

Review Article

The Geometry of Black Hole Singularities

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Recent results show that important singularities in General Relativity can be naturally described in terms of finite and invariant canonical geometric objects. Consequently, one can write field equations which are equivalent to Einstein's at nonsingular points but, in addition remain well-defined and smooth at singularities. The black hole singularities appear to be less undesirable than it was thought, especially after we remove the part of the singularity due to the coordinate system. Black hole singularities are then compatible with global hyperbolicity and do not make the evolution equations break down, when these are expressed in terms of the appropriate variables. The charged black holes turn out to have smooth potential and electromagnetic fields in the new atlas. Classical charged particles can be modeled, in General Relativity, as charged black hole solutions. Since black hole singularities are accompanied by dimensional reduction, this should affect Feynman's path integrals. Therefore, it is expected that singularities induce dimensional reduction effects in Quantum Gravity. These dimensional reduction effects are very similar to those postulated in some approaches to make Quantum Gravity perturbatively renormalizable. This may provide a way to test indirectly the effects of singularities, otherwise inaccessible.

1. Introduction

For millennia, space was considered the fixed background where physical phenomena took place. Special Relativity changed this, by proposing spacetime as the new arena. Then, while trying to extend the success of Special Relativity to noninertial frames and gravity, Einstein realized that one should let go the idea of an immutable background, and General Relativity (GR) was born. There is a very deep interdependence between matter and the geometry of spacetime, encoded in Einstein's equation. Its predictions were tested with high accuracy and confirmed.

However, the task of decoding the way our universe works from something as abstract as Einstein's equation is not easy, and we are far from grasping all of its consequences. For instance, even from the beginning, when Schwarzschild proposed his model for the exterior of a spherically symmetric object, Einstein's equations led to infinities [1, 2]. The Schwarzschild metric tensor becomes infinite at $r = 0$ and on the event horizon, where $r = 2m$. The big bang also exhibited a singularity [3–10].

The first reaction to the singularities was to somehow minimize their importance, on the grounds that they are exceptions due to the perfect symmetry of the solutions. This hope was ruined by the theorems of Penrose [11, 12] and Hawking [13–16], showing that the singularities are predicted to occur in GR under very general conditions and are not caused by the perfect symmetry.

Singularities, hidden by the event horizon or naked, are very well researched in the literature (e.g., [12, 17–25] and references therein).

Interesting results concerning singularities were obtained in some modified gravity theories, for example, $f(R)$ gravity ([26–30] and references therein). Another way to avoid singularities was proposed in nonlinear electrodynamics [31].

In addition to the singularities, infinities occur in GR when we try to quantize gravity, because gravity is perturbatively nonrenormalizable [32, 33]. It is expected by many that a solution to the problem of quantization will also remove the singularities. For example, *Loop quantum cosmology* obtained significant positive results in showing that quantum effects may prevent the occurrence of singularities [34–37].

There is another possibility: the problem of singularities may be in fact not due to GR but to our limited understanding of GR. Therefore, it would be useful to better understand singularities, even in the eventuality that a better theory will replace GR. In the following we review some recent results showing that by confronting singularities, we realize that they are not that undesirable [38]. Moreover, new possibilities open also for the Quantum Gravity problem.

2. The Problem of Singularities in General Relativity

2.1. Two Types of Singularities. Not all singularities are born equal. We can roughly classify the singularities in two types:

- (1) *Malign singularities:* some of the components of the metric are divergent: $g_{ab} \rightarrow \infty$.
- (2) *Benign singularities:* g_{ab} are smooth and finite but $\det g \rightarrow 0$.

Benign singularities turn out to be, in many cases, manageable [39–41]. The infinities simply disappear, if we use different geometric objects to write the equations and describe the phenomena. At points where the metric is nondegenerate, the proposed description is equivalent to the standard one. But, in addition, it works also at the points where the metric becomes degenerate.

Malign singularities appear in the black hole solutions. They appear to be malign because the coordinates in which they are represented are singular. In nonsingular coordinates, they become benign [42–44]. This is somewhat similar to the case of the apparent singularity on the event horizon, which turned out to be a coordinate singularity and not a genuine one [45, 46].

2.2. What Is Wrong with Singularities? The geometry of spacetime is encoded in the metric tensor. To write down field equations, we have to use partial derivatives. In curved spaces, partial derivatives are replaced by *covariant derivatives*. They are defined with the help of the *Levi-Civita connection*, which takes into account the parallel translations, to compare fields at infinitesimally closed points. The covariant derivative is written using the Christoffel symbol of the second kind, obtained from the metric tensor by

$$\Gamma_{ab}^c = \frac{1}{2} g^{cs} (\partial_a g_{bs} + \partial_b g_{sa} - \partial_s g_{ab}). \quad (1)$$

It can be used to define the Riemann curvature tensor:

$$R_{abc}^d = \Gamma_{ac,b}^d - \Gamma_{ab,c}^d + \Gamma_{bs}^d \Gamma_{ac}^s - \Gamma_{cs}^d \Gamma_{ab}^s. \quad (2)$$

It plays a major part in the Einstein equation:

$$G_{ab} + \Lambda g_{ab} = \kappa T_{ab}, \quad (3)$$

since

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}, \quad (4)$$

TABLE 1: Singular objects and their nonsingular equivalents.

Singular	Nonsingular	When g is
Γ_{ab}^c (2nd)	Γ_{abc} (1st)	Smooth
R_{abc}^d	R_{abcd}	Semiregular
R_{ab}	$R_{ab} \sqrt{ \det g }^W, W \leq 2$	Semiregular
R	$R \sqrt{ \det g }^W, W \leq 2$	Semiregular
Ric	$\text{Ric} \circ g$	Quasi-regular
R	$Rg \circ g$	Quasi-regular

where $R_{ab} = R_{asb}^s$ is the Ricci tensor and $R = R_s^s$ is the scalar curvature.

In the case of malign singularities, since some of metric's components are singular, the geometric objects like the Levi-Civita connection and the Riemann curvature tensor are singular too. Therefore, it seems that the situation of malign singularities is hopeless.

