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Research Article

Axially Symmetric-dS Solution in Teleparallel $f(T)$ Gravity Theories

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We apply a tetrad field with six unknown functions to Einstein field equations. Exact vacuum solution, which represents axially symmetric-dS spacetime, is derived. We multiply the tetrad field of the derived solution by a local Lorentz transformation which involves a generalization of the angle ϕ and get a new tetrad field. Using this tetrad, we get a differential equation from the scalar torsion $T = T^{\alpha}{}_{\mu\nu} S_{\alpha}{}^{\mu\nu}$. Solving this differential equation we obtain a solution to the $f(T)$ gravity theories under certain conditions on the form of $f(T)$ and its first derivatives. Finally, we calculate the scalars of Riemann Christoffel tensor, Ricci tensor, Ricci scalar, torsion tensor, and its contraction to explain the singularities associated with this solution.

1. Introduction

The discovery of the acceleration of the universe through the SNeIa Hubble diagram has been later confirmed by wide range of data, from more recent SNeIa data to BAOs and CMBR anisotropies [1–12]. Such overwhelming abundance of observational evidences in favor of the cosmic speed up does not fit in the framework of GR making clear that our theoretical background is seriously flawed [13].

The idea of unifying the gravitation and electromagnetism was made by Einstein [14] in 1928. This attempt was based on the mathematical structure of teleparallelism, also referred to as distant or absolute parallelism. In other words, the idea was the introduction of a tetrad field, a field of orthonormal bases on the tangent spaces at each point of the four-dimensional spacetime. The tetrad has sixteen components whereas the gravitational field, represented by the spacetime metric, has only ten. The six additional degrees of freedom of the tetrad were then supposed by Einstein to be related to the six components of the electromagnetic field [15–20]. This attempt of unification did not succeed because the additional six degrees of freedom of the tetrad are actually eliminated by the 6-parameter local Lorentz invariance of the theory. However, Einstein introduced concepts that remain important to the present day. Teleparallelism could be considered by using the Weitzenböck connection that is curvatureless but has torsion, rather than the curvature defined by the Levi-Civita connection [21].

Similar to the exotic dark energy and other modified gravity models, it is found that the cosmic acceleration can be obtained successfully from another gravitational scenario, $f(T)$ theory [22]. It is based on the teleparallel equivalent of general relativity (TEGR) which is known as teleparallel gravity (TG). A scalar torsion T is the Lagrangian of teleparallel gravity. The teleparallel gravity is not a new theory of gravity but an alternative geometric formulation of general relativity (GR). In teleparallel gravity, the Levi-Civita connection used in Einstein's GR is replaced by the Weitzenböck connection with torsion. However, the torsion vanishes in the dark energy and modified gravity models. Moreover, $f(T)$ theories have several interesting features. They not only can explain the late accelerating expansion, but also always have second order differential equations, which is simpler than the $f(R)$ gravity. In addition, when certain conditions are satisfied, the behavior of $f(T)$ will be similar to quintessence [23]. Although $f(T)$ gravity has attracted

wide attention, a disadvantage which has been pointed out in [24, 25] is that the action and its field equations do not satisfy local Lorentz symmetry. However, the investigation about $f(T)$ theories is meaningful because it can provide some hints about Lorentz violation.

Up till now, a number of $f(T)$ theories have been proposed [22, 26–56]. Under these cases it is shown that $f(T)$ theories are not dynamically equivalent to teleparallel action plus a scalar field [50]. Like other gravity theories and models, the $f(T)$ theories also have been investigated using the popular observational data. Investigations show that the $f(T)$ theories are compatible with observations (see, e.g., [57, 58] and references therein). So, we note that the new type of $f(T)$ theory in [54] was proposed to explain the acceleration. It behaves like a cosmological constant but is free from the coincidence problem. In order to have a better study of $f(T)$ gravitational theories, we try to search for exact solutions for the field equations of $f(T)$. Specifically, we are interested in axially symmetric solutions which asymptotically behave as dS.

