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Research Article

Inflation and Topological Phase Transition Driven by Exotic Smoothness

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We will discuss a model which describes the cause of inflation by a topological transition. The guiding principle is the choice of an exotic smoothness structure for the space-time. Here we consider a space-time with topology $S^3 \times \mathbb{R}$. In case of an exotic $S^3 \times \mathbb{R}$, there is a change in the spatial topology from a 3-sphere to a homology 3-sphere which can carry a hyperbolic structure. From the physical point of view, we will discuss the path integral for the Einstein-Hilbert action with respect to a decomposition of the space-time. The inclusion of the boundary terms produces fermionic contributions to the partition function. The expectation value of an area (with respect to some surface) shows an exponential increase; that is, we obtain inflationary behavior. We will calculate the amount of this increase to be a topological invariant. Then we will describe this transition by an effective model, the Starobinski or R^2 model which is consistent with the current measurement of the Planck satellite. The spectral index and other observables are also calculated.

1. Introduction

General relativity (GR) has changed our understanding of space-time. In parallel, the appearance of quantum field theory (QFT) has modified our view of particles, fields, and the measurement process. The usual approach for the unification of QFT and GR, to a quantum gravity, starts with a proposal to quantize GR and its underlying structure, space-time. There is a unique opinion in the community about the relation between geometry and quantum theory: the geometry as used in GR is classical and should emerge from a quantum gravity in the limit (Planck's constant tends to zero). Most theories went a step further and try to get a space-time from quantum theory. Then, the model of a smooth manifold is not suitable to describe quantum gravity. But, there is no sign for a discrete space-time structure or higher dimensions in current experiments. Hence, quantum gravity based on the concept of a smooth manifold should also able to explain the current problems in the standard cosmological model (ΛCDM) like the appearance of dark energy/matter or the correct form of inflation. But before we are going in this direction we will motivate the usage of the smooth manifold as our basic concept.

When Einstein developed GR, his opinion about the importance of general covariance changed over the years. In 1914, he wrote a joint paper with Grossmann. There, he rejected general covariance by the now famous hole argument. But after a painful year, he again considered general covariance now with the insight that there is no meaning in referring to the *space-time point A* or the *event A*, without further specifications. Therefore the measurement of a point without a detailed specification of the whole measurement process is meaningless in GR. The reason is simply the diffeomorphism invariance of GR which has tremendous consequences. Furthermore, GR do not depend on the topology of space-time. All restrictions on the topology of the space-time were formulated using additional physical conditions like causality (see [1]). This ambiguity increases in the 80's when the first examples of exotic smoothness structures in dimension 4 were found. The (smooth) atlas of a smooth 4-manifold M is called the smoothness structure (unique up to diffeomorphisms). One would expect that there

is only one smooth atlas for M; all other possibilities can be transformed into each other by a diffeomorphism. But in contrast, the deep results of Freedman [2] on the topology of 4-manifolds combined with Donaldson's work [3] gave the first examples of nondiffeomorphic smoothness structures on 4-manifolds including the well-known $\mathbb{R}^4$. Much of the motivation can be found in the FQXI essay [4, 5]. Here we will discuss another property of the exotic smoothness structure: its quantum geometry in the path integral.

Diffeomorphism invariance is the most important property of the Einstein-Hilbert action with far reaching consequences [6]. One of our results is a close relation between geometry and foliation to exotic smoothness [7, 8]. In the particular example of the exotic $\mathbb{R}^4$, we discussed the exotic smoothness structures as a manifestation of quantum gravity (by using string theory [9, 10]). This exotic $\mathbb{R}^4$ has some interesting properties as first noted by Brans [11, 12]. More importantly as shown by Sładkowski [13], the exotic $\mathbb{R}^4$ has a nontrivial curvature in contrast to the flat standard $\mathbb{R}^4$. It was the first result that an exotic $\mathbb{R}^4$ can be seen as a source of gravity (or it must contain sources of gravity). Sładkowski [14–16] went further and showed a relation to particle physics also related to quantum gravity. But why there is a relation to quantum gravity? In [17] we presented the first idea to understand this relation which was further extended in [18]. An exotic 4-manifold like $S^3 \times \mathbb{R}$ is also characterized by the property that there is no smoothly embedded 3-sphere but a topological embedded one. This topological S^3 is wildly embedded; that is, the image of the embedding must be triangulated by an infinite polyhedron. In [18], we proved that the (deformation) quantization of a usual (or tame) embedding is a wild embedding which can be seen as a quantum state. But then any exotic 4-manifold can be interpreted as a quantum state of the 4-manifold with standard smoothness structure. From this point of view, the calculation of the path integral in quantum gravity has to include the exotic smoothness structures. Usually it is hopeless to make these calculations. But by using the close relation of exotic smoothness to hyperbolic geometry, one has a chance to calculate geometric expressions like the expectation value of the surface area. In this paper we will show that this expectation value has an inflationary behavior; that is, the area grows exponentially (along the time axis). Therefore quantum gravity (in the sense of exotic smoothness) can be the root of inflation.

2. Space-Time and Smoothness

From the mathematical point of view, the space-time is a smooth 4-manifold endowed with a (smooth) metric as basic variable for general relativity. The existence question for Lorentz structure and causality problems (see Hawking and Ellis [1]) give further restrictions on the 4-manifold: causality implies noncompactness; Lorentz structure needs a nonvanishing normal vector field. Both concepts can be combined in the concept of a global hyperbolic 4-manifold M having a Cauchy surface $\mathcal{S}$ so that $M = \mathcal{S} \times \mathbb{R}$.

All these restrictions on the representation of space-time by the manifold concept are clearly motivated by physical questions. Among these properties there is one distinguished element: the smoothness. Usually one starts with a topological 4-manifold M and introduces structures on them. Then one has the following ladder of possible structures:

$$\begin{aligned} \text{Topology} &\longrightarrow \text{piecewise-linear (PL)} \longrightarrow \text{Smoothness} \\ &\longrightarrow \text{bundles, Lorentz, Spin, etc.} \\ &\longrightarrow \text{metric, geometry, } \ldots . \end{aligned} \tag{1}$$

We do not want to discuss the first transition, that is, the existence of a triangulation on a topological manifold. But we remark that the existence of a PL structure implies uniquely a smoothness structure in all dimensions smaller than 7 [19]. Here we have to consider the following steps to define a space-time.

(i) Fix a topology for the space-time M.

(ii) Fix a smoothness structure, that is, a maximal differentiable atlas $\mathcal{A}$.

(iii) Fix a smooth metric or get one by solving the Einstein equation.

The choice of a topology never fixes the space-time uniquely; that is, there are two space-times with the same topology which are not diffeomorphic. The main idea of the paper is the introduction of exotic smoothness structures into space-time. If two manifolds are homeomorphic but nondiffeomorphic, they are *exotic* to each other. The smoothness structure is called an *exotic smoothness structure*.

In dimension four there are many examples of compact 4-manifolds with countable infinite nondiffeomorphic smoothness structures and many examples of noncompact 4-manifolds with uncountable infinite many nondiffeomorphic smoothness structures. But in contrast, the number of nondiffeomorphic smoothness structures is finite for any other dimension [19]. As an example, we will consider the space-time $S^3 \times \mathbb{R}$ having uncountable many nondiffeomorphic smoothness structures in the following.

3. The Path Integral in Exotic $S^3 \times \mathbb{R}$

For simplicity, we consider general relativity without matter (using the notation of topological QFT). Space-time is a smooth oriented 4-manifold M which is noncompact and without boundary. From the formal point of view (no divergences of the metric) one is able to define a boundary ∂M at infinity. The classical theory is the study of the existence and uniqueness of (smooth) metric tensors g on M that satisfy the Einstein equations subject to suitable boundary conditions. In the first order Hilbert-Palatini formulation, one specifies an $\mathrm{SO}(1,3)$-connection A together with a cotetrad field e rather than a metric tensor. Fixing $A|_{\partial M}$ at the boundary, one can derive first-order field equations in the interior (now called *bulk*) which are equivalent to the Einstein equations provided that the cotetrad is nondegenerate. The theory is invariant under space-time diffeomorphisms $M \to M$.