Even in the case of benign singularities, when the metric is smooth, but its determinant $\det g \rightarrow 0$, the usual Riemannian objects are singular. For example, the covariant derivative cannot be defined, because the inverse of the metric, g^{ab} , becomes singular ($g^{ab} \rightarrow \infty$ when $\det g \rightarrow 0$). This makes Christoffel's symbols of the second kind (1) and the Riemann curvature (2) singular.

It is therefore understandable why singularities were considered unsolvable problems for so many years.

2.3. From Singular to Nonsingular: A Dictionary. The main variables which appear in the equations are indeed singular. But we can replace them with new variables, which are equivalent to the original ones on the domain where both are defined. Sometimes, we can choose the new variables so that the equations remain valid at points where the original ones were singular.

The geometric objects of interest that become singular when the metric is degenerate are the Levi-Civita connection (1), the Riemann curvature (2), and the Ricci and the scalar curvatures. If the metric is nondegenerate, the Christoffel symbols of the first kind are equivalent to those of the second kind, in the sense that by knowing one of them, we can obtain the other one. Similarly, the Riemann curvature R_{bcd}^a is equivalent to R_{abcd} , and the Ricci and scalar curvatures are equivalent to their densitized versions and to their Kulkarni-Nomizu products (see (30)) with the metric. In some important cases, these equivalent objects remain nonsingular even when the metric is degenerate [39, 41]. We summarize these cases in Table 1.

3. The Mathematical Methods: Singular Semi-Riemannian Geometry

3.1. Singular Semi-Riemannian Geometry. We review the main mathematical tool on which the results presented here are based, named *Singular Semi-Riemannian Geometry* [39, 40]. Singular Semi-Riemannian Geometry is mainly

concerned with the study of singular semi-Riemannian manifolds.

Definition 1 (see [39, 47]). A *singular semi-Riemannian manifold* (M, g) consists in a differentiable manifold M and a symmetric bilinear form g on M , named *metric tensor* or *metric*.

If g is nondegenerate, then (M, g) is just a *semi-Riemannian manifold*. If in addition g is positive definite, (M, g) is named *Riemannian manifold*. In General Relativity semi-Riemannian manifolds are normally used, but when we are dealing with singularities, it is natural to use the Singular Semi-Riemannian Geometry, which is more general.

3.2. Properties of the Degenerate Inner Product. Let (V, g) be an inner product vector space. Let $\flat : V \rightarrow V^*$ be the morphism defined by $u \mapsto u^\bullet := \flat(u) = u^\flat = g(u, -)$. We define the *radical* of V as the set of isotropic vectors in V : $V_\circ := \ker \flat = V^\perp$. We define the *radical annihilator* space of V as the image of \flat , $V^\bullet := \text{im } \flat \subset V^*$. The inner product g induces on V^\bullet an inner product, defined by $g_\bullet(u_1^\bullet, u_2^\bullet) := g(u_1, u_2)$. This one is the inverse of g if and only if $\det g \neq 0$. The *coannihilator* is the quotient space $V_\bullet := V/V_\circ$, given by the equivalence classes of the form $u + V_\circ$. On the *coannihilator* V_\bullet , the metric g induces an inner product $g^\bullet(u_1 + V_\circ, u_2 + V_\circ) := g(u_1, u_2)$.

Let $p \in M$. In the following, we will denote by $T_{\circ,p}M \leq T_pM$ the radical of the tangent space at p , by $T_p^\bullet M \leq T_p^*M$ the radical annihilator and by $T_{\bullet,p}M$ the coannihilator.

We have seen that one important problem which appears when the metric becomes degenerate is that it does not admit an inverse g^{ab} , and fundamental tensor operations like raising indices and contractions between covariant indices are no longer defined. But we can use the reciprocal metric g_\bullet to define metric contraction between covariant indices, for tensors that live in tensor products between T_pM and the subspace $T_p^\bullet M$. This turned out to be enough for some important singularities in General Relativity.

3.3. Covariant Derivative. Because at points where the metric is degenerate there is no inverse metric, the Levi-Civita connection is not defined. Then, how can we derivate? We will see that in some cases, which turn out to be enough for our purposes, we still can derivate.

3.3.1. The Koszul Object. Let X, Y, Z be vector fields on M . We define the *Koszul object* as

$$\begin{aligned} \mathcal{K}(X, Y, Z) := & \frac{1}{2} \{X \langle Y, Z \rangle + Y \langle Z, X \rangle - Z \langle X, Y \rangle \\ & - \langle X [Y, Z] \rangle + \langle Y [Z, X] \rangle + \langle Z [X, Y] \rangle\}. \end{aligned} \quad (5)$$

Its components in local coordinates are just Christoffel's symbols of the first kind:

$$\mathcal{K}_{abc} = \mathcal{K}(\partial_a, \partial_b, \partial_c) = \frac{1}{2} (\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab}) = \Gamma_{abc}. \quad (6)$$

If the metric is nondegenerate, one defines the Levi-Civita connection uniquely, by raising an index of the Koszul object:

$$\nabla_X Y = \mathcal{K}(X, Y, \cdot)^\sharp. \quad (7)$$

But if the metric is degenerate, one cannot raise the index, and we will have to avoid the usage of the Levi-Civita connection. Luckily, we can do what we do with the Levi-Civita connection and more, just by using the Koszul object instead.

3.3.2. The Covariant Derivatives. We define the *lower covariant derivative* of a vector field Y in the direction of a vector field X by

$$(\nabla_X^\flat Y)(Z) := \mathcal{K}(X, Y, Z). \quad (8)$$

This is not quite a true covariant derivative, because it does not map vector fields to vector fields but to 1-forms. However, we can use it to replace the covariant derivative of vector fields, and it is equivalent to it if the metric is nondegenerate.