Within $f(T)$ gravitational theory there are many solutions, spherically symmetric [59, 60], spherically symmetric charged [61], homogenous anisotropic [62], and stability of the Einstein static closed and open universe [63]. Some cosmological features of the ΛCDM model in the framework of the $f(T)$ are investigated [64]. However, up till now, no axially symmetric-dS solution derived in this theory. It is the aim of the present study to find an analytic, axially symmetric-dS solution in $f(T)$ gravitational theories. In Section 2, preliminaries of $f(T)$ gravitational theory are presented. In Section 3, a tetrad field with six unknown functions is applied to the field equation of $f(T)$. New analytic solution, vacuum one with two constants of integration, for Einstein field equation is derived. In Section 4, we multiply this solution by a local Lorentz transformation with a generalization of the angle ϕ, that is, $N(\phi)$. Using the calculations carried out in Section 3, we succeed to derive special solution for $f(T)$ gravitational theories under certain conditions on $f(T)$ and its first derivative. Final section is devoted to sum up the results obtained.

2. Preliminaries of $f(T)$

In a spacetime with absolute parallelism the parallel vector field $e_i{}^\mu$ [21] defines the nonsymmetric affine connection:

$$\Gamma^\mu{}_{\nu\rho} \overset{\text{def.}}{=} e_i{}^\mu e^i{}_{\nu,\rho}, \tag{1}$$

where $e_{i\mu,\nu} = \partial_\nu e_{i\mu}$. "We use the Greek indices $\mu, \nu, \ldots$ for local holonomic spacetime coordinates and the Latin indices $i, j, \ldots$ label (co)-frame components."

The curvature tensor defined by $\Gamma^\mu{}_{\nu\rho}$, given by (1), is identically vanishing. The metric tensor $g_{\mu\nu}$ is defined by

$$g_{\mu\nu} \overset{\text{def.}}{=} \eta_{ij} e^i{}_\mu e^j{}_\nu, \tag{2}$$

with $\eta_{ij} = (-1, +1, +1, +1)$ being the metric of Minkowski spacetime. Defining the torsion and the contortion components as

$$\begin{aligned} T^\mu{}_{\nu\rho} &\overset{\text{def.}}{=} \Gamma^\mu{}_{\rho\nu} - \Gamma^\mu{}_{\nu\rho} = e_i{}^\mu \left(\partial_\nu e^i{}_\rho - \partial_\rho e^i{}_\nu\right), \\ K^{\mu\nu}{}_\rho &\overset{\text{def.}}{=} -\frac{1}{2}\left(T^{\mu\nu}{}_\rho - T^{\nu\mu}{}_\rho - T_\rho{}^{\mu\nu}\right), \end{aligned} \tag{3}$$

where the contortion equals the difference between Weitzenböck and Levi-Civita connection; that is, $K^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\nu\rho} - \left\{{}^{\mu}_{\nu\rho}\right\}$.

The skew symmetric tensor $S_\mu{}^{\nu\rho}$ is defined as

$$S_\mu{}^{\nu\rho} \overset{\text{def.}}{=} \frac{1}{2}\left(K^{\nu\rho}{}_\mu + \delta^\nu_\mu T^{\lambda\rho}{}_\lambda - \delta^\rho_\mu T^{\lambda\nu}{}_\lambda\right). \tag{4}$$

The torsion scalar is defined as

$$T \overset{\text{def.}}{=} T^\mu{}_{\nu\rho} S_\mu{}^{\nu\rho}. \tag{5}$$

The action of $f(T)$ theory is given by

$$\mathscr{L}\left(e^a{}_\mu, \Phi_A\right) = \int d^4x e \left[\frac{1}{16\pi}\left(f(T) - 2\Lambda\right) + \mathscr{L}_{\text{Matter}}\left(\Phi_A\right)\right], \tag{6}$$

where $e = \sqrt{-g} = \det(e^a{}_\mu)$, $\mathscr{L}_{\text{Matter}}$ is the Lagrangian of matter field, Λ is the cosmological constant, and Φ_A are matter fields.

