Research Article

Exact Axially Symmetric Solution in $f(T)$ Gravity Theory

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Received 22 August 2013; Revised 29 January 2014; Accepted 12 February 2014; Published 16 March 2014

Academic Editor: George Siopsis

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A general tetrad field with sixteen unknown functions is applied to the field equations of $f(T)$ gravity theory. An analytic vacuum solution is derived with two constants of integration and an angle Φ that depends on the angle coordinate ϕ and radial coordinate r . The tetrad field of this solution is axially symmetric and the scalar torsion vanishes. We calculate the associated metric of the derived solution and show that it represents Kerr spacetime. Finally, we show that the derived solution can be described by two local Lorentz transformations in addition to a tetrad field that is the square root of the Kerr metric. One of these local Lorentz transformations is a special case of Euler's angles and the other represents a boost when the rotation parameter vanishes.

1. Introduction

The main postulate of the present model of cosmology is the truth of Einstein's theory of general relativity (GR) in all ranges: beginning from observed phenomena in the solar system to the large scale structure of the universe. Nevertheless, recent data from distant supernovae Ia [1–5] are explained as a guide of overdue time acceleration in the enlargement average of our universe. If GR is to be true in all ranges, the best fix to the observational data demands the presence of a positive cosmological constant. There are different theoretical problems linked to the presence of a cosmological constant [6–9] and the most substantial of which is the lack of a quantum theoretical way to compute the inferred value from cosmological data. Many researchers attempted to bypass the issues of cosmological constant with different demonstrations. To elucidate overdue time accelerated enlargement of the universe one can either modify “lumber” part, that is, matter part, or modify “marble” part, that is, the geometric construction of Einstein's field equations. In the previous process one adds the energy-momentum component of dark energy whose equation of state is given by $p/\rho \approx 1$ to the lumber part of Einstein's equation of motion. Alternatively, in the latter process, one modifies the GR, and this changes the marble part of Einstein's equations of motion.

Such modifications could be done in two ways: one can either increase the number of degrees of freedom by adding new gravitational fields into the theory or change the form of the gravity action without adding new fields. In the latter case it is shown that, after using a conformal mapping, the dependence of the effective Lagrangian on the curvature is not only singular but also bifurcates into several almost Einsteinian spaces, distinguished only by a different effective gravitational strength and cosmological constant [10, 11]. Metric or vierbein field still remains the only gravitational degree of freedom in the second process.

The most important set of modifications of Einstein-Hilbert Lagrangian are the $f(R)$ theories of gravity [12–15]. In those theories of gravity one employs a function of curvature scalar as the Lagrangian density. However, the field equations of $f(R)$ gravity models turn out to be fourth order differential equations in the metric formalism and therefore they are difficult to analyze. With a similar line of thought, one can also modify teleparallel equivalent of general relativity (TEGR). This theory is defined on a Wittenböck spacetime, which has vanishing curvature but has a nonvanishing value of torsion. Lagrangian density is equivalent to the torsion scalar and the field equations of teleparallel gravity which are exactly the same as Einstein's equations in any background metric [16–24]. One can modify teleparallel gravity by having

a Lagrangian density equivalent to a function of torsion scalar. This is first done in the context of Born-Infeld [25–27]. However, it is possible to have any function $f(T)$. Then one has $f(T)$ theories of gravity [28]. These theories are more manageable compared to $f(R)$ theories, because their field equations are second order differential equations. It is important that searching for exact solutions is a fundamental step to obtain a new field theory. Exact solutions allow full control of the systems and can contribute to the well-formulation and well-position of the Cauchy problem (for a discussion on this point see [29]).

It is the aim of the present work, within the framework of higher-torsion theories, $f(T)$, to find an analytic, axially symmetric solution. In Section 2, a brief review of the $f(T)$ theory is provided. In Section 3, a general tetrad field with sixteen unknown functions, is provided and application to the field equation of $f(T)$ is demonstrated. A new analytic, vacuum, axially symmetric solution with two constants of integration is derived in Section 3. In Section 4, we rewrite the tetrad field of the solution as two local Lorentz transformations and a diagonal tetrad of the Kerr metric spacetime. The final section is devoted to a summary of the main results.