If the Koszul object satisfies the condition that $\mathcal{K}(X, Y, W) = 0$ for any $W \in \Gamma(T_pM)$, then the singular semi-Riemannian manifold (M, g) is named *radical stationary*. In this case, it makes sense to contract in the third slot of the Koszul object and define by this covariant derivatives of differential forms. The covariant derivative of differential forms is defined by

$$(\nabla_X \omega)(Y) := X(\omega(Y)) - g_\bullet(\nabla_X^\flat Y, \omega), \quad (9)$$

if $\omega \in \mathcal{A}^*(M) := \Gamma(T^*M)$. More general,

$$\begin{aligned} \nabla_X(\omega_1 \otimes \cdots \otimes \omega_s) \\ := \nabla_X(\omega_1) \otimes \cdots \otimes \omega_s + \cdots + \omega_1 \otimes \cdots \otimes \nabla_X(\omega_s). \end{aligned} \quad (10)$$

The covariant derivative of a tensor $T \in \Gamma(\otimes_M^k T^*M)$ is defined as

$$\begin{aligned} (\nabla_X T)(Y_1, \dots, Y_k) = & X(T(Y_1, \dots, Y_k)) \\ & - \sum_{i=1}^k \mathcal{K}(X, Y_i, \cdot) T(Y_1, \dots, \cdot, \dots, Y_k). \end{aligned} \quad (11)$$

3.4. Riemann Curvature Tensor: Semi-Regular Manifolds. Let (M, g) be a radical stationary manifold. Then, the *Riemann curvature tensor* is defined as

$$\begin{aligned} R(X, Y, Z, T) \\ = (\nabla_X \nabla_Y^\flat Z)(T) - (\nabla_Y \nabla_X^\flat Z)(T) - (\nabla_{[X, Y]}^\flat Z)(T). \end{aligned} \quad (12)$$

The components of the Riemann curvature tensor in local coordinates are

$$R_{abcd} = \partial_a \mathcal{K}_{bcd} - \partial_b \mathcal{K}_{acd} + (\mathcal{K}_{ac} \mathcal{K}_{bd} - \mathcal{K}_{bc} \mathcal{K}_{ad}). \quad (13)$$

The Riemann curvature tensor has the same symmetry properties as in Riemannian geometry and is radical annihilator in each of its slots.

A singular semi-Riemannian manifold is called *semiregular* [39] if

$$\nabla_X \nabla_Y^b Z \in \mathcal{A}^*(M). \quad (14)$$

An equivalent condition is

$$\mathcal{K}(X, Y, \cdot) \mathcal{K}(Z, T, \cdot) \in \mathcal{F}(M). \quad (15)$$

It is easy to see that the Riemann curvature of semiregular manifolds is smooth.

3.5. Examples of Semiregular Semi-Riemannian Manifolds. We present some examples of semi-Riemannian manifolds [39, 40].

3.5.1. Isotropic Singularities. Isotropic singularities have the form

$$g = \Omega^2 \tilde{g}, \quad (16)$$

where \tilde{g} is a nondegenerate bilinear form on M .

Such singularities were studied in connection to some cosmological models [48–56].

3.5.2. Degenerate Warped Products. Warped products are products of two semi-Riemannian manifolds (B, g_B) and (F, g_F) , so that the metric on the manifold F is scaled by a scalar function f defined on the manifold B [57]. The warped product has the form

$$ds^2 = ds_B^2 + f^2(p) ds_F^2. \quad (17)$$

Normally, the warping function f is taken to be strictly positive at all points of B . However, it may happen to vanish at some points, and in this case the result is a singular semi-Riemannian manifold. The resulting manifold is semiregular [40]. Moreover, if the manifolds B and F are radical stationary and if $df \in \mathcal{A}^*(M)$, their warped product is radical stationary. If B and F are semiregular, $df \in \mathcal{A}^*(M)$, and $\nabla_X df \in \mathcal{A}^*(M)$ for any vector field X , and then $B \times_f F$ is semiregular [40].

4. Einstein Equations at Singularities

We discuss now two equations which are equivalent to Einstein's when the metric is nondegenerate but remains smooth and finite also at some singularities. The first equation remains smooth at semiregular singularities, while the second at quasi-regular singularities.

4.1. Einstein's Equation on Semi-regular Spacetimes

4.1.1. The Densitized Einstein Equation. Consider the following densitized version of the Einstein equation:

$$G \det g + \Lambda g \det g = \kappa T \det g, \quad (18)$$

or, in coordinates or local frames:

$$G_{ab} \det g + \Lambda g_{ab} \det g = \kappa T_{ab} \det g. \quad (19)$$

If the metric is nondegenerate, this equation is equivalent to the Einstein equation, the only difference is the factor $\det g \neq 0$. But what happens if the metric becomes degenerate? In this case, it is not allowed to divide by $\det g$, because this is 0.

On four-dimensional semi-regular spacetimes Einstein tensor density $G \det g$ is smooth [39]. Hence, the proposed densitized Einstein equation (18) is smooth, and non-singular. If the metric is regular, this equation is equivalent to the Einstein equation.

4.1.2. FLRW Spacetimes. To better understand black hole singularities, which will be discussed later, we start by taking a look at the Friedmann-Lemaître-Robertson-Walker (FLRW) singularities, which are benign. Black hole singularities are malign but can be made benign by removing the coordinate singularity (see Sections 5, 6, and 7).

FLRW spacetimes are examples of degenerate warped products, with the metric defined by

$$ds^2 = -dt^2 + a^2(t) d\Sigma^2, \quad (20)$$

where

$$d\Sigma^2 = \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (21)$$

where $k = 1$ for S^3 , $k = 0$ for \mathbb{R}^3 , and $k = -1$ for H^3 . It follows that they are semiregular.

Since the FLRW singularities are warped products, they are semiregular. Therefore, we can expect that the densitized Einstein equation holds. In fact, in [58] more is shown than that, as we will see now.