In the particular case of the space-time $M = S^3 \times \mathbb{R}$ (topologically), we have to consider a smooth 4-manifolds $M_{i,f}$ as parts of M whose boundary $\partial M_{i,f} = \Sigma_i \sqcup \Sigma_f$ is the disjoint union of two smooth 3-manifolds Σ_i and Σ_f to which we associate Hilbert spaces $\mathscr{H}_j$ of 3-geometries, $j = i, f$. These contain suitable wave functionals of connections $A|_{\Sigma_j}$. We denote the connection eigenstates by $|A|_{\Sigma_j}\rangle$. The path integral,

$$\begin{aligned}&\left\langle A|_{\Sigma_f} \left|T_M\right| A|_{\Sigma_i}\right\rangle \\ &\quad = \int_{A|\partial M_{i,f}} DA\, De\, \exp\left(\frac{i}{\hbar} S_{EH}\left[e, A, M_{i,f}\right]\right),\end{aligned} \tag{2}$$

is the sum over all connections A matching $A|_{\partial M_{i,f}}$ and over all e. It yields the matrix elements of a linear map $T_M : \mathscr{H}_i \to \mathscr{H}_f$ between states of 3-geometry. Our basic gravitational variables will be cotetrad e_a^I and connection A_a^{IJ} on space-time M with the index a to present it as 1-forms and the indices I, J for an internal vector space V (used for the representation of the symmetry group). Cotetrads e are "square-roots" of metrics and the transition from metrics to tetrads is motivated by the fact that tetrads are essential if one is to introduce spinorial matter. e_a^I is an isomorphism between the tangent space $T_p(M)$ at any point p and a fixed internal vector space V equipped with a metric η_{IJ} so that $g_{ab} = e_a^I e_b^J \eta_{IJ}$. Here we used the action

$$\begin{aligned}&S_{EH}\left[e, A, M_{i,f}, \partial M_{i,f}\right] \\ &\quad = \int_{M_{i,f}} \epsilon_{IJKL}\left(e^I \wedge e^J \wedge (dA + A \wedge A)^{KL}\right) \\ &\qquad + \int_{\partial M_{i,f}} \epsilon_{IJKL}\left(e^I \wedge e^J \wedge A^{KL}\right),\end{aligned} \tag{3}$$

in the notation of [20, 21]. Here the boundary term $\epsilon_{IJKL}(e^I \wedge e^J \wedge A^{KL})$ is equal to twice the trace over the extrinsic curvature (or the mean curvature). For fixed boundary data, (2) is a diffeomorphism invariant in the bulk. If $\Sigma_i = \Sigma_f$ are diffeomorphic, we can identify $\Sigma = \Sigma_i = \Sigma_f$ and $\mathscr{H} = \mathscr{H}_i = \mathscr{H}_f$; that is, we close the manifold $M_{i,f}$ by identifying the two boundaries to get the closed 4-manifold M'. Provided that the trace over $\mathscr{H}$ can be defined, the partition function,

$$Z\left(M'\right) = \mathrm{tr}_{\mathscr{H}} T_M = \int DA\, De\, \exp\left(\frac{i}{\hbar} S_{EH}\left[e, A, M, \partial M\right]\right), \tag{4}$$

where the integral is now unrestricted, is a dimensionless number which depends only on the diffeomorphism class of the smooth manifold M'. In case of the manifold $M_{i,f}$, the path integral (as transition amplitude) $\langle A|_{\Sigma_f}|T_M|A|_{\Sigma_i}\rangle$ is the diffeomorphism class of the smooth manifold relative to the boundary. But the diffeomorphism class of the boundary is unique and the value of the path integral depends on the topology of the boundary as well on the diffeomorphism class of the interior of $M_{i,f}$. Therefore we will shortly write

$$\left\langle \Sigma_f \left|T_M\right| \Sigma_i\right\rangle = \left\langle A|_{\Sigma_f} \left|T_M\right| A|_{\Sigma_i}\right\rangle \tag{5}$$

and consider the sum of manifolds like $M_{i,h} = M_{i,f} \cup_{\Sigma_f} M_{f,h}$ with the amplitudes

$$\left\langle \Sigma_h \left|T_M\right| \Sigma_i\right\rangle = \sum_{A|_{\Sigma_f}} \left\langle \Sigma_h \left|T_M\right| \Sigma_f\right\rangle \left\langle \Sigma_f \left|T_M\right| \Sigma_i\right\rangle, \tag{6}$$

where we sum (or integrate) over the connections and frames on Σ_h (see [22]). Then the boundary term

$$S_\partial\left[\Sigma_f\right] = \int_{\Sigma_f} \epsilon_{IJKL}\left(e^I \wedge e^J \wedge A^{KL}\right) = \int_{\Sigma_f} H\sqrt{h} d^3x \tag{7}$$

is needed where H is the mean curvature of Σ_f corresponding to the metric h at Σ_f (as restriction of the 4-metric). Therefore we have to divide the path integration into two parts: the contribution by the boundary (boundary integration) and the contribution by the interior (bulk integration).

3.1. Boundary Integration. The boundary Σ of a 4-manifold M can be understood as embedding (or at least as immersion). Let $\iota : \Sigma \hookrightarrow M$ be an immersion of the 3-manifold Σ into the 4-manifold M with the normal vector $\vec{N}$. The spin bundle S_M of the 4-manifold splits into two subbundles $S_M^\pm$, where one subbundle, say S_M^+, can be related to the spin bundle S_Σ of the 3-manifold. Then the spin bundles are related by $S_\Sigma = \iota^* S_M^+$ with the same relation $\phi = \iota_* \Phi$ for the spinors ($\phi \in \Gamma(S_\Sigma)$ and $\Phi \in \Gamma(S_M^+)$). Let $\nabla_X^M, \nabla_X^\Sigma$ be the covariant derivatives in the spin bundles along a vector field X as section of the bundle $T\Sigma$. Then we have the formula

$$\nabla_X^M(\Phi) = \nabla_X^\Sigma \phi - \frac{1}{2}\left(\nabla_X \vec{N}\right) \cdot \vec{N} \cdot \phi, \tag{8}$$

with the obvious embedding $\phi \mapsto \binom{\phi}{0} = \Phi$ of the spinor spaces. The expression $\nabla_X \vec{N}$ is the second fundamental form of the immersion, where the trace $\mathrm{tr}(\nabla_X \vec{N}) = 2H$ is related to the mean curvature H. Then from (8) one obtains a similar relation between the corresponding Dirac operators

$$D^M \Phi = D^{3D}\phi - H\phi, \tag{9}$$

with the Dirac operator D^{3D} of the 3-manifold Σ. Near the boundary Σ, the 4-manifolds looks like $\Sigma \times [0, 1]$ and a spinor Φ on this 4-manifold is a parallel spinor and has to fulfill the following equation:

$$D^M \Phi = 0; \tag{10}$$

that is, ϕ yields the eigenvalue equation

$$D^{3D}\phi = H\phi, \tag{11}$$

with the mean curvature H of the embedding ι as eigenvalue. See our previous work [23] for more details.

Now we will use this theory to get rid of the boundary integration. At first we will discuss the deformation of an immersion using a diffeomorphism. Let $I : \Sigma \hookrightarrow M$ be an immersion of Σ (3-manifold) into M (4-manifold). A deformation of an immersion $I' : \Sigma' \hookrightarrow M'$ is diffeomorphisms $f : M \to M'$ and $g : \Sigma \to \Sigma'$ of M and Σ, respectively, so that

$$f \circ I = I' \circ g. \tag{12}$$

One of the diffeomorphisms (say f) can be absorbed into the definition of the immersion and we are left with one diffeomorphism $g \in \mathrm{Diff}(\Sigma)$ to define the deformation of the immersion I. But as stated above, the immersion is directly given by an integral over the spinor ϕ on Σ fulfilling the Dirac equation (11). Therefore we have to discuss the action of the diffeomorphism group $\mathrm{Diff}(\Sigma)$ on the Hilbert space of L^2-spinors fulfilling the Dirac equation. This case was considered in the literature [24]. The spinor space $S_{g,\sigma}(\Sigma)$ on Σ depends on two ingredients: a (Riemannian) metric g and a spin structure σ (labeled by the number of elements in $H^1(\Sigma, \mathbb{Z}_2)$). Let us consider the group of orientation preserving diffeomorphism $\mathrm{Diff}^+(\Sigma)$ acting on g (by pullback f^*g) and on σ (by a suitable defined pullback $f^*\sigma$). The Hilbert space of L^2-spinors of $S_{g,\sigma}(\Sigma)$ is denoted by $H_{g,\sigma}$. Then according to [24], any $f \in \mathrm{Diff}^+(\Sigma)$ leads in exactly two ways to a unitary operator U from $H_{g,\sigma}$ to $H_{f^*g,f^*\sigma}$. The (canonically) defined Dirac operator is equivariant with respect to the action of U and the spectrum is invariant under (orientation preserving) diffeomorphisms. In particular we obtain for the boundary term

$$S_\partial\left[\Sigma_f, h\right] = \int_{\Sigma_f} H\sqrt{h}d^3x = \int_{\Sigma_f} \overline{\phi}\, D^{3D}\phi d^3x, \tag{13}$$

with $|\phi|^2 = \text{const.}$ (see [25]). But then we can change the integration process from the integration over the metric class h on the 3-manifold Σ_f with mean curvature to an integration over the spinor ϕ on Σ_f. Then we obtain

$$\begin{aligned} Z\left(\Sigma_f\right) &= \int Dh \exp\left(\frac{i}{\hbar}S\left[\Sigma_f, h\right]\right) \\ &= \int D\phi D\overline{\phi}\ \exp\left(\frac{i}{\hbar}\int_{\Sigma_f} \overline{\phi}\, D^{3D}\phi d^3x\right) \\ &= \sqrt{\det\left(D^{3D}D^{*3D}\right)}e^{i\pi\eta(\Sigma_f)/2}, \end{aligned} \tag{14}$$

where $\eta(\Sigma_f)$ is the Eta invariant of the Dirac operator at the 3-manifold Σ_f (here we use a result of Witten; see [26]).