2. Brief Review of $f(T)$

In a spacetime with absolute parallelism, the parallel vector fields h_a^μ [30] define the nonsymmetric affine connection

$$\Gamma_{jk}^i \stackrel{\text{def.}}{=} h_\mu^i h_{j,k}^\mu = -h_j^\mu h_{\mu,i,k}, \quad (1)$$

where $h_{\mu i,j} = \partial_j h_{\mu i}$ (we use the Latin indices $i, j, \dots$ for local holonomic spacetime coordinates and the Greek indices $\alpha, \beta, \dots$ to label (co)frame components). The curvature tensor defined by Γ_{jk}^i given by (1) is identically vanishing. The metric tensor g_{ij} is defined by

$$g_{ij} \stackrel{\text{def.}}{=} O_{\mu\nu} h_i^\mu h_j^\nu, \quad (2)$$

where $O_{\mu\nu} = (-1, +1, +1, +1)$ is the metric of Minkowski spacetime. The torsion components and the contortion are defined as

$$\begin{aligned} T_{jk}^i &\stackrel{\text{def.}}{=} \Gamma_{kj}^i - \Gamma_{jk}^i = h_\mu^i (\partial_j h_{k,\mu} - \partial_k h_{j,\mu}), \\ K_k^{ij} &\stackrel{\text{def.}}{=} -\frac{1}{2} (T_k^{ij} - T_k^{ji} - T_k^{ij}), \end{aligned} \quad (3)$$

where the contortion equals the difference between Weitzenböck and Levi-Civita connection; that is, $K_{jk}^i = \Gamma_{jk}^i - \left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\}$.

The tensor S_i^{jk} is defined as

$$S_i^{jk} \stackrel{\text{def.}}{=} \frac{1}{2} (K^{jk}_i + \delta_i^j T^{ak}_a - \delta_i^k T^{aj}_a). \quad (4)$$

The torsion scalar is defined as

$$T \stackrel{\text{def.}}{=} T^i_{jk} S_i^{jk}. \quad (5)$$

Similar to the $f(R)$ theory, one can define the action of $f(T)$ theory as

$$\mathcal{L}(h^\mu_i, \Phi_A) = \int d^4 x h \left[\frac{1}{16\pi} f(T) + \mathcal{L}_{\text{Matter}}(\Phi_A) \right], \quad (6)$$

$$\text{where } h = \sqrt{-g} = \det(h^\mu_i),$$

and we assumed the units in which $G = c = 1$ and Φ_A are the matter fields.

Considering action (6) as a function of the fields h^μ_i and Φ_A and putting the variation of the function with respect to the field h^μ_i to be vanishing one can obtain the following equation of motion [28]:

$$\begin{aligned} S_i^{aj} T_{,a} f(T)_{TT} + [h^{-1} h^\mu_i \partial_b (h h_\mu^a S_a^{bj}) - T^a_{bi} S_a^{jb}] f(T)_T \\ - \frac{1}{4} \delta_i^j f(T) = -4\pi \mathcal{T}_i^j, \end{aligned} \quad (7)$$

where $T_{,a} = \partial T / \partial x^a$, $f(T)_T = \partial f(T) / \partial T$, $f(T)_{TT} = \partial^2 f(T) / \partial T^2$, and $\mathcal{T}_\mu^\nu$ is the energy-momentum tensor. In this study we are interested in studying the vacuum case of $f(T)$ theory; that is, $\mathcal{T}_i^j = 0$.

3. Axially Symmetric Solution in $f(T)$ Gravity Theory

We assume that the manifold possesses sixteen unknown functions and has the form

$$(h^\alpha_i) = \begin{pmatrix} A_1(r, \theta, \phi) & A_2(r, \theta, \phi) & A_3(r, \theta, \phi) & A_4(r, \theta, \phi) \\ B_1(r, \theta, \phi) \sin \theta \cos \phi & B_2(r, \theta, \phi) \sin \theta \cos \phi & B_3(r, \theta, \phi) \cos \theta \cos \phi & B_4(r, \theta, \phi) \sin \phi \sin \theta \\ C_1(r, \theta, \phi) \sin \theta \sin \phi & C_2(r, \theta, \phi) \sin \theta \sin \phi & C_3(r, \theta, \phi) \cos \theta \sin \phi & C_4(r, \theta, \phi) \cos \phi \sin \theta \\ D_1(r, \theta, \phi) \cos \theta & D_2(r, \theta, \phi) \cos \theta & D_3(r, \theta, \phi) \sin \theta & D_4(r, \theta, \phi) \cos \theta \end{pmatrix}, \quad (8)$$

where $A_i(r, \theta, \phi)$, $B_i(r, \theta, \phi)$, $C_i(r, \theta, \phi)$ and $D_i(r, \theta, \phi)$, and $i = 1, \dots, 4$ are sixteen unknown functions of the radial

coordinate, r , the azimuth coordinate θ , and the angle ϕ . Applying tetrad field given by (8) to field equations (7),

