad P_y = \frac{1}{i}\frac{d}{dy} \tag{2}$$

and recalling that the vector potential is written as $\vec{A} = (-(B/2)y, (B/2)x, 0)$, where B is magnetic field, (1) takes the form

$$\left(-\left(P_x^{(\mathrm{NC})} + \frac{qB}{2}y^{(\mathrm{NC})}\right)^2 - \left(P_y^{(\mathrm{NC})} - \frac{qB}{2}x^{(\mathrm{NC})}\right)^2 + 2ME_{n,m}\right)\psi(x,y) = 0. \tag{3}$$

In NCPS the position operators are satisfied in the following relations [25–27]:

$$\left[x^{\mu(\mathrm{NC})}, x^{\nu(\mathrm{NC})}\right] = i\theta^{\mu\nu}, \tag{4}$$

where $\theta^{\mu\nu}$ is an antisymmetric tensor and is of space dimension $(L)^2$. The other commutation relations appear as

$$\left[x^{i(\mathrm{NC})}, x^{j(\mathrm{NC})}\right] = i\theta^{ij}, \qquad \left[P^{i(\mathrm{NC})}, P^{j(\mathrm{NC})}\right] = i\eta^{ij}, \qquad \left[x^{i(\mathrm{NC})}, P^{j(\mathrm{NC})}\right] = i\hbar_{\mathrm{eff}}\delta^{ij} \tag{5}$$

with the effective plank constant being

$$\hbar_{\mathrm{eff}} = \left(1 + \frac{\theta\eta}{4}\right), \qquad x^{i(\mathrm{NC})} = \hat{x}_i - \frac{\theta^{ij}\hat{p}_j}{2}, \qquad P^{i(\mathrm{NC})} = \hat{p}_i + \frac{\eta^{ij}\hat{x}_j}{2}, \tag{6}$$

where $\theta^{ij} = \varepsilon^{ijk}\theta_k, \theta_k = (0,0,\theta), \eta^{ij} = \varepsilon^{ijk}\eta_k$, and $\eta_k = (0,0,\eta)$ are the noncommutative parameters. The NCPS parameters are related to the commutative space parameters via

$$P_x^{(\mathrm{NC})} = \hat{P}_x^{(\mathrm{C})} + \frac{\eta\hat{y}^{(\mathrm{C})}}{2}, \qquad x^{(\mathrm{NC})} = \hat{x}^{(\mathrm{C})} - \frac{\theta\hat{P}_y^{(\mathrm{C})}}{2}, \qquad P_y^{(\mathrm{NC})} = \hat{P}_y^{(\mathrm{C})} - \frac{\eta\hat{x}^{(\mathrm{C})}}{2}, \qquad y^{(\mathrm{NC})} = \hat{y}^{(\mathrm{C})} + \frac{\theta\hat{P}_x^{(\mathrm{C})}}{2}. \tag{7}$$

By inserting (7) into (3), we obtain the equation in commutative space as

$$\left\{2ME_{n,m} - \left(1 + \frac{qB\theta}{4}\right)^2\left(\hat{P}_x^{(\mathrm{C})2} + \hat{P}_y^{(\mathrm{C})2}\right) - \left(\frac{\eta}{2} + \frac{qB}{2}\right)^2\left(\hat{x}^{(\mathrm{C})2} + \hat{y}^{(\mathrm{C})2}\right) + \left(\frac{\eta}{2} + \frac{qB}{2}\right) \times \left(1 + \frac{qB\theta}{4}\right)L_z\right\}\psi_{n,m}(\hat{x},\hat{y}) = 0, \tag{8}$$

in which $L_z = i(x(\partial/\partial y) - y(\partial/\partial x))$ is angular momentum.