From the physical point of view, we obtain fermions at the boundary. The additional term with the Eta invariant reflects also an important fact. The state space of general relativity is the space of the (Lorentzian) metric tensor up to the group of coordinate transformations. This group of coordinate transformations is not the full diffeomorphism group; it is only one connected component of the diffeomorphism. That is the group of diffeomorphisms connected to the identity. In addition, there is also the (discrete) group of global diffeomorphisms which is in our case detected by the Eta invariant. For 3-manifolds there is a deep relation to the Chern-Simons invariant [27] which will be further studied at our forthcoming work.

3.2. Bulk Integration. Now we will discuss the path integral of the action

$$S_{EH}[e, A, M] = \int_M \epsilon_{IJKL}\left(e^I \wedge e^J \wedge (dA + A \wedge A)^{KL}\right) \tag{15}$$

in the interior of the 4-manifold M. The contribution of the boundary was calculated in the previous subsection. In the (formal) path integral (2) we will ignore all problems (ill-definiteness, singularities, etc.) of the path integral approach. Next we have to discuss the measure De of the path integral. Currently there is no rigorous definition of this measure and as usual we assume a product measure.

Then we have two possible parts which are more or less independent from each other:

(i) integration De_G over geometries;

(ii) integration De_{DS} over different differential structures parametrized by some structure (see below).

Now we have to consider the following path integral:

$$\begin{aligned} &Z(M) \\ &= \int_{\text{Diff structures}} De_{DS}\left(\int_{\text{Geometries}} De_G\ \exp\left(\frac{i}{\hbar}S_{EH}[e, M]\right)\right), \end{aligned} \tag{16}$$

and we have to calculate the influence of the differential structures first. At this level we need an example, an exotic $S^3 \times \mathbb{R}$.

3.3. Constructing Exotic $S^3 \times \mathbb{R}$. In [28], Freedman constructed the first example of an exotic $S^3 \times \mathbb{R}$ of special type. There are also uncountable many different exotic $\mathbb{R}^4$ having an end homeomorphic to $S^3 \times \mathbb{R}$ but not diffeomorphic to it. But Freedman's first example is not of this type (as an end of an exotic $\mathbb{R}^4$). Therefore to get an infinite number of different exotic $S^3 \times \mathbb{R}$ one has to see $S^3 \times \mathbb{R}$ as an end of $\mathbb{R}^4$ also expressible as complement $\mathbb{R}^4 \setminus D^4$ of the 4-disk. A second possibility is the usage of the end-sum technique of Gompf, so that the standard $S^3 \times \mathbb{R}$ can be transformed into an exotic $S^3 \times \mathbb{R}$ by end-sum with an exotic $\mathbb{R}^4$. Here we will concentrate on the first construction; that is, the exotic $S^3 \times \mathbb{R}$ is an end of an exotic $\mathbb{R}^4$.

Furthermore we will restrict on a subclass of exotic $\mathbb{R}^4$ called small exotic $\mathbb{R}^4$ (exotic $\mathbb{R}^4$ which can be embedded in a 4-sphere S^4). For this class there is an explicit handle decomposition. Small exotic $\mathbb{R}^4$'s are the result of an anomalous behavior in 4-dimensional topology. In 4-manifold topology [2], a homotopy equivalence between two compact, closed, simply connected 4-manifolds implies a homeomorphism between them (the so-called h cobordism). But Donaldson [29] provided the first smooth counterexample that this homeomorphism is not a diffeomorphism;

that is, both manifolds are generally not diffeomorphic to each other. The failure can be localized at some contractible submanifold (Akbulut cork) so that an open neighborhood of this submanifold is a small exotic $\mathbb{R}^4$. The whole procedure implies that this exotic $\mathbb{R}^4$ can be embedded in the 4-sphere S^4. The idea of the construction is simply given by the fact that every smooth h-cobordism between nondiffeomorphic 4-manifolds can be written as a product cobordism except for a compact contractible sub-h-cobordism V, the Akbulut cork. An open subset $U \subset V$ homeomorphic to $[0,1] \times \mathbb{R}^4$ is the corresponding sub-h-cobordism between two exotic $\mathbb{R}^4$'s. These exotic $\mathbb{R}^4$'s are called ribbon $\mathbb{R}^4$'s. They have the important property of being diffeomorphic to open subsets of the standard $\mathbb{R}^4$. In [30] Freedman and DeMichelis constructed also a continuous family of small exotic $\mathbb{R}^4$. Now we are ready to discuss the decomposition of a small exotic $\mathbb{R}^4$ by Bižaca and Gompf [31] using special pieces, the handles forming a handle body. Every 4-manifold can be decomposed (seen as handle body) using standard pieces such as $D^k \times D^{4-k}$, the so-called k-handle attached along $\partial D^k \times D^{4-k}$ to the boundary $S^3 = \partial D^4$ of a 0-handle $D^0 \times D^4 = D^4$. The construction of the handle body for the small exotic $\mathbb{R}^4$, called R^4 in the following, can be divided into two parts:

$$R^4 = A_{\text{cork}} \bigcup_{D^2} \text{CH decomposition of small exotic } \mathbb{R}^4. \tag{17}$$

The first part is known as the Akbulut cork, a contractible 4-manifold with boundary a homology 3-sphere (a 3-manifold with the same homology as the 3-sphere). The Akbulut cork A_{cork} is given by a linking between a 1-handle and a 2-handle of framing 0. The second part is the Casson handle CH which will be considered now.

Let us start with the basic construction of the Casson handle CH. Let M be a smooth, compact, simple-connected 4-manifold and $f : D^2 \to M$ a (codimension-2) mapping. By using diffeomorphisms of D^2 and M, one can deform the mapping f to get an immersion (i.e., injective differential) generically with only double points (i.e., $\#|f^{-1}(f(x))| = 2$) as singularities [32]. But to incorporate the generic location of the disk, one is rather interesting in the mapping of a 2-handle $D^2 \times D^2$ induced by $f \times id : D^2 \times D^2 \to M$ from f. Then every double point (or self-intersection) of $f(D^2)$ leads to self-plumbings of the 2-handle $D^2 \times D^2$. A self-plumbing is an identification of $D_0^2 \times D^2$ with $D_1^2 \times D^2$, where $D_0^2, D_1^2 \subset D^2$ are disjoint subdisks of the first factor disk. In complex coordinates the plumbing may be written as $(z,w) \mapsto (w,z)$ or $(z,w) \mapsto (\overline{w},\overline{z})$ creating either a positive or negative (resp.,) double point on the disk $D^2 \times 0$ (the core). Consider the pair $(D^2 \times D^2, \partial D^2 \times D^2)$ and produce finitely many self-plumbings away from the attaching region $\partial D^2 \times D^2$ to get a kinky handle $(k, \partial^- k)$, where $\partial^- k$ denotes the attaching region of the kinky handle. A kinky handle $(k, \partial^- k)$ is a one-stage tower $(T_1, \partial^- T_1)$ and an $(n+1)$-stage tower $(T_{n+1}, \partial^- T_{n+1})$ is an n-stage tower union kinky handles $\cup_{\ell=1}^{n} (T_\ell, \partial^- T_\ell)$, where two towers are attached along $\partial^- T_\ell$. Let T_n^- be (interior T_n) $\cup\, \partial^- T_n$ and the Casson handle,

$$\text{CH} = \bigcup_{\ell=0} T_\ell^-, \tag{18}$$

is the union of towers (with direct limit topology induced from the inclusions $T_n \hookrightarrow T_{n+1}$).

The main idea of the construction above is very simple: an immersed disk (disk with self-intersections) can be deformed into an embedded disk (disk without self-intersections) by sliding one part of the disk along another (embedded) disk to kill the self-intersections. Unfortunately the other disk can be immersed only. But the immersion can be deformed to an embedding by a disk again and so forth. In the limit of this process one "shifts the self-intersections into infinity" and obtains the standard open 2-handle $(D^2 \times \mathbb{R}^2, \partial D^2 \times \mathbb{R}^2)$. In the proof of Freedman [2], the main complications come from the lack of control about this process.

A Casson handle is specified up to (orientation preserving) diffeomorphism (of pairs) by a labeled finitely branching tree with base-point $*$, having all edge paths infinitely extendable away from $*$. Each edge should be given a label $+$ or $-$. Here is the construction: tree $\to$ CH. Each vertex corresponds to a kinky handle; the self-plumbing number of that kinky handle equals the number of branches leaving the vertex. The sign on each branch corresponds to the sign of the associated self-plumbing. The whole process generates a tree with infinitely many levels. In principle, every tree with a finite number of branches per level realizes a corresponding Casson handle. Each building block of a Casson handle, the "kinky" handle with n kinks, is diffeomorphic to the n-times boundary connected sum $\natural_n (S^1 \times D^3)$ (see Appendix A) with two attaching regions. The number of end connected sums is exactly the number of self-intersections of the immersed two handle. One region is a tubular neighborhood of band sums of Whitehead links connected with the previous block. The other region is a disjoint union of the standard open subsets $S^1 \times D^2$ in $\#_n S^1 \times S^2 = \partial(\natural_n S^1 \times D^3)$ (this is connected with the next block).