we get sixteen lengthy nonlinear partial differential equations (these calculations are checked using Maple software 15). These equations are not consistent with each other due to the appearance of the terms of f_T and f_{TT} . If these terms are vanishing (i.e., $f_T = 0$ and $f_{TT} = 0$), we can solve the nonlinear partial differential equations. If they are not zero, it is not easy to solve these equations. Therefore, we are going to use some constraints. These constraints are the coefficients of f_T and f_{TT} , which are also lengthy. We here write the solution that satisfies these constraints (briefly we will write $A_i \equiv A_i(r, \theta, \phi)$, $B_i \equiv B_i(r, \theta, \phi)$, $C_i \equiv C_i(r, \theta, \phi)$, and $D_i \equiv D_i(r, \theta, \phi)$):

$$\begin{aligned} A_1 &= 1 - \frac{c_1}{\Sigma}, & A_2 &= \frac{c_1}{\Delta}, & A_3 &= 0, \\ A_4 &= -\frac{c_1 c_2 \sin^2 \theta}{\Sigma}, & B_1 &= \frac{c_1 \cos \Phi}{\cos \phi \Sigma}, & B_2 &= \frac{\omega - c_1 \cos \Phi}{\Delta \cos \phi}, \\ B_3 &= \frac{\omega}{\cos \phi}, & B_4 &= -\frac{\omega_1 \Sigma - c_1 c_2 \sin^2 \theta \cos \Phi}{\sin \phi \Sigma}, \\ C_1 &= \frac{c_1 \sin \Phi}{\sin \phi \Sigma}, & C_2 &= \frac{\omega_1 - c_1 \sin \Phi}{\Delta \sin \phi}, & C_3 &= \frac{\omega_1}{\sin \phi}, \\ C_4 &= \frac{\omega \Sigma + c_1 c_2 \sin^2 \theta \sin \Phi}{\cos \phi \Sigma}, & D_1 &= \frac{c_1}{\Sigma}, \\ D_2 &= 1 + \frac{c_1}{\Delta}, & D_3 &= -r, & D_4 &= \frac{c_1 c_2 \sin^2 \theta}{\Sigma}, \end{aligned} \quad (9)$$

where ω , ω_1 , Σ , Δ , and Φ are defined by

$$\begin{aligned} \omega &\stackrel{\text{def.}}{=} r \cos \Phi + c_2 \sin \Phi, & \omega_1 &\stackrel{\text{def.}}{=} r \sin \Phi - c_2 \cos \Phi, \\ \Sigma &= r + \frac{c_2^2 \cos^2 \theta}{r}, & \Delta &= r - 2c_1 + \frac{c_2^2}{r}, \\ \Phi &= \phi - \left(c_2 \tan^{-1} \left(\frac{r - c_1}{\sqrt{c_2^2 - c_1^2}} \right) \left(\sqrt{c_2^2 - c_1^2} \right)^{-1} \right), \end{aligned} \quad (10)$$

where c_1 and c_2 are two constants of integration. Tetrad field (8) using (10) is axially symmetric; that is, it is invariant under the transformation:

$$\begin{aligned} \bar{\Phi} &\longrightarrow \Phi + \delta\Phi, & \bar{h}_i^0 &\longrightarrow h_i^0, \\ \bar{h}_i^1 &\longrightarrow h_i^1 \cos \delta\Phi - h_i^2 \sin \delta\Phi, \\ \bar{h}_i^2 &\longrightarrow h_i^1 \sin \delta\Phi + h_i^2 \cos \delta\Phi, & \bar{h}_i^3 &\longrightarrow h_i^3. \end{aligned} \quad (11)$$

Using (8) and (9) in (5) we get a vanishing value of the scalar torsion. Using (8) and (9) in (7) we get an exact vacuum solution to the field equations of $f(T)$ provided that

$$f(0) = 0, \quad f_T(0) \neq 0, \quad f_{TT} \neq 0. \quad (12)$$

The metric associated with tetrad (8) after using solution (9) can be written as

$$\begin{aligned} ds^2 &= \alpha dt^2 - \frac{\Sigma}{\Delta} dr^2 - \Sigma d\theta^2 - \alpha_1 d\phi^2 - \frac{2c_1 c_2 \sin^2 \theta}{r \Sigma} dt d\phi, \\ &\text{where } \alpha = 1 - \frac{c_1}{\Sigma}, \\ \alpha_1 &= \sin^2 \theta \left(r^2 + c_2^2 + \frac{2c_1 c_2^2 \sin^2 \theta}{\Sigma} \right). \end{aligned} \quad (13)$$

This is the Kerr spacetime [31] provided that $c_1 = 2M$ and $c_2 = a$.