3. The Problem with a Minimal Length

For the sake of simplicity, we bring the problem into the momentum space. Recalling that $\hat{x}_i = i\hbar(1+\beta P^2)\partial/\partial P_i$, $\hat{P}_i = P_i$ [28],

$$\hat{y} = i\left(1+\beta P^2\right)\frac{\partial}{\partial P_y}, \qquad \hat{x} = i\left(1+\beta P^2\right)\frac{\partial}{\partial P_x}, \qquad \hat{P}_x = P_x, \qquad \hat{P}_y = P_y, \tag{9}$$

as well as writing

$$\hat{P}_y^{(\mathrm{C})} = P\sin\varphi, \qquad \hat{P}_x^{(\mathrm{C})} = P\cos\varphi, \qquad P^2 = \hat{P}_x^{(\mathrm{C})2} + \hat{P}_y^{(\mathrm{C})2}, \tag{10}$$

and $\psi_{n,m} = \psi_{n,m}(P,\varphi) = U_n'(P)Q_m(\varphi)$, where $Q_m(\varphi) = e^{im\varphi}$, the Schrödinger equation appears as

$$\left\{\frac{d^2}{dP^2} + \frac{1}{P}\frac{d}{dP} + \frac{-m^2}{P^2} + \frac{2ME_{n,m}}{((\eta/2)+(qB/2))^2(1+\beta P^2)^2} - \frac{\beta}{1+\beta P^2} - \frac{(1+(qB\theta/4))^2P^2}{((\eta/2)+(qB/2))^2(1+\beta P^2)^2} + \frac{2\beta P}{1+\beta P^2}\frac{d}{dP} + \frac{(1+(qB\theta/4))m}{((\eta/2)+(qB/2))(1+\beta P^2)}\right\}U_n' = 0 \tag{11}$$

or, via $U_n' = P^{-1/2}U_n$, in the form

$$\left\{\frac{d^2}{dP^2} + \frac{(1/4)-m^2}{P^2} + \frac{2ME_{n,m}}{((\eta/2)+(qB/2))^2(1+\beta P^2)^2} - \frac{\beta}{1+\beta P^2} - \frac{(1+(qB\theta/4))^2P^2}{((\eta/2)+(qB/2))^2(1+\beta P^2)^2} + \frac{2\beta P}{1+\beta P^2}\frac{d}{dP} + \frac{(1+(qB\theta/4))m}{((\eta/2)+(qB/2))(1+\beta P^2)}\right\}U_n = 0. \tag{12}$$

A transformation of the form $p = (1/\sqrt{\beta})tg((x/2)+(\pi/4))$ brings (12) into

$$\left\{\frac{d^2}{dx^2} + \frac{\left((1/4)-m^2\right)\left(1+tg^2((x/2)+(\pi/4))\right)^2}{4tg^2((x/2)+(\pi/4))} + \frac{2ME_{n,m}}{4\beta((\eta/2)+(qB/2))^2} - \frac{\left(1+tg^2((x/2)+(\pi/4))\right)}{4} - \frac{(1+(qB\theta/4))^2tg^2((x/2)+(\pi/4))}{4\beta^2((\eta/2)+(qB/2))^2}, + \frac{\left(1+tg^2((x/2)+(\pi/4))\right)(1+(qB\theta/4))m}{4\beta((\eta/2)+(qB/2))}\right\}U_n = 0. \tag{13}$$

By simplifying (13), we obtain the following equation:

$$\left\{\frac{d^2}{dx^2}+\frac{\left((1/4)-m^2\right)}{4\sin^2\left((x/2)+(\pi/4)\right)\cos^2\left((x/2)+(\pi/4)\right)}\right.$$
$$+\frac{2ME_{n,m}}{4\beta\left((\eta/2)+(qB/2)\right)^2}-\frac{1}{4\cos^2\left((x/2)+(\pi/4)\right)}$$
$$-\frac{\left(1+(qB\theta/4)\right)^2\sin^2\left((x/2)+(\pi/4)\right)}{4\beta^2\left((\eta/2)+(qB/2)\right)^2\cos^2\left((x/2)+(\pi/4)\right)}$$
$$\left.+\frac{\left(1+(qB\theta/4)\right)m}{4\beta\left((\eta/2)+(qB/2)\right)\cos^2\left((x/2)+(\pi/4)\right)}\right\}U_n=0, \tag{14}$$