For the construction of an exotic $S^3 \times \mathbb{R}$, denoted by $S^3 \times_\theta \mathbb{R}$, we consider the complement $R^4 \setminus D^4$ or the decomposition:

$$S^3 \times_\theta \mathbb{R} = R^4 \setminus D^4 = \left(A_{\text{cork}} \setminus D^4\right) \bigcup_{D^2} \text{CH}. \tag{19}$$

The first part $A_{\text{cork}} \setminus D^4$ contains a cobordism between the 3-sphere S^3 and the boundary of the Akbulut cork ∂A_{cork} (a homology 3-sphere). The complement $\mathbb{R}^4 \setminus D^4$ is conformally equivalent to $S^3 \times \mathbb{R}$. Equivalently, the complement $R^4 \setminus D^4$ is diffeomorphic to an exotic $S^3 \times \mathbb{R}$. But the exoticness is not confined to a compact subset but concentrated at infinity (for instance at $+\infty$). In our case we choose a decomposition like

$$S^3 \times_\theta \mathbb{R} = M\left(S^3, \partial A_{\text{cork}}\right) \bigcup_{D^2} \text{CH}, \tag{20}$$

where $M(S^3, \partial A_{\text{cork}})$ is a cobordism between S^3 and ∂A_{cork}. For the Casson handle we need another representation

obtained by using Morse theory (see [33]). Every kinky handle $(k, \partial^- k)$ is given by n pairs of 1-/2-handle pairs, where n is the number of kinks (or self-intersections). These handles are given by the level sets of the Morse functions

$$\begin{aligned} f_1 &= x^2 + y^2 + z^2 - t^2 \quad \text{for the 1-handle,} \\ f_2 &= x^2 + y^2 - z^2 - t^2 \quad \text{for the 2-handle,} \end{aligned} \tag{21}$$

that is, by the sets $L(f_i, C) = \{(x, y, z, t) \mid f_i(x, y, z, t) = C = \text{const.}\}$ for $i = 1, 2$. Now we represent the Casson handle by the union of all n-stage towers

$$\mathrm{CH} = \bigcup_{\text{level } \ell \text{ of tree } \mathcal{T}} \left(\operatorname{int}(T_\ell) \cup \partial^- T_\ell\right), \tag{22}$$

arranged along the tree $\mathcal{T}$. But every tower T_ℓ is given by the union of pairs (f_1, f_2). But what is the geometry of T_ℓ (and better of $\operatorname{int}(T_\ell)$)? Every level set $L(f_1, C)$ and $L(f_2, C)$ is a hyperbolic 3-manifold (i.e., with negative curvature) and the union of all level sets is a hyperbolic 4-manifold. A central point in our argumentation is Mostow rigidity, a central property of all hyperbolic 3-manifolds (or higher) with finite volume explained in the next subsection.

3.4. The Hyperbolic Geometry of CH. The central element in the Casson handle is a pair of 1- and 2-handles representing a kinky handle. As we argued above this pair admits a hyperbolic geometry (or it is a hyperbolic 3-manifold) having negative scalar curvature. A 3-manifold admits a hyperbolic structure in the interior if there is a diffeomorphism to $\mathbb{H}^3/\Gamma$, where Γ is a discrete subgroup $\Gamma \subset \mathrm{SO}(3,1)$ of the Lorentz group and we have a representation of the fundamental group $\pi_1(M)$ into $\mathrm{SO}(3,1)$ (the isometry group of the hyperbolic space $\mathbb{H}^3$). One property of hyperbolic 3- and 4-manifolds is central: Mostow rigidity. As shown by Mostow [34], every hyperbolic n-manifold $n > 2$ with finite volume has this property: *Every diffeomorphism (especially every conformal transformation) of a hyperbolic n-manifold with finite volume is induced by an isometry.* Therefore one cannot scale a hyperbolic 3-manifold with finite volume. Then the volume vol() and the curvature are topological invariants but for later usages we combine the curvature and the volume into the Chern-Simons invariant CS(). But more is true: in a hyperbolic 3-manifold there are special surfaces which cannot be contracted, called incompressible surface. A properly embedded connected surface $S \subset N$ in a 3-manifold N is called 2-sided if its normal bundle is trivial and 1-sided if its normal bundle is nontrivial. The "sides" of S then correspond to the components of the complement of S in a tubular neighborhood $S \times [0,1] \subset N$. A 2-sided connected surface S other than S^2 or D^2 is called incompressible if for each disk $D \subset N$ with $D \cap S = \partial D$ there is a disk $D' \subset S$ with $\partial D' = \partial D$; that is, the boundary of the disk D can be contracted in the surface S. The boundary of a 3-manifold is an incompressible surface. More importantly, this surface can be detected in the fundamental group $\pi_1(N)$ of the 3-manifold, that is; there is an injective homomorphism $\pi_1(S) \to \pi_1(N)$. The consequence of all properties is the following conclusion.

The tower T_ℓ has a hyperbolic geometry (with finite volume) and therefore fixed size; that is, it cannot be scaled by any diffeomorphism or conformal transformation. Then we obtain an invariant decomposition of the Casson handle into towers arranged with respect to a tree. Secondly, inside of every tower T_ℓ there is (at least one) an incompressible surface also of fixed size.

In case of the tower T_ℓ, one knows two incompressible surfaces, the two tori coming from the complement of the Whitehead link (with two components) used in the construction.

3.5. The Path Integral of the Exotic $S^3 \times \mathbb{R}$. Now we will discuss the path integral using the decomposition

$$\begin{aligned} &S^3 \times_\theta \mathbb{R} \\ &\quad = M\left(S^3, \partial A_{\text{cork}}\right) \bigcup_{D^2} \left(\bigcup_{\text{level } \ell \text{ of tree } \mathcal{T}} \left(\operatorname{int}(T_\ell) \cup \partial^- T_\ell\right) \right), \end{aligned} \tag{23}$$

and we remark that the construction of the cobordism $M(S^3, \partial A_{\text{cork}})$ requires the usage of a Casson handle again, denoted by $M(S^3, \partial A_{\text{cork}}) \cup \mathrm{CH}_{\text{cork}}$. Therefore we have to clarify the role of the Casson handle. In the previous Section 3.4, we discussed the strong connection between geometry and topology for hyperbolic manifolds. The topology of $S^3 \times_\theta \mathbb{R}$ is rather trivial but the smoothness structure (and therefore the differential topology) can be very complicate.

As stated above, the boundary terms can be factorized from the terms in the interior. Formally we obtain

$$\begin{aligned} Z\left(S^3 \times_\theta \mathbb{R}\right) = &\left\{ \prod_\ell Z\left(\partial^- T_\ell\right) Z\left(\partial A_{\text{cork}}\right) Z\left(S^3\right) \right\} \\ &\times \left(\prod_\ell Z\left(\operatorname{int}(T_\ell)\right) \right) Z\left(\mathrm{CH}_{\text{cork}}\right), \end{aligned} \tag{24}$$

and for an expectation value of the observable $\mathcal{O}$

$$\left\langle S^3 \times_\theta \mathbb{R} \,|\mathcal{O}|\, S^3 \times_\theta \mathbb{R} \right\rangle, \tag{25}$$

but for the following we have to discuss it more fully. To understand the time-like evolution of a disk (or a surface), we have to describe a disk inside of a Casson handle as pioneered by Bižaca [35]. With the same arguments, one can also describe the modification of the 3-sphere into homology 3-spheres Σ. But then we obtain (formally) an infinite sequence of homology 3-spheres $\Sigma_1 \to \Sigma_2 \to \cdots$ with amplitudes

$$Z\left(S^3 \times_\theta \mathbb{R}\right) = \langle \Sigma_1 | T_M | \Sigma_2 \rangle \langle \Sigma_2 | T_M | \Sigma_3 \rangle \cdots, \tag{26}$$

including the boundary terms. Every spatial section Σ_n can be seen as an element of the phase space in quantum gravity. Therefore this change of transitions is a *topological phase transition* which will be further investigated in our work.

The choice of the boundary term has a kind of arbitrariness. We can choose the decomposition much finer to get

more boundary terms. Therefore the path integral (14) must be extended away from the boundary. We will discuss this extension also in our forthcoming work.