The FLRW stress-energy tensor is

$$T^{ab} = (\rho + p) u^a u^b + p g^{ab}, \quad (22)$$

where u^a is the time-like vector field ∂_t , normalized. The scalar ρ represents the mass density and p the pressure density. From the stress-energy tensor (22), in the case of a homogeneous and isotopic universe, follow the *Friedmann equation*:

$$\rho = \frac{3 \dot{a}^2 + k}{\kappa a^2}, \quad (23)$$

and the *acceleration equation*:

$$\rho + 3p = -\frac{6 \ddot{a}}{\kappa a}. \quad (24)$$

Equations (23) and (24) show that the scalars ρ and p are singular for $a = 0$. But ρ and p represent the mass and pressure densities the orthonormal frame obtained by normalizing the comoving frame $(\partial_t, \partial_x, \partial_y, \partial_z)$, where (x, y, z) are coordinates on the space manifold S . The mass and pressure density can be identified with the scalars ρ and p only in an orthogonal frame. But at the singularity $a = 0$ there is no orthonormal frame, so we should not normalize the comoving frame. In general, nonnormalized case, the actual densities contain in fact the factor $\sqrt{-g}(= a^3 \sqrt{g_\Sigma})$:

$$\begin{aligned}\tilde{\rho} &= \rho \sqrt{-g} = \rho a^3 \sqrt{g_\Sigma}, \\ \tilde{p} &= p \sqrt{-g} = p a^3 \sqrt{g_\Sigma}.\end{aligned}\quad (25)$$

The Friedmann and the acceleration equations become

$$\begin{aligned}\tilde{\rho} &= \frac{3}{\kappa} a (\dot{a}^2 + k) \sqrt{g_\Sigma}, \\ \tilde{\rho} + 3\tilde{p} &= -\frac{6}{\kappa} a^2 \dot{a} \sqrt{g_\Sigma}.\end{aligned}\quad (26)$$

We see that $\tilde{\rho}$ and \tilde{p} are smooth and so is the densitized stress-energy tensor:

$$T_{ab} \sqrt{-g} = (\tilde{\rho} + \tilde{p}) u_a u_b + \tilde{p} g_{ab}. \quad (27)$$

We obtain a densitized Einstein equation, from which (18) follows by multiplying with $\sqrt{-g}$.

Hence, the FLRW solution is described by smooth densities even at the big bang singularity. Moreover, the solution extends beyond the singularity.

4.2. Einstein's Equation on Quasi-Regular Spacetimes

4.2.1. The Ricci Decomposition. Let (M, g) be an n -dimensional semi-Riemannian manifold. The Riemann curvature decomposes algebraically [59–61] as

$$R_{abcd} = S_{abcd} + E_{abcd} + C_{abcd}, \quad (28)$$

where

$$\begin{aligned}S_{abcd} &= \frac{1}{n(n-1)} R(g \circ g)_{abcd}, \\ E_{abcd} &= \frac{1}{n-2} (S \circ g)_{abcd}, \\ S_{ab} &:= R_{ab} - \frac{1}{n} R g_{ab},\end{aligned}\quad (29)$$

where \circ denotes the Kulkarni-Nomizu product:

$$(h \circ k)_{abcd} := h_{ac} k_{bd} - h_{ad} k_{bc} + h_{bd} k_{ac} - h_{bc} k_{ad}. \quad (30)$$

If the Riemann curvature tensor on a semiregular manifold (M, g) admits such a decomposition so that all of its terms are smooth, (M, g) is said to be *quasi-regular*.

4.2.2. The Expanded Einstein Equation. For dimension $n = 4$, in [41] we introduced the *expanded Einstein equation*:

$$(G \circ g)_{abcd} + \Lambda (g \circ g)_{abcd} = \kappa (T \circ g)_{abcd} \quad (31)$$

or, equivalently,

$$2E_{abcd} - 6S_{abcd} + \Lambda (g \circ g)_{abcd} = \kappa (T \circ g)_{abcd}. \quad (32)$$

It is equivalent to Einstein's equation if the metric is nondegenerate but in addition extends smoothly at quasi-regular singularities.

4.2.3. Examples of Quasi-Regular Singularities. As shown in [41], the following are examples of quasi-regular singularities:

- (i) isotropic singularities,
- (ii) degenerate warped products $B \times_f F$ with $\dim B = 1$ and $\dim F = 3$,
- (iii) FLRW singularities, as a particular case of degenerate warped products [62],
- (iv) Schwarzschild singularities (after removing the coordinates singularity, see Section 5). The question whether the Reissner-Nordström and Kerr-Newman singularities are quasi-regular, or at least semi-regular, is still open.

4.2.4. The Weyl Curvature Hypothesis and Quasi-Regular Singularities. To explain the low entropy at the big bang and the high homogeneity of the universe, Penrose emitted the Weyl curvature hypothesis, stating that the Weyl curvature tensor vanishes at the big bang singularity [18].

From (28), the *Weyl curvature tensor* is

$$C_{abcd} = R_{abcd} - S_{abcd} - E_{abcd}. \quad (33)$$

In [63] it was shown that when approaching a quasi-regular singularity, $C_{abcd} \rightarrow 0$ smoothly. Because of this, any quasi-regular big bang satisfies the Weyl curvature hypothesis. In [63] it has also been shown that a very large class of big bang singularities, which are not homogeneous or isotropic, are quasi-regular.

4.3. Taming a Malign Singularity. We have seen that when the singularity is benign; that is, the singularity is due to the degeneracy of the metric tensor, which is smooth; there are important cases when we can obtain a complete description of the fields and their evolution, in terms of finite quantities.

But what can we do if the singularities are malign? This case is important, since all black hole singularities are malign. In [42–44] we show that although the black hole singularities appear to be malign, we can make them benign, by a proper choice of coordinates. This is somewhat analog to the method used in [45, 46] to show that the event horizon singularity is not a true singularity, being due to coordinates. In the following sections, we will review these results.

5. Schwarzschild Singularity Is Semi-Regular

The Schwarzschild metric is given in Schwarzschild coordinates by

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\sigma^2, \quad (34)$$

where

$$d\sigma^2 = d\theta^2 + \sin^2\theta d\phi^2. \quad (35)$$

Let us change the coordinates to

$$r = \tau^2, \quad t = \xi\tau^4. \quad (36)$$

The four-metric becomes

$$ds^2 = -\frac{4\tau^4}{2m - \tau^2} d\tau^2 + (2m - \tau^2) \tau^4 (4\xi d\tau + \tau d\xi)^2 + \tau^4 d\sigma^2, \quad (37)$$

which is analytic and semiregular at $r = 0$ [42].

The problems were fixed by a coordinate change. Does not this mean that the singularity depends on the coordinates? Well, this deserves an explanation. Changing the coordinates does not make a singularity appear or disappear, if the coordinate transformation is a local diffeomorphism. But a regular tensor can become singular or a singular tensor can become regular, if the coordinate transformation itself is singular. This situation is very similar to that of the event horizon singularity $r = 2m$ of the Schwarzschild metric, in Schwarzschild coordinates (34). This singularity vanishes when we go to the Eddington-Finkelstein coordinates. This proves that the Eddington-Finkelstein coordinates are from the correct atlas, while the original Schwarzschild coordinates were in fact singular at $r = 2m$. In our case, the coordinate transformation (36) allows us to move to an atlas in which the metric is analytic and semiregular, showing that the Schwarzschild coordinates were in fact singular at $r = 0$.