Similar to the $f(R)$ theory, one can define the action of $f(T)$ theory as a function of the fields $e^a{}_\mu$ and by putting the variation of the function with respect to the field $e^a{}_\mu$ to be vanishing, one can obtain the following equations of motion:

$$\begin{aligned} &S_\mu{}^{\nu\rho} T_{,\rho} f(T)_{TT} + \left[e^{-1} e^a{}_\mu \partial_\rho \left(e e_a{}^\alpha S_\alpha{}^{\nu\rho}\right) - T^\alpha{}_{\mu\lambda} S_\alpha{}^{\nu\lambda}\right] f(T)_T \\ &\quad - \frac{1}{4}\delta^\nu_\mu \left(f(T) - 2\Lambda\right) = 8\pi \mathscr{T}_\mu{}^\nu, \end{aligned} \tag{7}$$

where

$$T_{,\rho} = \frac{\partial T}{\partial x^\rho}, \qquad f(T)_T = \frac{\partial f(T)}{\partial T}, \qquad f(T)_{TT} = \frac{\partial^2 f(T)}{\partial T^2}, \tag{8}$$

and $\mathscr{T}_\mu{}^\nu$ is the energy momentum tensor.

Now we are going to rewrite (7) in another form: the field equations (7) are written in terms of the tetrad and its partial derivatives. These equations appear to be different from Einstein's field equations. Following [24, 25], one can obtain an equation relating T with the Ricci scalar of the metric R. These will make the equivalence between TG and GR clear. On the other hand, the tetrad cannot be eliminated completely in favor of the metric in (7), because of the lack of local Lorentz symmetry, but the latter can be brought in a form that closely resembles Einstein's equation. This form is more suitable for constructing analytic solutions in the $f(T)$ theory. To start writing the field equations in a covariant version, one must replace partial derivatives in the tensors by

covariant derivatives compatible with the metric $g_{\mu\nu}$, that is, ∇_σ where $\nabla_\sigma g_{\mu\nu} = 0$. Thus, (3) can be written as

$$T^{\mu}{}_{\nu\rho} = e_a{}^{\mu}\left(\nabla_\nu e^a{}_\rho - \nabla_\rho e^a{}_\nu\right). \tag{9}$$

Using (9) in (3) and (4) one can get

$$\begin{aligned} K^{\mu}{}_{\nu\rho} &= e_a{}^{\mu}\nabla_\rho e^a{}_\nu, \\ S^{\mu\nu}{}_{\rho} &= \eta^{ij}e_i{}^{\mu}\nabla_\rho e_j{}^{\nu} + \delta^{\nu}{}_{\rho}\eta^{ij}e_i{}^{\sigma}\nabla_\sigma e_j{}^{\mu} - \delta^{\mu}{}_{\rho}\eta^{ij}e_i{}^{\sigma}\nabla_\sigma h_j{}^{\nu}. \end{aligned} \tag{10}$$

On the other hand, from the relation between Weitzenböck connection and the Levi-Civita connection one can write the Riemann tensor for the Levi-Civita connection, $\left\{{}^{\rho}_{\mu\nu}\right\}$, in terms of the nonsymmetric connection, $\Gamma^{\rho}{}_{\mu\nu}$, in the form

$$\begin{aligned} R^{\rho}{}_{\mu\lambda\nu} &= \partial_\lambda\Gamma^{\rho}{}_{\mu\nu} - \partial_\nu\Gamma^{\rho}{}_{\mu\lambda} + \Gamma^{\rho}{}_{\sigma\lambda}\Gamma^{\sigma}{}_{\mu\nu} - \Gamma^{\rho}{}_{\sigma\nu}\Gamma^{\sigma}{}_{\mu\lambda} \\ &= \nabla_\lambda K^{\rho}{}_{\mu\nu} - \nabla_\nu K^{\rho}{}_{\mu\lambda} + K^{\rho}{}_{\sigma\lambda}K^{\sigma}{}_{\mu\nu} - K^{\rho}{}_{\sigma\nu}K^{\sigma}{}_{\mu\lambda}. \end{aligned} \tag{11}$$

The associated Ricci tensor can then be written as

$$R_{\mu\nu} = \nabla_\nu K^{\rho}{}_{\mu\rho} - \nabla_\rho K^{\rho}{}_{\mu\nu} + K^{\rho}{}_{\sigma\nu}K^{\sigma}{}_{\mu\rho} - K^{\rho}{}_{\sigma\rho}K^{\sigma}{}_{\mu\nu}. \tag{12}$$