which, via the change of variable $z=\cos^2((x/2)+(\pi/4))$, takes the more familiar form

$$\frac{d^2U_n(z)}{dz^2}+\frac{1/2-z}{z(1-z)}\frac{dU_n(z)}{dz}+\frac{1}{z^2(1-z)^2}\times\left\{\alpha_1z^2+\alpha_2z+\alpha_3\right\}U_n(z)=0, \tag{15}$$

with

$$\alpha_1=\frac{2ME_{n,m}}{4\beta\left((\eta/2)+(qB/2)\right)^2}+\frac{\left(1+(qB\theta/4)\right)^2}{4\beta^2\left((\eta/2)+(qB/2)\right)^2},$$
$$\alpha_2=\frac{1}{4}+\frac{2ME_{n,m}}{4\beta\left((\eta/2)+(qB/2)\right)^2}+\frac{\left(1+(qB\theta/4)\right)^2}{2\beta^2\left((\eta/2)+(qB/2)\right)^2}-\frac{m\left(1+(qB\theta/4)\right)}{4\beta\left((\eta/2)+(qB/2)\right)},$$
$$\alpha_3=-\frac{1}{16}+\frac{1}{4}+\frac{m^2}{4}+\frac{\left(1+(qB\theta/4)\right)^2}{4\beta^2\left((\eta/2)+(qB/2)\right)^2}-\frac{m\left(1+(qB\theta/4)\right)}{4\beta\left((\eta/2)+(qB/2)\right)}. \tag{16}$$

Equation (15) possesses the eigenfunctions

$$U_n(z)=z^{(1/4)+\sqrt{(1/16)+\alpha_3}}(1-z)^{(1/4)+\sqrt{(1/16)+\alpha_1+\alpha_3-\alpha_2}}\times P_n^{\left(2\sqrt{(1/16)+\alpha_3},2\sqrt{(1/16)+\alpha_1+\alpha_3-\alpha_2}\right)}(1-2z), \tag{17}$$

and its energy eigenvalues can be derived from

$$n(n+1)+(2n+1)\times\left(\sqrt{\frac{1}{16}+\alpha_1+\alpha_3-\alpha_2}+\sqrt{\frac{1}{16}+\alpha_3}\right)-\alpha_2+\frac{3}{8}+2\alpha_3+2\sqrt{\left(\frac{1}{16}+\alpha_3\right)\left(\frac{1}{16}+\alpha_1+\alpha_3-\alpha_2\right)}=0, \tag{18}$$

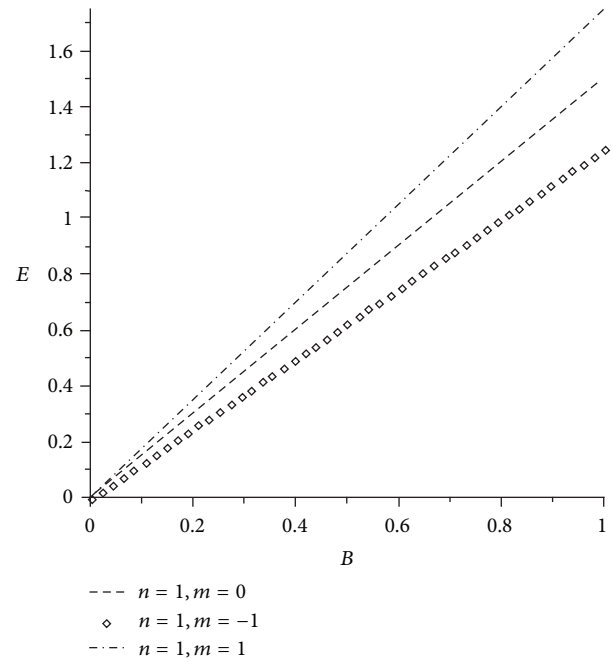

Figure 1: The energy eigenvalues versus B ($\theta=0.000003, \eta=0.000001, M=1$).