4. Decomposition of Derived Solution

In this section we are going to study the internal properties of tetrad (8) using (9). For this purpose we use the fact that any tetrad can be written as

$$(h^\alpha_i) = (\Lambda^\alpha_\beta) (h^\beta_i), \quad (14)$$

where (Λ^α_β) is a local Lorentz transformation satisfying

$$(\Lambda^\alpha_\beta) \eta_{\alpha\gamma} (\Lambda^\gamma_\rho) = \eta_{\beta\rho}, \quad (15)$$

and $(h^\beta_i)_1$ is another tetrad field which reproduces the same metric of the tetrad field (h^α_i) .

Now we use (14) to rewrite tetrad field (8).

The general form of Euler's angles in three dimensions has the form [32]

$$(h^\alpha_i)_{\text{Euler}} = \begin{pmatrix} -\cos \psi \cos \phi_1 - \sin \psi \cos \theta_1 \sin \phi_1 & \cos \psi \sin \phi_1 - \sin \psi \cos \theta_1 \cos \phi_1 & -\sin \psi \sin \theta_1 \\ \sin \psi \cos \phi_1 + \cos \psi \cos \theta_1 \sin \phi_1 & -\sin \psi \sin \phi_1 + \cos \psi \cos \theta_1 \cos \phi_1 & \cos \psi \sin \theta_1 \\ -\sin \theta_1 \sin \phi_1 & -\sin \theta_1 \cos \phi_1 & \cos \theta_1 \end{pmatrix}. \quad (16)$$

Therefore, tetrad field (8) can be rewritten as

$$(h^\alpha_i) = (\Lambda^\alpha_\beta) (\Lambda^\beta_\gamma)_1 (h^\gamma_i)_d, \quad (17)$$

where

$$(\Lambda^\alpha_\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{r\omega \sin \theta}{\sqrt{\Sigma(r^2 + a^2)}} & \frac{\omega \cos \theta}{\sqrt{\Sigma}} & -\frac{\omega_1}{\sqrt{r^2 + a^2}} \\ 0 & \frac{r\omega_1 \sin \theta}{\sqrt{\Sigma(r^2 + a^2)}} & \frac{\omega_1 \cos \theta}{\sqrt{\Sigma}} & \frac{\omega}{\sqrt{r^2 + a^2}} \\ 0 & \frac{\sqrt{r^2 + a^2} \cos \theta}{\sqrt{\Sigma}} & \frac{-r \sin \theta}{\sqrt{\Sigma}} & 0 \end{pmatrix}, \quad (18)$$

$$(\Lambda^\alpha_\beta)_1 = \begin{pmatrix} \frac{A_5}{2\sqrt{\Delta\Sigma A_6}} & \frac{rM}{\sqrt{\Delta\Sigma}} & 0 & -\frac{rMa \sin \theta}{\sqrt{\Sigma A_6}} \\ \frac{rM\sqrt{r^2 + a^2}}{\sqrt{\Delta A_6}} & \frac{\sqrt{\Delta}}{\sqrt{r^2 + a^2}} & 0 & \frac{rMa \sin \theta}{\sqrt{(r^2 + a^2) A_6}} \\ 0 & 0 & 1 & 0 \\ -\frac{rMa\sqrt{r^2 + a^2} \sin \theta}{\sqrt{\Delta\Sigma A_6}} & -\frac{rMa \sin \theta}{\sqrt{(r^2 + a^2) \Delta\Sigma}} & 0 & \frac{A_7}{2\sqrt{(r^2 + a^2) \Sigma A_6}} \end{pmatrix}, \quad (19)$$

where

$$\begin{aligned} A_5 &= \Delta a^2 \cos^2 \theta + 2(r^2 + 2rM)(r^2 + a^2), \\ A_6 &= \Delta a^2 \cos^2 \theta + (r^2 + 2rM)a^2 + r^4, \\ A_7 &= (2r^2 - 2rM + 2a^2)a^2 \cos^2 \theta + 2(r^2 + 2rM)r^4, \end{aligned} \quad (20)$$

$$(h^\gamma_i)_d = \begin{pmatrix} \sqrt{\frac{A_5}{\Sigma}} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{\Sigma}{\Delta}} & 0 & 0 \\ 0 & 0 & \sqrt{\Sigma} & 0 \\ \frac{2rMa \sin^2 \theta}{\sqrt{\Sigma A_5}} & 0 & 0 & -\sin \theta \sqrt{\frac{\Sigma \Delta}{A_5}} \end{pmatrix}. \quad (21)$$

Equations (18) and (19) are local Lorentz transformations; that is, they satisfy (15).