6. Charged and Nonrotating Black Holes

Charged nonrotating black holes are described by the Reissner-Nordström metric:

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) dt^2 + \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 + r^2 d\sigma^2. \quad (38)$$

To make the singularity benign, we choose the new coordinates ρ and τ [43]; so that

$$t = \tau\rho^T, \quad r = \rho^S. \quad (39)$$

In the new coordinates, the metric has the following form:

$$ds^2 = -\Delta\rho^{2T-2S-2}(\rho d\tau + T\tau d\rho)^2 + \frac{S^2}{\Delta}\rho^{4S-2}d\rho^2 + \rho^{2S}d\sigma^2, \quad (40)$$

where

$$\Delta := \rho^{2S} - 2m\rho^S + q^2. \quad (41)$$

To remove the infinity of the metric at $r = 0$ and ensure analyticity, we have to choose

$$S \geq 1, \quad T \geq S + 1. \quad (42)$$

In the Reissner-Nordström coordinates (t, r, ϕ, θ) , the electromagnetic potential is singular at $r = 0$,

$$A = -\frac{q}{r} dt. \quad (43)$$

But in the new coordinates $(\tau, \rho, \phi, \theta)$, the electromagnetic potential is

$$A = -q\rho^{T-S-1}(\rho d\tau + T\tau d\rho), \quad (44)$$

and the electromagnetic field is

$$F = q(2T - S)\rho^{T-S-1}d\tau \wedge d\rho, \quad (45)$$

and they are analytic everywhere, including at the singularity $\rho = 0$ [43].

The proposed coordinates define a space + time foliation only if $T \geq 3S$ [43].

7. Rotating Black Holes

Electrically neutral rotating black holes are represented by the Kerr solution. If they are also charged, they are described by the very similar Kerr-Newman solution.

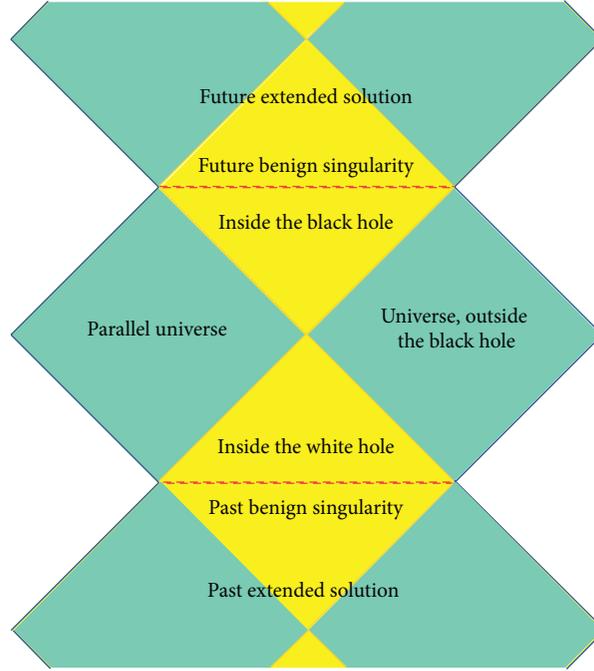
Consider the space $\mathbb{R} \times \mathbb{R}^3$, where \mathbb{R} represents the time coordinate and \mathbb{R}^3 the space, parameterized by the spherical coordinates (r, ϕ, θ) . The rotation is characterized by the parameter $a \geq 0$, $m \geq 0$ is the mass, and $q \in \mathbb{R}$ the charge. The following notations are useful:

$$\Sigma(r, \theta) := r^2 + a^2 \cos^2\theta, \quad (46)$$

$$\Delta(r) := r^2 - 2mr + a^2 + q^2.$$

The nonvanishing components of the Kerr-Newman metric are [64]

$$\begin{aligned} g_{tt} &= -\frac{\Delta(r) - a^2 \sin^2\theta}{\Sigma(r, \theta)}, \\ g_{rr} &= \frac{\Sigma(r, \theta)}{\Delta(r)}, \\ g_{\theta\theta} &= \Sigma(r, \theta), \\ g_{\phi\phi} &= \frac{(r^2 + a^2)^2 - \Delta(r) a^2 \sin^2\theta}{\Sigma(r, \theta)} \sin^2\theta, \\ g_{t\phi} = g_{\phi t} &= -\frac{2a \sin^2\theta (r^2 + a^2 - \Delta(r))}{\Sigma(r, \theta)}. \end{aligned} \quad (47)$$


 FIGURE 1: Schwarzschild solution, analytically extended beyond the $r = 0$ singularity.

In [44] it was shown that in the coordinates τ , ρ , and μ , are defined by

$$\begin{aligned} t &= \tau \rho^{\mathbb{T}}, & r &= \rho^{\mathbb{S}}, \\ \phi &= \mu \rho^{\mathbb{M}}, & \theta &= \theta, \end{aligned} \quad (48)$$

where $\mathbb{S}, \mathbb{T}, \mathbb{M} \in \mathbb{N}$ are positive integers so that

$$\begin{aligned} \mathbb{S} &\geq 1, \\ \mathbb{T} &\geq \mathbb{S} + 1, \\ \mathbb{M} &\geq \mathbb{S} + 1, \end{aligned} \quad (49)$$

and the metric is analytic.

Not only the metric becomes analytic in the proposed coordinates, but also the electromagnetic potential and electromagnetic field. The electromagnetic potential of the Kerr-Newman solution is, in the standard coordinates, the 1-form:

$$A = -\frac{qr}{\Sigma(r, \theta)} (dt - a \sin^2 \theta d\phi). \quad (50)$$

In the proposed coordinates

$$A = -\frac{q\rho^{\mathbb{S}}}{\Sigma(r, \theta)} (\rho^{\mathbb{T}} d\tau + \mathbb{T}\tau\rho^{\mathbb{T}-1} d\rho - a \sin^2 \theta \rho^{\mathbb{M}} d\mu). \quad (51)$$

which is smooth [44]. The electromagnetic field $F = dA$ is smooth too.