Before we go ahead we have to discuss the foliation structure of $S^3\times_\theta\mathbb{R}$ or the appearance of different time variables. As stated above, our space-time has the topology of $S^3\times\mathbb{R}$ with equal slices parametrized by a *topological time* t_{TOP}; that is,

$$\begin{aligned}\left(S^3\times_\theta\mathbb{R}\right)_{\mathrm{TOP}} &= \left\{\left(p,t_{\mathrm{TOP}}\right)\mid p\in S^3, t_{\mathrm{TOP}}\in\mathbb{R}\right\}\\ &= \left\{S^3\times\left\{t_{\mathrm{TOP}}\right\}\mid t_{\mathrm{TOP}}\in\{-\infty,\ldots,+\infty\}\right\},\end{aligned}\tag{27}$$

defined by the topological embedding $S^3\hookrightarrow S^3\times_\theta\mathbb{R}$. It is the defining property of exotic smoothness that S^3 inside of $S^3\times_\theta\mathbb{R}$ is only a topological 3-sphere; that is, it is wildly embedded and so only represented by an infinite polyhedron. There is another possibility to introduce t_{TOP} which will point us to the smooth case. For that purpose we define a map $F: S^3\times\mathbb{R}\to\mathbb{R}$ by $(x,t)\mapsto t$ so that $t_{\mathrm{TOP}}=F(p)$ for $p\in S^3\times\mathbb{R}$. In contrast one also has the smooth time t_{Diff} which we have to define now. Locally it is the smooth (physical coordinate) time. We know also that the exotic $S^3\times_\theta\mathbb{R}$ is composed by a sequence $\Sigma_1\to\Sigma_2\to\Sigma_3\to\cdots$ of homology 3-spheres or better by a sequence $M(\Sigma_1,\Sigma_2)\cup_{\Sigma_2}M(\Sigma_2,\Sigma_3)\cup_{\Sigma_3}\ldots$ of (homology) cobordism between the homology 3-spheres. All sequences are ordered and so it is enough to analyze one cobordism $M(\Sigma_1,\Sigma_2)$. Every cobordism between two homology 3-spheres Σ_1 and Σ_2 is characterized by the existence of a finite number of 1-/2-handle pairs (or dually 2-/3-handle pairs). Now we define a smooth map $F_{\mathrm{cob}}: M(\Sigma_1,\Sigma_2)\to[0,1]$ which must be a Morse function (i.e., it has isolated critical points) [33]. The number of critical points N of F is even, say $N=2k$, where k is the number of 1-/2-handle pairs. These critical points are also denoted as naked singularities in GR (but of bounded curvature). Like in the case of topological time t_{TOP} we introduce the smooth time by $t_{\mathrm{Diff}}=F_{\mathrm{cob}}(p)$ for all $p\in M(\Sigma_1,\Sigma_2)\subset S^3\times_\theta\mathbb{R}$. The extension of t_{Diff} to the whole $S^3\times_\theta\mathbb{R}$ by the Morse function $F: S^3\times_\theta\mathbb{R}\to\mathbb{R}$ is straightforward $t_{\mathrm{Diff}}=F(p)$ for all $p\in S^3\times_\theta\mathbb{R}$. The cobordism $M(\Sigma_1,\Sigma_2)$ is part of the exotic $S^3\times_\theta\mathbb{R}$ and can be embedded to make it $S^3\times[0,1]$ topologically. Therefore function F_{cob} is a continuous function which is strictly increasing on future directed causal curves, so it is a time function (see [36, 37]). But there is also another method to construct t_{Diff} by using codimension-1 foliations. In [7] we uncovered a strong relation between codimension-1 foliations (also used to construct a Lorentz structure on a manifold) and exotic smoothness structures for a small exotic $\mathbb{R}^4$. The coordinate of this codimension-1 submanifold is also the smooth time t_{Diff}. This approach will be more fully discussed in our forthcoming work.

4. The Expectation Value of the Area and Inflation

In Section 3.4 we described the hyperbolic geometry originated in the exotic smoothness structure of $S^3\times_\theta\mathbb{R}$. Because of this hyperbolic geometry, there are incompressible surfaces inside of the hyperbolic manifold as the smallest possible units of geometry. Then Mostow rigidity determines the behavior of this incompressible surface. At first we will concentrate on the first cobordism $M(S^3,\partial A_{\mathrm{cork}})$ between S^3 and the boundary $\partial A_{\mathrm{cork}}$ of the Akbulut cork. The area of a surface is given by

$$A(e,S)=\int_S d^2\sigma\sqrt{E^aE^bn_an_b},\tag{28}$$

with the normal vector n_a and the densitized frame $E^a=\det(e)e^a$. The expectation value of the area A,

$$\begin{aligned}&\left\langle S^3\,|A(e,S)|\,\partial A_{\mathrm{cork}}\right\rangle\\ &\quad=\frac{1}{Z\left(M\left(S^3,\partial A_{\mathrm{cork}}\right)\right)}\\ &\qquad\times\int De\,A(e,S)\exp\left(\frac{i}{\hbar}S_{EH}\left[e,M\left(S^3,\partial A_{\mathrm{cork}}\right]\right)\right),\end{aligned}\tag{29}$$

depends essentially on the hyperbolic geometry. As argued above, this cobordism has a hyperbolic geometry but in the simplest case, the boundary of the Akbulut cork is the homology 3-sphere $\partial A_{\mathrm{cork}}=\Sigma(2,5,7)$, a Brieskorn homology 3-sphere. Now we study the area of a surface where one direction is along the time axis. Then we obtain a decomposition of the surface into a sum of small surfaces so that every small surface lies in one component of the cobordism. Remember, that the cobordism $M(S^3,\partial A_{\mathrm{cork}})$ is decomposed into the trivial cobordism $S^3\times[0,1]$ and a Casson handle $\mathrm{CH}=\cup_\ell T_\ell$. Then the decomposition of the surface

$$S=\cup_\ell S_\ell,\tag{30}$$

corresponds to the decomposition of the expectation value of the area

$$A_\ell(e,S_\ell)=\int_{S_\ell}d^2\sigma\sqrt{E^aE^bn_an_b},\tag{31}$$

so that

$$\begin{aligned}\left\langle\partial^-T_\ell\left|A_{\ell'}(e,S_{\ell'})\right|\partial^-T_{\ell+1}\right\rangle&=\left\langle\partial^-T_\ell\left|A_\ell(e,S_\ell)\right|\partial^-T_{\ell+1}\right\rangle\delta_{\ell\ell'},\\ \left\langle S^3\,|A(e,S)|\,\partial A_{\mathrm{cork}}\right\rangle&=\sum_\ell\left\langle\partial^-T_\ell\left|A_\ell(e,S_\ell)\right|\partial^-T_{\ell+1}\right\rangle.\end{aligned}\tag{32}$$

The initial value for $\ell=0$ is the expectation value

$$\left\langle\partial^-T_0\left|A_0(e,S_0)\right|\partial^-T_1\right\rangle=a_0^2,\tag{33}$$

where a_0 is the radius of the 3-sphere S^3. But because of the hyperbolic geometry (with constant curvature because of Mostow rigidity) every further level scales this expectation value by a constant factor. Therefore, to calculate the expectation value, we have to study the scaling behavior.

Consider a cobordism $M(\Sigma_0, \Sigma_1)$ between the homology 3-spheres Σ_0, Σ_1. As shown by Witten [38–40], the action,

$$\int_{\Sigma_{0,1}} {}^3R\sqrt{h}d^3x = L \cdot \mathrm{CS}\left(\Sigma_{0,1}\right), \tag{34}$$

for every 3-manifold (in particular for Σ_0 and Σ_1 denoted by $\Sigma_{0,1}$) is related to the Chern-Simons action $\mathrm{CS}(\Sigma_{0,1})$ (defined in Appendix B). The scaling factor L is related to the volume by $L = \sqrt[3]{\mathrm{vol}(\Sigma_{0,1})}$ and we obtain formally

$$L \cdot \mathrm{CS}\left(\Sigma_{0,1}, A\right) = L^3 \cdot \frac{\mathrm{CS}\left(\Sigma_{0,1}\right)}{L^2} = \int_{\Sigma_{0,1}} \frac{\mathrm{CS}\left(\Sigma_{0,1}\right)}{L^2}\sqrt{h}d^3x, \tag{35}$$

by using

$$L^3 = \mathrm{vol}\left(\Sigma_{0,1}\right) = \int_{\Sigma_{0,1}} \sqrt{h}d^3x. \tag{36}$$

Together with

$${}^3R = \frac{3k}{a^2}, \tag{37}$$

one can compare the kernels of the integrals of (34) and (35) to get for a fixed time

$$\frac{3k}{a^2} = \frac{\mathrm{CS}\left(\Sigma_{0,1}\right)}{L^2}. \tag{38}$$

This gives the scaling factor

$$\vartheta = \frac{a^2}{L^2} = \frac{3}{\mathrm{CS}\left(\Sigma_{0,1}\right)}, \tag{39}$$

where we set $k = 1$ in the following. The hyperbolic geometry of the cobordism is best expressed by the metric

$$ds^2 = dt^2 - a(t)^2 h_{ik}dx^i dx^k, \tag{40}$$

also called the Friedmann-Robertson-Walker metric (FRW metric) with the scaling function $a(t)$ for the (spatial) 3-manifold. But Mostow rigidity enforces us to choose