Equation (18) is a special case of Euler's angle provided that

$$\begin{aligned} \theta_1 &= \frac{\pi}{2}, & \phi_1 &= \cos^{-1} \left(\frac{r \sin \theta}{\sqrt{\Sigma}} \right), \\ \psi &= \cos^{-1} \left(\frac{\omega}{\sqrt{r^2 + a^2}} \right). \end{aligned} \quad (22)$$

The values of (22) reproduce (16). Equation (21) has the same metric of tetrad field (8) after using (9). We call (21) the diagonal form of tetrad field (8) with the help of (9). Finally (19) is the boost of tetrad (8). Equation (19) shows that the boost transformation depends on the gravitational mass, the charge parameter, and the rotation parameter. When the charge and rotation parameters are vanishing, the boost transformation depends on the mass only. This is acceptable in a physical sense because in curved spacetime we do not have a uniform velocity (as in special relativity) and therefore the acceleration is produced by the gravitational mass.

To understand this, we examine the methodology of the acceleration calculation. We will take advantage of "generalized acceleration" [33, 34], taking the frame $e_\alpha = h^i_\alpha \partial_i$, which is dual to the coframe $\vartheta^\alpha = h^i_\alpha dx^i$. The zero component of the frame $e_{\hat{0}} = u$ usually explained as the 4-velocity components of an observer and the frame then plays a comoving reference system of the observer. The "generalized acceleration" is usually given by

$$\Psi_\alpha{}^\beta = h^\beta_i \frac{\tilde{D}h^i_\alpha}{ds}, \quad \text{where } \tilde{D}h^i_\alpha = dh^i_\alpha + \tilde{\Gamma}^i_j h^j_\alpha \quad (23)$$

Which represents a covariant derivative with respect to Christoffel connection. It operates on the vector index "i," while the local tetrad index is a label of the four components of the frame.

The quantity $\Psi_\alpha{}^\beta$ as shown by (23) is not a tensor. In fact, employing the link of the components of the connection in

different frames, one can show that $h_{i\beta}\tilde{D}h^i_\alpha = h_{i\beta}dh^i_\alpha + h_{i\beta}\tilde{\Gamma}^i_j h^j_\alpha = \tilde{\Gamma}^\alpha_\beta$. Explicitly,

$$\Psi_\alpha^\beta = u] \tilde{\Gamma}^\alpha_\beta = \tilde{\Gamma}^\beta_{\hat{0}\alpha}. \quad (24)$$

The condition that

$$\Psi_\alpha^\beta = 0 \quad (25)$$

gives an inertial reference system. One can show that $\Psi_0^\beta = A^\beta = h_i^\beta A^i$, where $A^i = u^k \tilde{\nabla}_i u^i$ is the acceleration. Therefore, if $\Psi_\alpha^\beta = 0$, then the observer has no acceleration, and the vanishing of the spatial components Ψ_a^b ($a, b, \dots = 1, 2, 3$) means that the comoving system of an observer is not rotating.

Using (25) in (21) we get the following nonvanishing components of the generalized acceleration Φ_a^b :

$$\begin{aligned} \Psi_0^1 &= -\frac{\sqrt{r^2 + a^2} A_8}{4\Sigma\Delta^{3/2} A_6}, & \Psi_0^3 &= -\frac{\sqrt{r^2 + a^2} a \sin\theta A_9}{4\Sigma^{3/2} \Delta^{3/2} A_6}, \\ \Psi_1^3 &= \frac{-rMa \sin\theta A_{10}}{2\Sigma\Delta^{3/2} [a^2 + r^2] A_6}, \end{aligned} \quad (26)$$

where

$$\begin{aligned} A_8 &= -rM \left(2[a^2 + rM] Ma^2 \cos^2\theta + 2rM [Ma^2 + r^3] \right), \\ A_9 &= -rM \left(2[rM - a^2] Ma^2 \cos^2\theta \right. \\ &\quad \left. + 2r^2 M [2r^2 - 3rM + a^2] \right), \\ A_{10} &= -rM \left(\Delta [r^2 - a^2] Ma^2 \cos^2\theta \right. \\ &\quad \left. + 2Mr^4 [2r^2 - 3rM + 2a^2] \right). \end{aligned} \quad (27)$$

Equation (26) shows that the boost transformation has two nonvanishing components, A^1 and A^2 , and one spatial acceleration. The two components A^1 and A^2 depend on the gravitational mass and the rotation parameter. When the gravitational mass vanishes, then the two components of acceleration vanish and we return to special relativity. On the other hand the component of the spatial acceleration depends on the gravitational mass and the rotation parameter and when the gravitational mass or the rotation parameter vanishes, those accelerations are zero.