8. Global Hyperbolicity and Information Loss

8.1. Foliations with Cauchy Hypersurfaces. While Einstein's equation describes the relation between geometry and matter in a block-world view of the universe, there are equivalent formulations which express this relation from the perspective of the time evolution. Einstein's equation can be expressed in terms of a Cauchy problem [65–70].

The standard black hole solutions pose two main problems to the Cauchy problem. First, the solutions have malign singularities. Second, they have in general Cauchy horizons. Luckily, there is more than one way to skin a black hole.

The evolution equations make sense at least locally, if the singularities are benign. The black hole singularities appear to be malign in the coordinates used so far, but by removing the coordinate's contribution to the singularity, they become benign. Even so, to formulate initial value problems globally, spacetime has to admit space + time foliations. The space-like hypersurfaces have to be Cauchy surfaces; in other words, the global hyperbolicity condition has to be true. The topology of the space-like hypersurfaces must remain independent on the time t , although the metric is allowed to become degenerate. This seems to be prevented in the case of Reissner-Nordström and Kerr-Newman black holes, by the existence of Cauchy horizons. As shown in [71], the stationary black hole singularities admit such foliations and are therefore compatible with the condition of global hyperbolicity.

8.2. Schwarzschild Black Holes. In the proposed coordinates for the Schwarzschild black hole, the metric extends analytically beyond the $r = 0$ singularity (Figure 1).

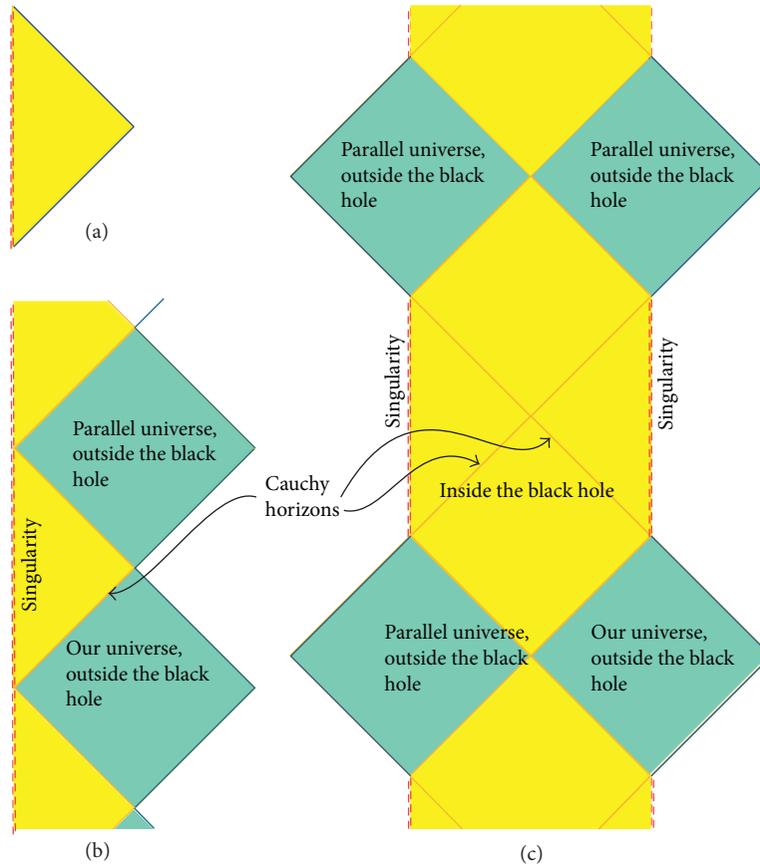


FIGURE 2: Reissner-Nordström black holes. (a) Naked solutions ($q^2 > m^2$). (b) Extremal solution ($q^2 = m^2$). (c) Solutions with $q^2 < m^2$.

This solution can be foliated in space + time and therefore is globally hyperbolic.

8.3. Space-Like Foliation of the Reissner-Nordström Solution. Figure 2 shows the standard Penrose diagrams for the Reissner-Nordström spacetimes [72].

The Penrose diagram 3 shows how our extensions beyond the singularities allow the Reissner-Nordström solutions to be foliated in Cauchy hypersurfaces. In Figures 3(b) and 3(c), in addition to extending the solution beyond the singularity, we cut out the spacetime along the Cauchy horizons. This is justified if the black holes form by collapse at a finite time and then evaporate after a finite lifetime [43, 71].

For the Kerr-Newman black holes, the foliations are similar to those for the Reissner-Nordström solutions [71], especially because the extension proposed in [44] can be chosen so that the closed time-like curves disappear.

8.4. Black Hole Information Paradox. Bekenstein and Hawking discovered that black holes obey laws similar to those of thermodynamics and proposed that these laws are in fact thermodynamics (see [73–75], also [76, 77], and references therein). Hawking realized that black holes evaporate, and the radiation is thermal. This led him to the idea that after evaporation, the information is lost [78–80]. Many solutions were proposed, such as [81–94]. It was proposed

that Quantum Gravity would naturally cure this problem, but it has been suggested that in fact it would make the problem exist even in the absence of black holes [95].

Since the extended Schwarzschild solution can be foliated in space + time (Sections 5 and 8.2), it can be used to represent evaporating electrically neutral nonrotating black holes. The solution can be analytically extended beyond $r = 0$, and hence the affirmation that the information is lost at the singularity is no longer supported. In Figure 4, it can be seen that our solution extends through the singularity and allows the existence of globally hyperbolic spacetimes containing evaporating black holes.

9. Possible Experimental Consequences and Quantum Gravity

9.1. Can We Do Experiments with Singularities? We reviewed the foundations of Singular General Relativity (SGR) and its applications to black hole singularities. SGR is a natural extension of GR, but, nevertheless, it would be great to be able to submit it to experimental tests. We have seen that the solutions are the same as those predicted by Einstein's equation, as long as the metric is nondegenerate. The only differences appear where the metric is degenerate, at singularities. But how can we go to the singularities, or how can we generate singularities, and test the results at the singularities?