Now, by using $K^{\sigma}{}_{\mu\nu}$ given by (3) along with the relations $K^{(\mu\nu)\sigma} = T^{\mu(\nu\sigma)} = S^{\mu(\nu\sigma)} = 0$ and considering that $S^{\mu}{}_{\rho\mu} = 2K^{\mu}{}_{\rho\mu} = -T^{\mu}{}_{\rho\mu}$ one has [24–53]

$$\begin{aligned} R_{\mu\nu} &= -\nabla^\lambda S_{\nu\lambda\mu} - g_{\mu\nu}\nabla^\lambda T^{\rho}{}_{\lambda\rho} - S^{\rho\sigma}_{\mu}K_{\sigma\rho\nu}, \\ R &= -T - 2\nabla^\mu T^{\rho}{}_{\mu\rho} = -T - \frac{2}{e}\partial^\mu\left(eT^{\rho}{}_{\mu\rho}\right). \end{aligned} \tag{13}$$

Equation (13) implies that the T and R differ only by a covariant divergence of a spacetime vector. Therefore, the Einstein-Hilbert action and the teleparallel action (i.e., $\mathscr{L}_{\mathrm{TEGR}} = \int d^4x|e|T$) will both lead to the same field equations and are dynamically equivalent theories. In [24, 25] the authors have shown that this equivalence is directly at the level of the field equations. By using the equations listed above and after some algebraic manipulations, one can get

$$G_{\mu\nu} - \frac{g_{\mu\nu}}{2}T = -\nabla^\rho S_{\nu\rho\mu} - S^{\sigma\rho}{}_{\mu}K_{\rho\sigma\nu}, \tag{14}$$

where $G_{\mu\nu} = R_{\mu\nu} - (1/2)g_{\mu\nu}R$ is the Einstein tensor.

With the aid of the equations listed above, it can be shown, after some algebraic manipulations, that [24, 25]

$$G_{\mu\nu} = e^{-1}e^a{}_\mu\partial_\rho\left(ee_a{}^\alpha S_{\alpha\nu}{}^\rho\right) - T^\alpha{}_{\mu\lambda}S_{\alpha\nu}{}^\lambda + \frac{1}{2}g_{\mu\nu}T. \tag{15}$$

Finally, by using (14) and (15), the field equations for $f(T)$ gravity (7) can be rewritten in the form

$$\begin{aligned} &f(T)_T G_{\mu\nu} - \frac{1}{2}\left(f(T) - 2\Lambda - Tf(T)_T\right)g_{\mu\nu} \\ &\quad + S_\nu{}^\rho{}_\mu T_{,\rho} f(T)_{TT} = 8\pi\mathscr{T}_{\mu\nu}. \end{aligned} \tag{16}$$

Equation (16) can be taken as the starting point of the $f(T)$ modified gravity model, and it has a structure similar to the field equation of $f(R)$ gravity. Note that in the more general case with $f(T) \neq T$ the field equations are covariant form. Nevertheless, the theory is not local Lorentz invariant. In case of $f(T) = T$ and constant scalar torsion, $f(T_0)$, GR is recovered and field equations are covariant and the theory is Lorentz invariant.

3. Axially Symmetric Solution in $f(T)$ Gravity Theory

The tetrad field that is stationary and has axial symmetry takes the form

$$\left(h^i{}_\mu\right) = \begin{bmatrix} S_1(r,\theta) & 0 & 0 & S_2(r,\theta) \\ 0 & S_3(r,\theta) & 0 & 0 \\ 0 & 0 & S_4(r,\theta) & 0 \\ S_5(r,\theta) & 0 & 0 & S_6(r,\theta) \end{bmatrix}, \tag{17}$$

where $S_i(r,\theta)$, $i = 1\cdots 6$, are unknown functions of the radial coordinate, r, and the azimuthal angle θ. Applying tetrad field (17) to the field equations of GR, $R_{\mu\nu} - (1/2)g_{\mu\nu}(R - 2\Lambda)$, we get a system of lengthy partial nonlinear differential equations. The solution of these system has the form