5. Main Results and Discussion

The $f(T)$ gravitational theory is a modification of TEGR that aims to resolve some recent observation problems. It is not easy to find analytic solutions within this theory. We have studied the vacuum case of $f(T)$. The field equations have been applied to a nondiagonal, axially symmetric, tetrad field having sixteen unknown functions. The resulting

nonlinear partial differential equations are complicated and we derived the explicate form of the sixteen functions that satisfy these field equations. This solution contains an angle Φ which depends on the coordinate angle ϕ and on a function of the radial coordinate. This radial coordinate function depends on two constants of integration. When one of these constants vanishes, the radial coordinate function also vanishes. An important property of this solution is that it has a zero scalar torsion (i.e.; $T = 0$), it also satisfies the field equations of $f(T)$ provided that

$$f_{TT}(0) \neq 0, \quad f_T(0) \neq 0, \quad f(0) = 0. \quad (28)$$

In addition, this solution has axial symmetry: the components of its tetrad fields are invariant under the change of the angle Φ . We have shown that the associated metric of this solution gives Kerr spacetime.

To understand the internal construction of the derived solution we wrote it as two local Lorentz transformations and a diagonal tetrad (diagonal means the square root of the metric). We have shown that one of the local Lorentz transformations is related to Euler's angles under certain conditions. This local Lorentz transformation depends on the rotation parameter a and it reduces to the one studied in [35, 36] when this parameter vanishes. In this case, the radial coordinate function (that appears in the definition of the angle Φ) will also vanish. This ensures that the special form of Euler's angle is axially symmetric [37]. The other local Lorentz transformation depends on both the gravitational mass and the rotational parameter. When the rotational parameter is zero this transformation reduces to a boost transformation (i.e., inertia). This inertia contributes to physics in TEGR [38–40]. It was shown [33] by a regularization procedure that one can remove such inertia.

However, it seems that this inertia plays a crucial role in $f(T)$ gravitational theories. The main reason for this is the special transformation of Euler's angles which give zero value for the scalar torsion. The dependency of the second local Lorentz transformation on the rotation parameter makes this transformation not a boost in straightforward. The components of this matrix are not only time-spatial components but also spatial-spatial components. These depend on the rotation parameter and when this parameter is zero, this results in a boost transformation with only time-spatial components.

The way of writing tetrad field (8) as two local Lorentz transformations and diagonal one as given by (18), (19), and (21) is not unique. For example, If the form of tetrad field (21) is changed, then the form of (18) and (19) also changes.

Let us explain this by rewriting tetrad (8) with the help of (9) in the form

$$(h^\alpha_i) = (\Lambda^\alpha_\beta)_2 (\Lambda^\beta_\gamma)_3 (h^\gamma_i)_{dd}, \quad (29)$$

where

$$\begin{aligned}
 (\Lambda^\alpha_\beta)_2 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{r\omega \sin \theta}{\sqrt{\Sigma(r^2 + a^2)}} & \frac{\omega \cos \theta}{\sqrt{\Sigma}} & -\frac{\omega_1 \cos \theta}{\sqrt{r^2 + a^2}} \\ 0 & \frac{r\omega_1 \sin \theta}{\sqrt{\Sigma(r^2 + a^2)}} & \frac{\omega_1 \cos \theta}{\sqrt{\Sigma}} & \frac{\omega}{\sqrt{r^2 + a^2}} \\ 0 & \frac{\cos \theta \sqrt{r^2 + a^2}}{\sqrt{\Sigma}} & \frac{-r \sin \theta}{\sqrt{\Sigma}} & 0 \end{pmatrix}, \\
 (\Lambda^\alpha_\beta)_3 &= \begin{pmatrix} -\frac{A_{11}}{\sqrt{\Sigma}A_{12}} & -\frac{rM}{\sqrt{\Delta\Sigma}} & 0 & \frac{rMa \sin \theta A_{13}}{\Sigma^{3/2} \sqrt{\Delta}A_{12}} \\ \frac{rM}{\sqrt{(r^2 + a^2)\Delta}} & \frac{A_{14}}{\sqrt{(r^2 + a^2)\Delta}} & 0 & \frac{rMa \sin \theta A_{15}}{\Sigma \sqrt{A_{12}\Delta}(r^2 + a^2)} \\ 0 & 0 & 1 & 0 \\ \frac{rMa \sin \theta}{\sqrt{A_{12}\Sigma}(r^2 + a^2)} & -\frac{rMa \sin \theta}{\sqrt{\Delta\Sigma}(r^2 + a^2)} & 0 & \frac{A_{16}}{2\Sigma^{3/2} \sqrt{A_{12}\Delta}(r^2 + a^2)} \end{pmatrix},
 \end{aligned} \tag{30}$$