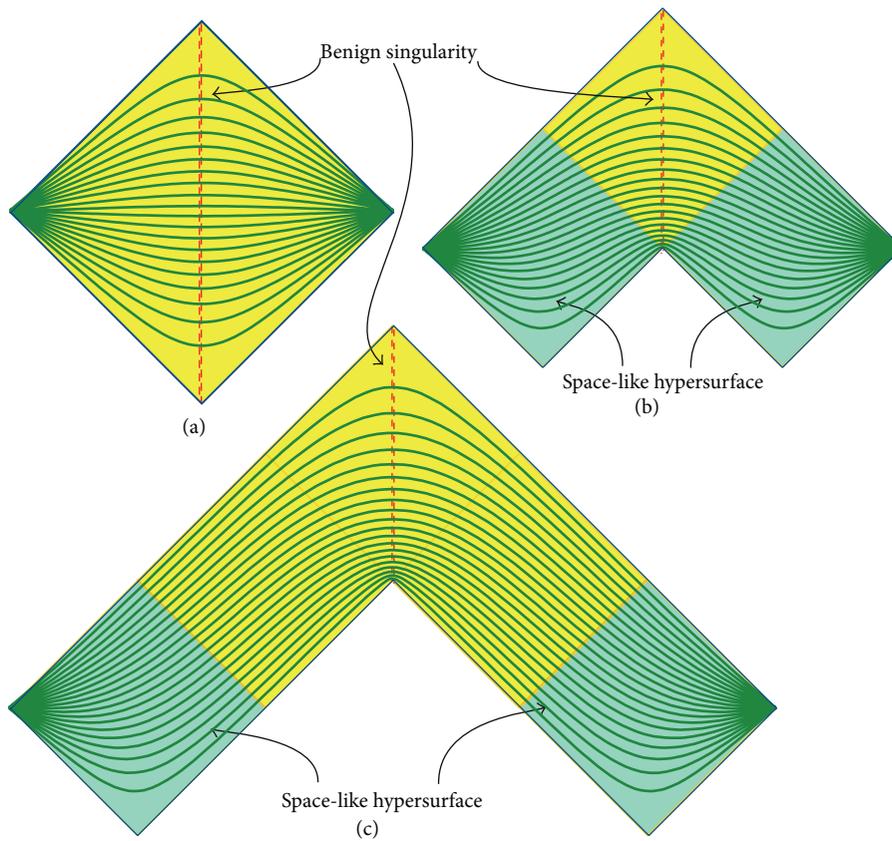


FIGURE 3: Reissner-Nordström black hole solutions, extended beyond the singularities and restricted to globally hyperbolic regions. (a) Naked solutions ($q^2 > m^2$). (b) Extremal solution ($q^2 = m^2$). (c) Solutions with $q^2 < m^2$.

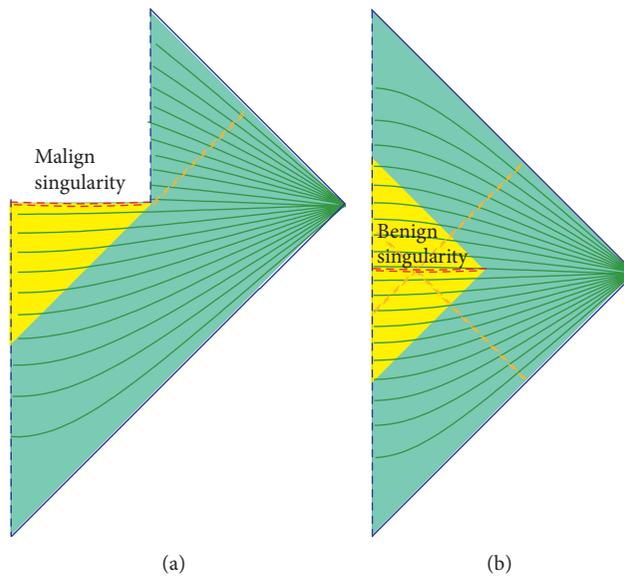


FIGURE 4: (a) Standard evaporating black hole, whose singularity destroys the information. (b) Evaporating black hole extended through the singularity preserves information and admits a space + time foliation.

How could we design an experimental apparatus which is not destroyed by the singularity? It seems that a direct experiment to test the predictions of SGR is not possible.

What about indirect tests? For example, if information is preserved, this would be evidence in favor of SGR. But how can we test this? Can we monitor a black hole, from the time when it is formed to the time when it evaporates completely, and check that the information is preserved during this entire process? The current knowledge predicts that this information will be anyway extremely scrambled. Even if we would be able to do this someday, the conservation of information is predicted by a long list of other approaches to Hawking's information loss paradox (see Section 8.4).

In General Relativity, classical elementary particles can be considered small black holes. If they are pointlike and have definite trajectories, then they are singularities, like the Schwarzschild, Reissner-Nordström, and Kerr-Newman singularities. To go from classical to quantum, one applies path integrals over the classical trajectories. In this way, possible effects of the singularities may also be present at the points where the metric is nonsingular.

In [96] we suggested that the geometric and topological properties we identified at singularities have implications to Quantum Gravity (QG), as we shall see in the following. This suggests that it might be possible to test our approach by QG effects. One feature that seems to be required by most, if not all approaches to QG, is dimensional reduction. Singular General Relativity shows that singularities are accompanied in a natural way by dimensional reduction.

9.2. Dimensional Reduction in QFT And QG. Various results obtained in Quantum Field Theory (QFT) and in QG suggest that at small scales a dimensional reduction should take place. The definition and the cause of this reduction differ from one approach to another. Here is just a small part of the literature using one form of dimensional reduction or another to obtain regularization in QFT and QG:

- (i) fractal universe [97, 98], based on a Lebesgue-Stieltjes measure or a fractional measure [99], fractional calculus, and fractional action principles [100–109];
- (ii) topological dimensional reduction [110–114];
- (iii) vanishing dimensions at LHC [115];
- (iv) dimensional reduction in QG [116–118];
- (v) asymptotic safety [119];
- (vi) Hořava-Lifschitz gravity [120];
- (vii) other approaches to Quantum Gravity based on dimensional reduction including [121–126].

Some of these types of dimensional reduction are very similar to those predicted by SGR to occur at benign singularities.