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{L^2}, \tag{41}$$

in the length scale L of the hyperbolic structure. But why is it possible to choose the FRW metric? At first we state that the FRW metric is not sensitive to the topology of the space-time. One needs only a space-time which admits a slicing with respect to a smooth time t_{Diff} and a metric of constant curvature for every spatial slice. Then for the cobordism $M(\Sigma_1, \Sigma_2)$ between Σ_1 and Σ_2 we have two cases: the curvature parameter $k(\Sigma_1)$ of Σ_1 (say $k(\Sigma_1) = +1$) jumps to the value $k(\Sigma_2)$ of Σ_2 (say $k(\Sigma_2) = -1$) or both curvatures remain constant. The second case is the usual one. Each homology 3-sphere Σ_1, Σ_2 has the same geometry (or geometric structure in the sense of Thurston [41]) which is hyperbolic in most case. The first case is more complicated. Here we need the smooth function to represent the jump in the curvature parameter k. Let us choose the function $k : \mathbb{R} \to \mathbb{R}$

$$t \longmapsto \begin{cases} +1 & 0 \le t \\ 1 - 2 \cdot \exp\left(-\lambda \cdot t^{-2}\right) & t > 0, \end{cases} \tag{42}$$

which is smooth and the parameter λ determines the slope of this function. Furthermore the metric (40) is also the metric of a hyperbolic space (which has to fulfill Mostow rigidity because the cobordism $M(\Sigma_1, \Sigma_2)$ is compact).

In the following we will switch to quadratic expressions because we will determine the expectation value of the area. Then we obtain

$$da^2 = \frac{a^2}{L^2}\, dt^2 = \vartheta dt^2, \tag{43}$$

with respect to the scale ϑ. By using the tree of the Casson handle, we obtain a countable infinite sum of contributions for (43). Before we start we will clarify the geometry of the Casson handle. The discussion of the Morse functions above uncovers the hyperbolic geometry of the Casson handle (see also Section 3.4). Therefore the tree corresponding to the Casson handle must be interpreted as a metric tree with hyperbolic structure in $\mathbb{H}^2$ and metric $ds^2 = (dx^2 + dy^2)/y^2$. The embedding of the Casson handle in the cobordism is given by the following rules.

(i) The direction of the increasing levels $n \to n+1$ is identified with dy^2 and dx^2 is the number of edges for a fixed level with scaling parameter ϑ.

(ii) The contribution of every level in the tree is determined by the previous level best expressed in the scaling parameter ϑ.

(iii) An immersed disk at level n needs at least one disk to resolve the self-intersection point. This disk forms the level $n+1$ but this disk is connected to the previous disk. So we obtain for $da^2|_{n+1}$ at level $n+1$

$$da^2|_{n+1} \sim \vartheta \cdot da^2\big|_n \tag{44}$$

up to a constant.

By using the metric $ds^2 = (dx^2 + dy^2)/y^2$ with the embedding $(y^2 \to n+1, dx^2 \to \vartheta)$ we obtain for the change dx^2/y^2 along the x-direction (i.e., for a fixed y) $\vartheta/(n+1)$. This change determines the scaling from the level n to $n+1$; that is,

$$da^2\big|_{n+1} = \frac{\vartheta}{n+1} \cdot da^2\big|_n = \frac{\vartheta^{n+1}}{(n+1)!} \cdot da^2\big|_0; \tag{45}$$

and after the whole summation (as substitute for an integral for the discrete values) we obtain for the relative scaling

$$\begin{aligned} a^2 &= \sum_{n=0}^{\infty}\left(da^2\big|_n\right) = a_0^2 \cdot \sum_{n=0}^{\infty}\frac{1}{n!}\vartheta^n \\ &= a_0^2 \cdot \exp(\vartheta) = a_0^2 \cdot l_{\mathrm{scale}} \end{aligned} \tag{46}$$

with $da^2|_0 = a_0^2$. With this result in mind, we consider the expectation value where we use the constant scalar curvature (Mostow rigidity). By using the normalization, many terms are neglected (like the boundary terms):

$$\begin{aligned}
&\left\langle S^3 \left| A(e,S) \right| \partial A_{\text{cork}} \right\rangle \\
&= \left(\left(\left\{ \prod_{\ell} Z(\partial^- T_\ell)\, Z(\partial A_{\text{cork}})\, Z(S^3) \right\} \right.\right. \\
&\qquad \left.\left. \times \sum_{n=0}^{\infty} \left\langle \partial^- T_n \left| A_n(e,S_n) \right| \partial^- T_{n+1} \right\rangle \right) \right. \\
&\qquad \left. \times \left(\left\{ \prod_{\ell} Z(\partial^- T_\ell)\, Z(\partial A_{\text{cork}})\, Z(S^3) \right\} \right)^{-1} \right) \\
&= \sum_{n=0}^{\infty} \left\langle \partial^- T_n \left| A_n(e,S_n) \right| \partial^- T_{n+1} \right\rangle .
\end{aligned} \tag{47}$$

Finally we obtain for the area a_0^2 for the first level $\ell = 0$,

$$\begin{aligned}
\left\langle S^3 \left| A(e,S) \right| \partial A_{\text{cork}} \right\rangle &= \sum_{n=0}^{\infty} \left\langle \partial^- T_n \left| A_n(e,S_n) \right| \partial^- T_{n+1} \right\rangle \\
&= a_0^2 \cdot \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{3}{\text{CS}(\partial A_{\text{cork}})} \right)^n \\
&= a_0^2 \cdot \exp\left(\frac{3}{\text{CS}(\partial A_{\text{cork}})} \right),
\end{aligned} \tag{48}$$

with the radius a_0 of Σ_0 and arrive at a for Σ_1. From the physical point of view we obtain an exponential increase of the area; that is, we get an inflationary behavior. This derivation can be also extended to the next Casson handle but we have to determine the 3-manifold in which ∂A_{cork} can change. It will be done below.

5. An Effective Theory

Now will ask for an effective theory where the influence of the exotic smoothness structure is contained in some moduli (or some field). As explained above, the main characteristics is given by a change of the (spatial) 4-manifold (but without changing the homology). Therefore let us describe this change (the so-called homology cobordism) between two homology 3-spheres Σ_0 and Σ_1. The situation can be described by a diagram

$$\begin{array}{ccc}
\Sigma_1 & \xrightarrow{\Psi} & \mathbb{R}, \\
\phi \downarrow & \circlearrowleft & \updownarrow id, \\
\Sigma_0 & \xrightarrow{\psi} & \mathbb{R},
\end{array} \tag{49}$$

which commutes. The two functions ψ and Ψ are the Morse function of Σ_0 and Σ_1, respectively, with $\Psi = \psi \circ \phi$. The Morse function over $\Sigma_{0,1}$ is a function $\Sigma_{0,1} \to \mathbb{R}$ having only isolated, nondegenerated, critical points (i.e., with vanishing first derivatives at these points). A homology 3-sphere has two critical points (located at the two poles). The Morse function looks like $\pm\|x\|^2$ at these critical points. The transition $y = \phi(x)$ represented by the (homology) cobordism $M(\Sigma_0, \Sigma_1)$ maps the Morse function $\psi(y) = \|y\|^2$ on Σ_0 to the Morse function $\Psi(x) = \|\phi(x)\|^2$ on Σ_1. The function $-\|\phi\|^2$ represents also the critical point of the cobordism $M(\Sigma_0, \Sigma_1)$. But as we learned above, this cobordism has a hyperbolic geometry and we have to interpret the function $\|\phi(x)\|^2$ not as Euclidean form but change it to the hyperbolic geometry so that

$$-\|\phi\|^2 = -\left(\phi_1^2 + \phi_2^2 + \phi_3^2\right) \longrightarrow -e^{-2\phi_1}\left(1 + \phi_2^2 + \phi_3^2\right); \tag{50}$$

that is, we have a preferred direction represented by a single scalar field $\phi_1 : \Sigma_1 \to \mathbb{R}$. Therefore, the transition $\Sigma_0 \to \Sigma_1$ is represented by a single scalar field $\phi_1 : \Sigma_1 \to \mathbb{R}$ and we identify this field as the moduli. Finally we interpret this Morse function in the interior of the cobordism $M(\Sigma_0, \Sigma_1)$ as the potential (shifted away from the point 0) of the scalar field ϕ with Lagrangian

$$L = R + \left(\partial_\mu \phi\right)^2 - \frac{\rho}{2}\left(1 - \exp(-\lambda\phi)\right)^2, \tag{51}$$

with two free constants ρ and λ. For the value $\lambda = \sqrt{2/3}$ and $\rho = 3M^2$ we obtain the Starobinski model [42] (by a conformal transformation using ϕ and a redefinition of the scalar field [43])

$$L = R + \frac{1}{6M^2} R^2, \tag{52}$$

with the mass scale $M \ll M_P$ much smaller than the Planck mass. From our discussion above, the appearance of this model is not totally surprising. It favors a surface to be incompressible (which is compatible with the properties of hyperbolic manifolds). In the next section we will determine this mass scale.