which yields

$$E_{n,m}=\left[\frac{\beta(\eta+qB)^2n(n+1)}{2M}+\frac{\beta(\eta+qB)^2m(2n+1)}{4M}+\frac{(\eta+qB)^2(2n+1+m)}{2M}\times\sqrt{\beta^2\left(\frac{1+m^2}{4}\right)+\frac{\left(1+(qB\theta/4)\right)^2}{(\eta+qB)^2}-\frac{\beta m\left(1+(qB\theta/4)\right)}{2(\eta+qB)}}+\frac{\beta(\eta+qB)^2\left(1+m^2\right)}{4M}-\frac{m\left(1+(qB\theta/4)\right)(\eta+qB)}{4M}\right]. \tag{19}$$

4. Special Cases

Let us now check the special cases. First, when the minimal length is absent, that is, $\beta=0$, the energy relation (19) reduces to

$$E_{n,m}=\left[\frac{(\eta+qB)(2n+1+m)\left(1+(qB\theta/4)\right)}{2M}-\frac{m\left(1+(qB\theta/4)\right)(\eta+qB)}{4M}\right]. \tag{20}$$

The energy has plotted the energy eigenvalues versus B in Figure 1. The energy relation in the special cases of $\theta=0$ and

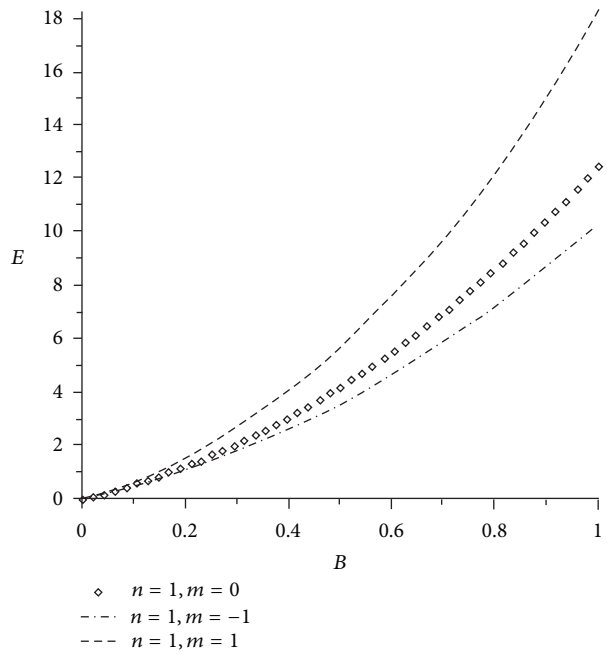

Figure 2: The energy eigenvalues versus B ($\beta = 1.5, \eta = 0.000001, M = 0.3$).

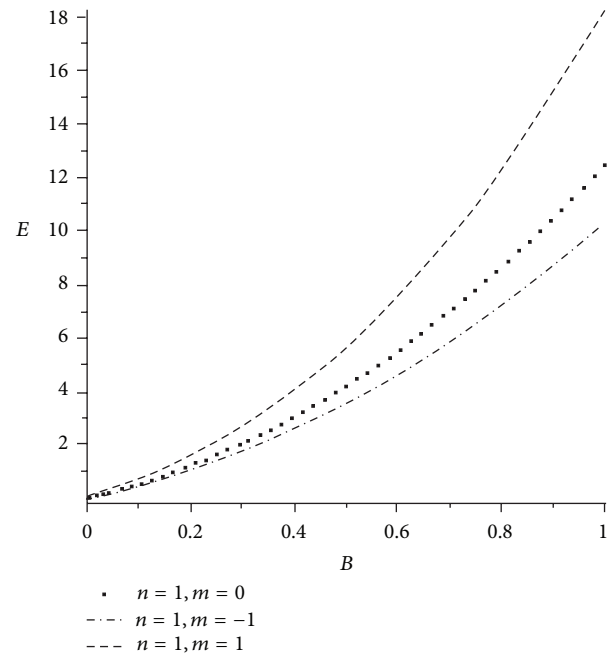

Figure 3: The energy eigenvalues versus B ($\beta = 1.5, \theta = 0.000001, M = 0.3$).