$$\begin{gathered} S_1(r,\theta) = \sqrt{\frac{\delta(r)}{\Sigma(r,\theta)}}, \qquad S_2(r,\theta) = S_7(r,\theta)S_1(r,\theta), \\ S_3(r,\theta) = \frac{1}{S_1(r,\theta)}, \qquad S_4(r,\theta) = \frac{\sqrt{\Sigma(r,\theta)}}{f(\theta)}, \\ S_5(r,\theta) = \frac{\beta\sin\theta}{\sqrt{\Sigma(r,\theta)}}, \qquad S_6(r,\theta) = \frac{S_8(r,\theta)}{\sqrt{\Sigma(r,\theta)}}, \\ \text{where } \delta(r) = \left(r^2 + c_2{}^2\right)\left(1 - \frac{\Lambda r^2}{3}\right) - 2c_1 r, \\ \Sigma(r,\theta) = r^2 + c_2{}^2\cos^2\theta, \qquad f(\theta) = 1 + \frac{c_2{}^2\Lambda\cos^2\theta}{3}, \\ S_7 = -\Omega c_2\sin^2\theta, \qquad \Omega = 1 + \frac{c_2{}^2\Lambda}{3}, \\ \beta = c_2\sqrt{f(\theta)}, \qquad S_8(r,\theta) = -\Omega\sin\theta\sqrt{\left(r^2 + c_2{}^2\right)f(\theta)}. \end{gathered} \tag{18}$$

The metric associated with tetrad field (17) after using (18) has the following form. "For brevity, we write $S_i(r,\theta) \equiv S_i$, $i = 1\cdots 6$":

$$\begin{aligned} ds^2 &= \left[S_1{}^2 - S_5{}^2\right]dt^2 - S_3{}^2dr^2 - S_4{}^2d\theta^2 \\ &\quad - \left[S_6{}^2 - S_2{}^2\right]d\phi^2 - \left[S_5S_6 - S_1S_2\right]dtd\phi, \end{aligned} \tag{19}$$

which is the Kerr-dS spacetime provided that $c_1 = M$ and $c_2 = a$ where M and a are the mass of the gravitational system and the rotation parameter [65]. Solution (18) is a solution to Einstein field equations; however the main task of the present study is to find an axially symmetric-dS solution for $f(T)$. To

do such aim, we multiply tetrad field (17) with the following $so(3)$ matrix:

$$\left(\Lambda^{i}{}_{j}\right)=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \sin\theta\cos\Phi & \cos\theta\sin\Phi & -\sin\Phi\\ 0 & \sin\theta\sin\Phi & \cos\theta\sin\Phi & \cos\Phi\\ 0 & \cos\theta & -\sin\theta & 0\end{pmatrix}, \quad (20)$$

where Φ is a function of azimuthal angle ϕ; that is, $\Phi = N(\phi)$. "Here we consider Φ to be function of ϕ only which is sufficient to carry out our aim. However, if Φ becomes a general function, that is, depends on r, θ and ϕ the calculations will be very complicated." The new tetrad field,

$$\left(e^{i}{}_{\mu}\right)_{1}=\Lambda^{i}{}_{j}e^{j}{}_{\mu}, \quad (21)$$

takes the form

$$\left(e^{i}{}_{\mu}\right)_{1}=\begin{pmatrix}\frac{\sqrt{\delta(r)}}{\sqrt{\Sigma(r,\theta)}} & 0 & 0 & \frac{a\sin^{2}\theta\sqrt{\delta(r)}}{\Omega\sqrt{\Sigma(r,\theta)}}\\ \frac{-a\sin\theta\sin\Phi\sqrt{f(\theta)}}{\sqrt{\Sigma(r,\theta)}} & \frac{\sin\theta\cos\Phi\sqrt{\Sigma(r,\theta)}}{\sqrt{\delta(r)}} & \frac{\cos\theta\cos\Phi\sqrt{\Sigma(r,\theta)}}{\sqrt{f(\theta)}} & -\frac{\left(r^{2}+a^{2}\right)\sin\theta\sin\Phi\sqrt{f(\theta)}}{\Omega\sqrt{\Sigma(r,\theta)}}\\ \frac{a\sin\theta\cos\Phi\sqrt{f(\theta)}}{\sqrt{\Sigma(r,\theta)}} & \frac{\sin\theta\sin\Phi\sqrt{\Sigma(r,\theta)}}{\sqrt{\delta(r)}} & \frac{\cos\theta\sin\Phi\sqrt{\Sigma(r,\theta)}}{\sqrt{f(\theta)}} & \frac{\left(r^{2}+a^{2}\right)\sin\theta\cos\Phi\sqrt{f(\theta)}}{\Omega\sqrt{\Sigma(r,\theta)}}\\ 0 & \frac{\cos\theta\sqrt{\Sigma(r,\theta)}}{\sqrt{\delta(r)}} & \frac{\sin\theta\sqrt{\Sigma(r,\theta)}}{\sqrt{f(\theta)}} & 0\end{pmatrix}. \quad (22)$$