6. A Cosmological Model Compared to the Planck Satellite Results

In this section we will go a step further and discuss the path integral for $S^3 \times \mathbb{R}$, where we sum over all smoothness structures. Furthermore we will assume that $S^3 \times \mathbb{R}$ is the end of a small exotic $\mathbb{R}^4$. But then we have to discuss the parametrization of all Casson handles. As discussed by Freedman [2], all Casson handles can be parametrized by a dual tree where the vertices are 5-stage towers (with three extra conditions). We refer to [2] or to [44] for the details of the well-known construction. This tree has one root from which two 5-stage towers branch. Every tower has an attaching circle of any framing. Using Bizacas technique [35], we obtain an attaching of a 5-tower along the sum $P\#P$ of two Poincare spheres P (for the two towers). Therefore for the universal case, we obtain two transitions

$$S^3 \xrightarrow{\text{cork}} \partial A_{\text{cork}} \xrightarrow{\text{tower}} P\#P, \tag{53}$$

with the scaling behavior,

$$a = a_0 \cdot \exp\left(\frac{3}{2 \cdot \mathrm{CS}\left(\partial A_{\mathrm{cork}}\right)} + \frac{3}{2 \cdot \mathrm{CS}\left(P\#P\right)}\right). \quad (54)$$

It can be expressed by the expectation value

$$\left\langle S^3 \left| A\left(e,S\right)\right| P\#P\right\rangle = a_0^2 \cdot \exp\left(\frac{3}{\mathrm{CS}\left(\partial A_{\mathrm{cork}}\right)} + \frac{3}{\mathrm{CS}\left(P\#P\right)}\right), \quad (55)$$

for the transition $S^3 \rightarrow P\#P$. It is important to note that this expectation value is the sum over all smoothness structures of $S^3 \times \mathbb{R}$ and we obtain also

$$\begin{aligned} &\left\langle S^3 \times \mathbb{R} \left| A\left(e,S\right)\right| S^3 \times \mathbb{R}\right\rangle \\ &= \sum_{\text{diff structures}} \left\langle S^3 \times_\theta \mathbb{R} \left| A\left(e,S\right)\right| S^3 \times_\theta \mathbb{R}\right\rangle \\ &= a_0^2 \cdot \exp\left(\frac{3}{\mathrm{CS}\left(\partial A_{\mathrm{cork}}\right)} + \frac{3}{\mathrm{CS}\left(P\#P\right)}\right), \end{aligned} \quad (56)$$

with a_0^2 as the size of the 3-sphere S^3 at $-\infty$. With the argumentation above, the smoothness structure has a kind of universality so that the two transitions above are generic.

In our model (using the exotic smoothness structure), we obtain two inflationary phases. In the first phase we have a transition

$$S^3 \longrightarrow \partial A_{\mathrm{cork}}, \quad (57)$$

and for the simplest case $\partial A_{\mathrm{cork}} = \Sigma(2,5,7)$, a Brieskorn homology 3-sphere. Now we will assume that the 3-sphere has Planck-size

$$a_0 = L_P = \sqrt{\frac{hG}{c^3}}; \quad (58)$$

then we obtain for the size

$$a = L_P \cdot \exp\left(\frac{3}{2 \cdot \mathrm{CS}(\Sigma\left(2,5,7\right)}\right). \quad (59)$$

We can use the method of Fintushel and Stern [45–47] to calculate the Chern-Simons invariants for the Brieskorn spheres. The calculation can be found in Appendix C. Note that the relation (34) is only true for the Levi-Civita connection. Then the Chern-Simons invariant is uniquely defined to be the minimum, denoted by $\tau()$ (see (B.4)). Then we obtain for the invariant (C.7) so that

$$L_P \cdot \exp\left(\frac{140}{3}\right) \approx 7.5 \cdot 10^{-15} m \quad (60)$$

is the size of the cosmos at the end of the first inflationary phase. This size can be related to an energy scale by using it as Compton length and one obtains 165 MeV, comparable to the energy scale of the QCD. For the two inflationary transitions

$$S^3 \longrightarrow \Sigma\left(2,5,7\right) \longrightarrow P\#P \quad (61)$$

one obtains the size

$$a = L_P \cdot \exp\left(\frac{140}{3} + 90\right) \approx 9.14 \cdot 10^{24} m \approx 10^9 Lj. \quad (62)$$

As explained above, the effective theory is the Starobinsky model. This model is in very good agreement with results of the Planck satellite [48] with the two main observables:

$n_s \sim 0.96$ spectral index for scalar perturbations,

$r \sim 0.004$ tensor-to-scalar ratio,

but one parameter of the model is open, the energy scale M in Planck units. In our model it is related to the second derivative of the Morse function, which is the curvature of the critical point. In our paper [49], we determined also the energy scale of the inflation by using a simple argument to incorporate only the first 3 levels of the Casson handle. For the scale

$$\vartheta = \frac{3}{2 \cdot \mathrm{CS}\left(\Sigma\left(2,5,7\right)\right)}, \quad (63)$$

of the first transition, we obtain the scaling of the Planck energy (associated with the Planck-sized 3-sphere at the beginning)

$$E_{\mathrm{Inflation}} = \frac{E_{\mathrm{Planck}}}{\left(1 + \vartheta + \left(\vartheta^2/2\right) + \left(\vartheta^3/6\right)\right)}, \quad (64)$$

with the relative scaling

$$\alpha = \frac{E_{\mathrm{Inflation}}}{E_{\mathrm{Planck}}} = \frac{1}{\left(1 + \vartheta + \left(\vartheta^2/2\right) + \left(\vartheta^3/6\right)\right)} \approx 5.5325 \cdot 10^{-5}, \quad (65)$$

leading to the energy scale of the inflation

$$E_{\mathrm{Inflation}} \approx 6.7547 \cdot 10^{14} GeV, \quad (66)$$

by using $E_{\mathrm{Planck}} \approx 1.2209 \cdot 10^{19} GeV$. We remark that the relative scaling $\alpha \approx 5.5325 \cdot 10^{-5}$ (in Planck units) above is an energy scale for the potential in the effective theory (52). Therefore we have to identify $M = \alpha$ in the parameter $1/6M^2$ of the Starobinsky model (in very good agreement with the measurements). Now we can go a step further and discuss the appearance of the cosmological constant.

Again we can use the hyperbolic geometry to state that the curvature is negative and we have the Mostow rigidity; that is, the scalar curvature of the 4-manifold has a constant value, the cosmological constant Λ. If we assume that the 3-sphere has the size of the Planck length (as above), then we obtain

$$\Lambda = \frac{1}{L_P^2} \cdot \exp\left(-\frac{3}{\mathrm{CS}\left(\Sigma\left(2,5,7\right)\right)} - \frac{3}{\mathrm{CS}\left(P\#P\right)}\right). \quad (67)$$

With the values of the Chern-Simons invariants (C.7), we obtain the value

$$\Lambda \cdot L_P^2 = \exp\left(-\frac{280}{3} - 180\right) \approx 5 \cdot 10^{-118}, \quad (68)$$

in Planck units. In cosmology, one usually relate the cosmological constant to the Hubble constant H_0 (expressing the critical density) leading to the length scale

$$L_c^2 = \frac{c^2}{3H_0^2}. \tag{69}$$

The corresponding variable is denoted by Ω_Λ and we obtain

$$\Omega_\Lambda = \frac{c^5}{3hGH_0^2} \cdot \exp\left(-\frac{3}{\mathrm{CS}(\Sigma(2,5,7))} - \frac{3}{\mathrm{CS}(P\#P)}\right), \tag{70}$$

in units of the critical density. This formula is in very good agreement with the WMAP results; that is, by using the value for the Hubble constant

$$H_0 = 74\ \frac{\mathrm{km}}{s \cdot \mathrm{Mpc}}, \tag{71}$$

we are able to calculate the dark energy density to be

$$\Omega_\Lambda = 0.729, \tag{72}$$

agreeing with the WMAP results. But it differs from the Planck results [50] of the Hubble constant

$$(H_0)_{\mathrm{Planck}} = 68\ \frac{\mathrm{km}}{s \cdot \mathrm{Mpc}}, \tag{73}$$

for which we obtain

$$\Omega_\Lambda \approx 0.88, \tag{74}$$

in contrast with the measured value of the dark energy

$$(\Omega_\Lambda)_{\mathrm{Planck}} = 0.683. \tag{75}$$

But there is another possibility for the size of the 3-sphere at the beginning and everything depends on this choice. But we can use the entropy formula of a Black hole in Loop quantum gravity

$$S = \frac{A \cdot \gamma_0}{4 \cdot \gamma \cdot L_P^2} \cdot 2\pi, \tag{76}$$

with

$$\gamma_0 = \frac{\ln(2)}{\pi \cdot \sqrt{3}}, \tag{77}$$

according to [51] with the Immirzi parameter γ, where the extra factor 2π is given by a different definition of L_P replacing h by $\hbar$. In the original approach of Ashtekar in Loop quantum gravity one usually set $\gamma = 1$. If we take it seriously then we obtain a reduction of the length in (70)

$$\frac{1}{L_P^2} \longrightarrow \frac{1}{L_P^2} \cdot \frac{2 \cdot \ln(2)}{\sqrt{3}} = \frac{2\pi\gamma_0}{L_P^2} \approx 0.80037 \cdot \frac{1}{L_P^2}, \tag{78}$$

or the new closed formula

$$\Omega_\Lambda = \frac{c^5}{3\hbar GH_0^2} \cdot \gamma_0 \cdot \exp\left(-\frac{3}{\mathrm{CS}(\Sigma(2,5,7))} - \frac{3}{\mathrm{CS}(P\#P)}\right), \tag{79}$$

correcting the value $\Omega_\Lambda \approx 0.88$ to

$$\Omega_\Lambda \approx 0.704. \tag{80}$$

But γ_0 depends on the gauge group and if one uses the value [52]

$$\gamma_0 = \frac{\ln(3)}{\pi \cdot \sqrt{8}}, \tag{81}$$

agreeing also with calculations in the spin foam models [53], then one gets a better fit

$$\Omega_\Lambda \approx 0.6836, \tag{82}$$

which is in good agreement with the measurements.