$\eta = 0$, that is, for vanishing NCPS parameters, respectively, yields

$$E_{n,m} = \left[\frac{\beta(\eta + qB)^2 n(n+1)}{2M} + \frac{\beta(\eta + qB)^2 m(2n+1)}{4M} + \frac{(\eta + qB)^2 (2n+1+m)}{2M} \times \sqrt{\beta^2 \left(\frac{1+m^2}{4}\right) + \frac{1}{(\eta + qB)^2} - \frac{\beta m}{2(\eta + qB)}} + \frac{\beta(\eta + qB)^2 (1+m^2)}{4M} - \frac{m(\eta + qB)}{4M} \right],$$

$$E_{n,m} = \left[\frac{2\beta q^2 B^2 n(n+1) + \beta q^2 B^2 m(2n+1)}{4M} + \frac{\beta q^2 B^2 (1+m^2) - mqB(1 + (qB\theta/4))}{4M} + \frac{q^2 B^2 (2n+1+m)}{2M} \times \left(\beta^2 \left(\frac{1+m^2}{4}\right) + \frac{(1 + (Bq\theta/4))^2}{q^2 B^2} - \frac{\beta m (1 + (qB\theta/4))}{2qB} \right)^{1/2} \right]. \quad (21)$$

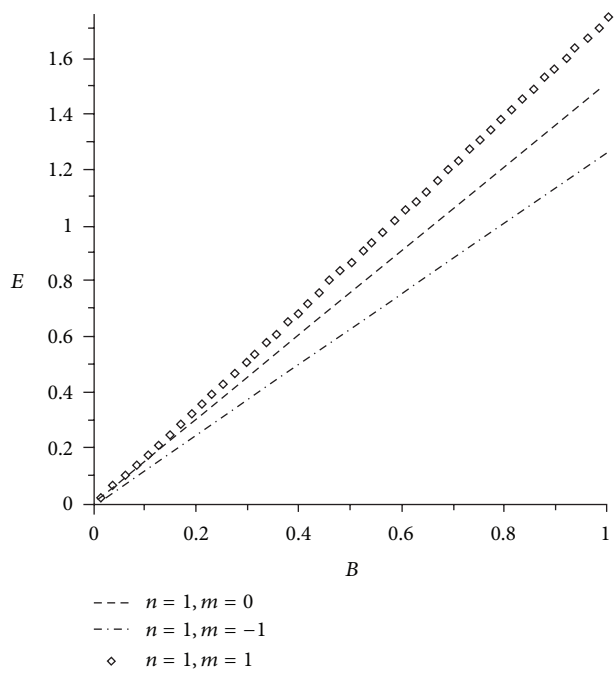

Figure 4: The energy eigenvalues versus B ($M = 1$).

We have depicted in Figures 2 and 3 the energy versus B. Finally, when both the noncommutative and minimal length parameters are absent, that is, when $\beta = \theta = \eta = 0$, and also $(q^2 B^2/8M) \to 1/2M\omega^2$, the energy spectrum takes the form

$$E_{n,m} = \omega_0 (2n + 1 + m). \quad (22)$$

This is the known form of energy in [29–31]. In Figure 4, we have shown the energy values versus B.

5. Conclusion

In this study, we considered the Schrödinger equation incorporated with two recent trends in the field: the noncommutative formulation and the minimal length uncertainty relation. After two novel transformations, we transformed the equation into a familiar form and reported the solutions in terms of the Jacobi polynomials. Our results, in the special cases of the engaged parameters, yield the previous results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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