where

$$\begin{aligned}
 A_{11} &= r^2 + a^2 \cos^2 \theta - rM, \\
 A_{12} &= r^2 + a^2 \cos^2 \theta - 2rM, \\
 A_{13} &= 3r^3 - 4rM + 3a^2 \cos^2 \theta, \\
 A_{14} &= r^2 - rM + a^2, \\
 A_{15} &= r^2 - 4rM + a^2 \cos^2 \theta,
 \end{aligned}$$

$$\begin{aligned}
 A_{16} &= A_{14}a^4 \cos^4 \theta \\
 &\quad + \{[4r^2 - 2rM]a^2 + 4r^4 \\
 &\quad - 6r^3M + 8r^2M^2\}a^2 \cos^2 \theta \\
 &\quad + \{2r^4 - 2r^3M - 8r^2M^2\}a^2 \\
 &\quad + 2r^6 - 4r^5M,
 \end{aligned} \tag{31}$$

$$(h^y_i)_{dd} = \begin{pmatrix} \sqrt{\frac{\Delta - a^2 \sin^2 \theta}{\Sigma}} & 0 & 0 & \frac{-a \sin^2 \theta [\Delta - (a^2 + r^2)]}{\sqrt{\Sigma}(\Delta - a^2 \sin^2 \theta)} \\ 0 & \sqrt{\frac{\Sigma}{\Delta}} & 0 & 0 \\ 0 & 0 & \sqrt{\Sigma} & 0 \\ 0 & 0 & 0 & \sin \theta \Sigma \sqrt{\frac{\Delta}{\Sigma(\Delta - a^2 \sin^2 \theta)}} \end{pmatrix}. \tag{32}$$

Equation (32) is a special case of Euler's angle provided that

$$\begin{aligned}
 \theta_1 &= \frac{\pi}{2}, \quad \phi_1 = \cos^{-1} \left(\frac{-r \sin \theta}{\Sigma} \right), \\
 \psi &= -\cos^{-1} \left(\frac{\omega}{\sqrt{r^2 + a^2}} \right).
 \end{aligned} \tag{33}$$

To sum up, solution (9) justifies that any GR solution remains valid in $f(T)$ theories having $f_T(0) = 1$ whenever the geometry admits a tetrad field with vanishing scalar torsion, T [27].

Appendix

Calculations of Torsion and Contortion Tensors of Solution (9)

Using solution (9) the nonvanishing components of the torsion tensor have the form

$$\begin{aligned}
 T^0_{01} &= - \left(M \left(a^2 \cos^2 \theta \left[r^2 \{ 4Mr - r^2 - 2a^2 \} \right. \right. \right. \\
 &\quad \left. \left. \left. + a^2 \{ 2Mr - a^2 \} \right] \right. \right. \\
 &\quad \left. \left. - r^4 \{ 2Mr - 2a^2 - r^2 \} \right) \right)
 \end{aligned}$$

$$\begin{aligned}
& +2a^2r^2[a^2-2Mr])) \\
& \times (r^4\Sigma^2\Delta^2)^{-1}, \\
T^0_{02} = & -\frac{Ma^2\sin 2\theta(Mr-r^2-a^2)}{r^2\Sigma^2\Delta}, \\
T^0_{03} = & -\frac{4M^3a\sin^2\theta}{\Sigma^2\Delta}, \quad T^1_{12} = \frac{Ma^2\sin 2\theta}{2r\Sigma\Delta}, \\
T^1_{13} = & \frac{Ma\sin^2\theta[a^2\Delta\cos^2\theta+2Mr^2]}{r^2\Sigma^2\Delta}, \\
T^1_{10} = & -\frac{M[a^2[Mr-a^2]\cos^2\theta-[2M-r]r^3]}{r^3\Sigma^2\Delta}, \\
T^1_{23} = & -\frac{Ma\sin 2\theta[a^2+2r^2+a^2\cos^2\theta]}{2r\Sigma^2}, \\
T^1_{20} = & \frac{Ma^2\sin 2\theta}{2r\Sigma^2}, \quad T^1_{30} = -\frac{Ma\sin^2\theta}{2\Sigma^2}, \\
T^2_{12} = & -\frac{M}{\Sigma\Delta}, \quad T^2_{13} = \frac{Ma(r^2+a^2)\sin 2\theta}{2r^2\Sigma^2\Delta}, \\
T^2_{10} = & \frac{Ma^2\sin 2\theta}{2r^2\Sigma^2\Delta}, \quad T^2_{23} = \frac{Ma\sin^2\theta}{\Sigma^2}, \\
T^2_{20} = & \frac{M}{\Sigma^2}, \quad T^2_{30} = -\frac{Ma\sin 2\theta}{2r\Sigma^2}, \\
T^3_{12} = & \frac{Ma\cos\theta(r[2M-r]-a^2\cos^2\theta)}{r^2\sin\theta\Sigma\Delta^2}, \\
T^3_{13} = & -\left(M(a^4r\Delta\cos^4\theta\right. \\
& +a^2(r^2+a^2)(Mr-a^2)\cos^2\theta \\
& -r^4[2Mr-2a^2-r^2] \\
& \left.+a^2r^2[a^2-4Mr])\right) \\
& \times (r^4\Sigma^2\Delta^2)^{-1}, \\
T^3_{10} = & -\frac{Ma(a^2(2Mr-a^2)\cos^2\theta-r^2[2Mr-a^2-2r^2])}{r^4\Sigma^2\Delta^2}, \\
T^3_{23} = & \frac{Ma^2\sin 2\theta[2Mr-3r^2-2a^2-a^2\cos^2\theta]}{r^2\Sigma^2\Delta}, \\
T^3_{20} = & -\frac{Ma\cos\theta[2Mr-r^2-2a^2+a^2\cos^2\theta]}{\sin\theta r^2\Sigma^2\Delta}, \\
T^3_{30} = & -\frac{M[r[2M-r]-a^2\cos^2\theta]}{r\Sigma^2\Delta}.
\end{aligned} \tag{A.1}$$