7. Conclusion

The strong relation between hyperbolic geometry (of the space-time) and exotic smoothness is one of the main results in this paper. Then using Mostow rigidity, geometric observables like area and volume or curvature are topological invariants which agree with the expectation values of these observables (calculated via the path integral). We compared the results with the recent results of the Planck satellite and found a good agreement. In particular as a direct result of the hyperbolic geometry, the inflation can be effectively described by the Starobinski model. Furthermore we also obtained a cosmological model which produces a realistic cosmological constant.

Appendices

A. Connected and Boundary Connected Sum of Manifolds

Now we will define the connected sum # and the boundary connected sum $\natural$ of manifolds. Let M, N be two n-manifolds with boundaries $\partial M, \partial N$. The *connected sum* $M\#N$ is the procedure of cutting out a disk D^n from the interior $\mathrm{int}(M) \setminus D^n$ and $\mathrm{int}(N) \setminus D^n$ with the boundaries $S^{n-1} \sqcup \partial M$ and $S^{n-1} \sqcup \partial N$, respectively, and gluing them together along the common boundary component S^{n-1}. The boundary $\partial(M\#N) = \partial M \sqcup \partial N$ is the disjoint sum of the boundaries $\partial M, \partial N$. The *boundary connected sum* $M\natural N$ is the procedure of cutting out a disk D^{n-1} from the boundary $\partial M \setminus D^{n-1}$ and $\partial N \setminus D^{n-1}$ and gluing them together along S^{n-2} of the boundary. Then the boundary of this sum $M\natural N$ is the connected sum $\partial(M\natural N) = \partial M\#\partial N$ of the boundaries $\partial M, \partial N$.

B. Chern-Simons Invariant

Let P be a principal G bundle over the 4-manifold M with $\partial M \neq 0$. Furthermore let A be a connection in P with the curvature,

$$F_A = dA + A \wedge A, \tag{B.1}$$

and Chern class,

$$C_2 = \frac{1}{8\pi^2}\int_M \operatorname{tr}\left(F_A \wedge F_A\right), \tag{B.2}$$

for the classification of the bundle P. By using the Stokes theorem we obtain

$$\int_M \operatorname{tr}\left(F_A \wedge F_A\right) = \int_{\partial M} \operatorname{tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right), \tag{B.3}$$

with the Chern-Simons invariant

$$\mathrm{CS}\left(\partial M, A\right) = \frac{1}{8\pi^2}\int_{\partial M} \operatorname{tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right). \tag{B.4}$$

Now we consider the gauge transformation $A \rightarrow g^{-1}Ag + g^{-1}dg$ and obtain

$$\mathrm{CS}\left(\partial M, g^{-1}Ag + g^{-1}dg\right) = \mathrm{CS}\left(\partial M, A\right) + k, \tag{B.5}$$

with the winding number

$$k = \frac{1}{24\pi^2}\int_{\partial M}\left(g^{-1}dg\right)^3 \in \mathbb{Z}, \tag{B.6}$$

of the map $g : M \rightarrow G$. Thus the expression,

$$\mathrm{CS}\left(\partial M, A\right) \bmod 1, \tag{B.7}$$

is an invariant, the Chern-Simons invariant. Now we will calculate this invariant. For that purpose we consider the functional (B.4) and its first variation vanishes

$$\delta\mathrm{CS}\left(\partial M, A\right) = 0, \tag{B.8}$$

because of the topological invariance. Then one obtains the equation

$$dA + A \wedge A = 0; \tag{B.9}$$

that is, the extrema of the functional are the connections of vanishing curvature. The set of these connections up to gauge transformations is equal to the set of homomorphisms $\pi_1(\partial M) \rightarrow \mathrm{SU}(2)$ up to conjugation. Thus the calculation of the Chern-Simons invariant reduces to the representation theory of the fundamental group into SU(2). In [45] the authors define a further invariant

$$\tau\left(\Sigma\right) = \min\left\{\mathrm{CS}\left(\alpha\right) \mid \alpha : \pi_1\left(\Sigma\right) \longrightarrow \mathrm{SU}\left(2\right)\right\}, \tag{B.10}$$

for the 3-manifold Σ. This invariants fulfills the relation

$$\tau\left(\Sigma\right) = \frac{1}{8\pi^2}\int_{\Sigma\times\mathbb{R}} \operatorname{tr}\left(F_A \wedge F_A\right), \tag{B.11}$$

which is the minimum of the Yang-Mills action

$$\left|\frac{1}{8\pi^2}\int_{\Sigma\times\mathbb{R}} \operatorname{tr}\left(F_A \wedge F_A\right)\right| \leq \frac{1}{8\pi^2}\int_{\Sigma\times\mathbb{R}} \operatorname{tr}\left(F_A \wedge *F_A\right), \tag{B.12}$$

that is, the solutions of the equation $F_A = \pm * F_A$. Thus the invariant $\tau(\Sigma)$ of Σ corresponds to the self-dual and anti-self-dual solutions on $\Sigma \times \mathbb{R}$, respectively. Or the invariant $\tau(\Sigma)$ is the Chern-Simons invariant for the Levi-Civita connection.

C. Chern-Simons Invariant of Brieskorn Spheres

In [45–47] an algorithm for the calculation of the Chern-Simons invariant for the Brieskorn sphere $\Sigma(p,q,r)$ is presented. According to that result, a representation $\alpha : \pi_1(\Sigma(p,q,r)) \rightarrow \mathrm{SU}(2)$ is determined by a triple of 3 numbers $\langle k,l,m\rangle$ with $0 < k < p$, $0 < l < q$, $0 < m < r$ and the further relations

$$\begin{aligned}
&\frac{l}{q} + \frac{m}{r} < 1 \quad l \bmod 2 = m \bmod 2,\\
&\frac{k}{p} + \frac{l}{q} + \frac{m}{r} > 1,\\
&\frac{k}{p} - \frac{l}{q} + \frac{m}{r} < 1,\\
&\frac{k}{p} + \frac{l}{q} - \frac{m}{r} < 1.
\end{aligned} \tag{C.1}$$

Then the Chern-Simons invariant is given by

$$\mathrm{CS}\left(\alpha\right) = \frac{e^2}{4\cdot p\cdot q\cdot r} \bmod 1, \tag{C.2}$$

with

$$e = k\cdot q\cdot r + l\cdot p\cdot r + m\cdot p\cdot q. \tag{C.3}$$

Now we consider the Poincaré sphere P with $p = 2$, $q = 3$, $r = 5$. Then we obtain

$$\begin{aligned}
\langle 1,1,1\rangle\ \mathrm{CS} &= \frac{1}{120},\\
\langle 1,1,3\rangle\ \mathrm{CS} &= \frac{49}{120},
\end{aligned} \tag{C.4}$$

and for the Brieskorn sphere $\Sigma(2,5,7)$

$$\begin{aligned}
\langle 1,1,3\rangle\ \mathrm{CS} &= \frac{81}{280},\\
\langle 1,3,1\rangle\ \mathrm{CS} &= \frac{9}{280},\\
\langle 1,2,2\rangle\ \mathrm{CS} &= \frac{169}{280},\\
\langle 1,2,4\rangle\ \mathrm{CS} &= \frac{249}{280}.
\end{aligned} \tag{C.5}$$

In [45] the authors define a further invariant

$$\tau\left(\Sigma\right) = \min\left\{\mathrm{CS}\left(\alpha\right) \mid \alpha : \pi_1\left(\Sigma\right) \longrightarrow \mathrm{SU}\left(2\right)\right\}, \tag{C.6}$$

for a homology 3-sphere Σ. For P and $\Sigma(2,5,7)$ one obtains

$$\tau\left(P\right) = \frac{1}{120}, \qquad \tau\left(\Sigma\left(2,5,7\right)\right) = \frac{9}{280}. \tag{C.7}$$

And we are done.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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