9.3. Is Dimensional Reduction due to the Benign Singularities? Quantum Gravity is perturbatively nonrenormalizable, but it can be made renormalizable by assuming one kind or another of dimensional reduction. The above mentioned approaches

did this, by modifying General Relativity. In this section we point that several types of dimensional reduction, which were postulated by various authors, occur naturally at our semiregular and quasi-regular singularities [96].

9.3.1. Geometric Dimensional Reduction. First, at each point where the metric becomes degenerate, a geometric or *metric reduction* takes place, because the rank of the metric is reduced:

$$\dim T_{p^*}M = \dim T_p^*M = \text{rank } g_p. \quad (52)$$

9.3.2. Topological Dimensional Reduction. From the Kupeli theorem [47] follows that for constant signature, the manifold is locally a *product* $M = P \times_0 N$ between a manifold of lower dimension P and another manifold N with metric 0. In other words, from the viewpoint of geometry, a region where the metric is degenerate and has constant signature can be identified with a lower dimensional space. This suggests a connection with the *topological dimensional reduction* explored by Shirkov and Fizev [110–114].

9.3.3. Vanishing of Gravitons. If the singularity is quasi regular, the Weyl tensor $C_{abcd} \rightarrow 0$ as approaching a quasi-regular singularity. This implies that the *local degrees of freedom*, that is, the gravitational waves for GR and the gravitons for QG, vanish, allowing by this the needed renormalizability [116].

9.3.4. Anisotropy between Space and Time. In [43] we obtained new coordinates, which make the Reissner-Nordström metric analytic at the singularity. In these coordinates, the metric is given by (40). A *charged particle* with spin 0 can be viewed, at least classically, as a Reissner-Nordström black hole. The above metric reduces its dimension to $\dim = 2$.

To admit space + time foliation in these coordinates, we should take $T \geq 3S$. An open research problem is whether this anisotropy is connected to the similar anisotropy from *Hořava-Lifschitz* gravity, introduced in [120].

9.3.5. Measure Dimensional Reduction. In the *fractal universe approach* [97, 98, 127], one expresses the measure in the integral

$$S = \int_{\mathcal{M}} d\varrho(x) \mathcal{L}, \quad (53)$$

in terms of some functions $f_{(\mu)}(x)$, some of them vanishing at low scales:

$$d\varrho(x) = \prod_{\mu=0}^{D-1} f_{(\mu)}(x) dx^\mu. \quad (54)$$

In *Singular General Relativity*,

$$d\varrho(x) = \sqrt{-\det g} dx^D. \quad (55)$$

If the metric is diagonal in the coordinates (x^μ) , then we can take

$$f_{(\mu)}(x) = \sqrt{|g_{\mu\mu}(x)|}. \quad (56)$$

This suggests that the results obtained by Calcagni by considering the universe to be fractal follow naturally from the benign metrics.

9.4. Dimensional Reduction and Quantum Gravity. The Singular General Relativity approach leads, as a side effect, to various types of dimensional reduction, which are similar to those proposed in the literature to make Quantum Gravity perturbatively renormalizable. By investigating the nonrenormalizability problems appearing when quantizing gravity, many researchers were led to the conclusion that the problem would vanish if one kind of dimensional reduction or another is postulated (sometimes ad hoc). By contrary, our approach led to this as a natural consequence of understanding the singularities.

Of course, in SGR the dimensional reduction appears at the singularity, while QG is expected to be perturbatively renormalizable everywhere. But if classical particles are singularities, quantum particles behave like sums over histories of classical particles. Thus, at any point there will be virtual singularities to contribute to the Feynman integrals. This means that the effects will be present everywhere. They are expected as a reduction of the determinant of the metric, and of the Weyl curvature tensor, which allows the desired regularization. Moreover, as the energy increases, the order of the Feynman diagrams in the same region increases, and we expect that the dimensional reduction effects induced by singularities become more significant too. It is an open question at this time whether this dimensional reduction is enough to regularize gravity, but this research is just at the beginning.

10. Conclusions

We reviewed some of our results of Singular General Relativity [38], concerning the black hole singularities. Some singularities allow the canonical and invariant construction of geometric objects which remain smooth and nonsingular. By using these objects, one can write equations which are equivalent to Einstein's equations outside singularities but in addition extend smoothly at singularities. The FLRW big bang singularities turn out to be of this type. The black hole singularities can be made so by removing the coordinate singularity for the charged black hole singularities, the electromagnetic potential and field become smooth. The singularities of the black hole having a finite life span are compatible with global hyperbolicity and conservation of information. Such singularities are accompanied by dimensional reduction, a feature which is desired by many approaches to Quantum Gravity. While in these approaches dimensional reduction is obtained by modifying General Relativity, these singularities lead naturally to it, within the framework of GR.

There is a rich literature concerning gravity, black holes, and singularities in lower or higher dimensions (see e.g., [76, 128–130] and references therein). While the geometric apparatus of Singular Semi-Riemannian Geometry reviewed in Section 3 works for other dimensions too, in this review we focused only on four-dimensional spacetimes, and some of the results do not work in more dimensions.

Conflict of Interests

The author declares that there is no conflict of interests section regarding the publication of this paper.

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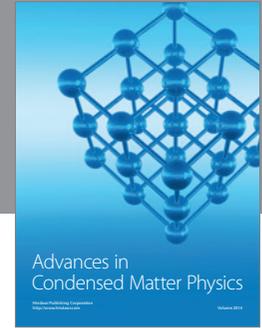
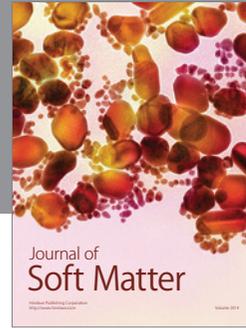
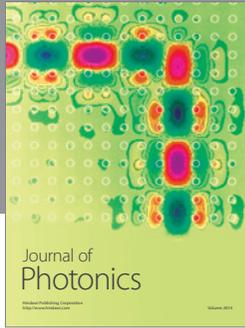
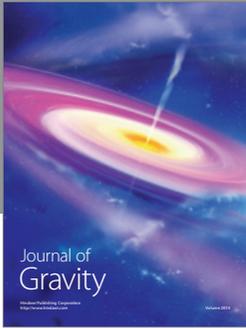
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