The nonvanishing components of the contortion tensor have the form

$$\begin{aligned}
K^{01}_1 = & \frac{M(r^2-a^2\cos^2\theta)}{r^2\Sigma\Delta}, \quad K^{01}_2 = -\frac{Ma^2\sin 2\theta}{r\Sigma^2}, \\
K^{01}_3 = & -\frac{Ma\sin^2\theta(a^4\cos^2\theta-2r^4-r^2a^2)}{r^3\Sigma^3},
\end{aligned}$$

$$\begin{aligned}
K^{01}_0 = & \frac{M(a^4\cos^2\theta-r^4-r^2a^2\sin^2\theta)}{r^3\Sigma^3}, \\
K^{02}_1 = & -\frac{Ma^2\sin 2\theta(\Delta-2M)}{2r^2\Sigma^2\Delta^2}, \quad K^{02}_2 = -\frac{M(r^2+a^2)}{r\Sigma^2\Delta}, \\
K^{02}_3 = & (Ma\sin 2\theta[r(\Delta-2M)a^2\cos^2\theta \\
& +a^2(2Mr-3a^2r^2-r^4-2a^2)]) \\
& \times (2r^3\Sigma^3\Delta)^{-1}, \\
K^{02}_0 = & -\frac{Ma^2\sin 2\theta}{r^2\Sigma^3}, \quad K^{03}_1 = \frac{Ma}{r\Sigma\Delta^2}, \\
K^{03}_2 = & \frac{Ma\cot\theta}{r\Sigma\Delta}, \quad K^{03}_3 = -\frac{M}{\Sigma^2\Delta}, \\
K^{12}_1 = & -\frac{Ma^2\sin 2\theta}{2r^2\Sigma^2\Delta}, \quad K^{12}_2 = -\frac{M}{\Sigma^2}, \\
K^{12}_3 = & \frac{Ma\sin 2\theta}{2r\Sigma^2}, \quad K^{13}_1 = -\frac{Ma^3\cos^2\theta}{r^3\Sigma^2\Delta}, \\
K^{13}_2 = & \frac{Ma\cot\theta}{r\Sigma^2}, \\
K^{13}_3 = & \frac{M(r^4+a^2r^2-a^4\sin^2\theta\cos^2\theta)}{r^3\Sigma^3}, \\
K^{13}_0 = & \frac{Ma(r^2-a^2\cos^2\theta)}{r^3\Sigma^3}, \\
K^{21}_2 = & -\frac{M}{\Sigma^2}, \quad K^{21}_3 = -\frac{Ma\sin 2\theta}{r\Sigma^2}, \\
K^{23}_1 = & -\frac{Ma\cot\theta(2r^2-4Mr+a^2[\cos^2\theta+1])}{r^3\Sigma^2\Delta^2}, \\
K^{23}_2 = & -\frac{Ma}{r\Sigma^2\Delta}, \\
K^{23}_3 = & \frac{Ma^2\sin 2\theta(4Mr-3r^2-a^2[2+\cos^2\theta])}{2r^3\Sigma^3\Delta}, \\
K^{23}_0 = & -\frac{2Ma\cot\theta}{r^2\Sigma^3}.
\end{aligned} \tag{A.2}$$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This work is partially supported by the Egyptian Ministry of Scientific Research under project ID 24-2-12.

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