We rewrite (16) as

$$\begin{aligned}&G_{\mu\nu}+\frac{1}{f(T)_{T}}\Lambda g_{\mu\nu}\\&=\frac{1}{f(T)_{T}}\left(\frac{1}{2}\left(f(T)-Tf(T)_{T}\right)g_{\mu\nu}\right.\\&\qquad\left.-S_{\nu}{}^{\rho}{}_{\mu}T_{,\rho}f(T)_{TT}+8\pi\mathcal{T}_{\mu\nu}\right),\\&\Longrightarrow G_{\mu\nu}+\frac{1}{f(T)_{T}}\Lambda g_{\mu\nu}=\frac{1}{f(T)_{T}}\left(4\pi\mathcal{T}_{\mu\nu}+T^{\text{eff}}{}_{\mu\nu}\right),\end{aligned} \quad (23)$$

where $T^{\text{eff}}{}_{\mu\nu}=(1/2)[f(T)-Tf(T)_{T}]g_{\mu\nu}-S_{\nu}{}^{\rho}{}_{\mu}T_{,\rho}f(T)_{TT}$ is the effective energy momentum tensor.

To find a vacuum solution within $f(T)$ theories, it is sufficient to find the condition that makes the effective energy momentum tensor $T^{\text{eff}}{}_{\mu\nu}$ vanishing. The simplest condition that satisfies this aim is the vanishing of the scalar torsion, T. The imposition of $T=0$ is just one (simplest) solution, and in principle there can exist many more; for example, when the torsion scalar is constant the effective energy momentum tensor $T^{\text{eff}}{}_{\mu\nu}$ will also be vanishing under some constraints on the form of f (constant) and its first derivative. Using (6) and tetrad field (22) one can obtain $h=\det(h^{a}{}_{\mu})=(r^{2}+a^{2}\cos^{2}\theta)\sin\theta/(1+\Lambda a^{2})$. With the use of (3), (4), and (5) we obtain the torsion scalar as

$$\begin{aligned}T=\sqrt{f(\theta)\delta(r)}\Big(&2\Omega\Sigma^{2}N'-2a^{6}\Lambda\cos^{6}\theta\\&-2\left[1+\frac{\Lambda}{3}\left(5r^{2}-a^{2}\right)\right]a^{4}\cos^{4}\theta\\&-2r\left(\frac{5\Lambda}{3}r^{3}+4M\right)a^{2}\cos^{2}\theta\\&+2r^{4}\left[1-\frac{\Lambda}{3}\left(3r^{2}+a^{2}\right)\right]\Big)\\&+2\Sigma^{2}\left(3+a^{2}\Lambda\cos^{2}\theta+3\Omega N'\right)\\&\cdot\left[\frac{r\Lambda}{3}\left(2r^{2}+a^{2}\right)-r+M\right],\end{aligned} \quad (24)$$

where $N'=dN(\phi)/d\phi$. The exact solution of (24) has the form

$$\begin{aligned}N(\phi)=&\frac{\phi\sqrt{f(\theta)}}{\Omega\Sigma^{2}\left(2\Lambda r^{3}-3r+ra^{2}\Lambda+3M+3\sqrt{f(\theta)\delta(r)}\right)}\\&\cdot\Big(a^{2}f(\theta)\left(2a^{2}\cos^{4}\theta+\cos^{2}\theta\left[3r^{2}+a^{2}\cos^{2}\theta\right]-r^{2}\right)\\&+\sqrt{\delta(r)}\Sigma\left\{\left[2r^{3}\Lambda+\left\{a^{2}\Lambda-3\right\}r+M\right]r\right.\\&\qquad\left.-a^{4}\sin^{2}\theta\cos^{2}\theta\Lambda\right\}\\&-9\sqrt{\delta^{3}(r)}\left[r^{2}-a^{2}\cos^{2}\theta\right]\\&-3\Sigma^{2}\sqrt{f(\theta)}\left(3M+2r^{3}\Lambda-3r+ra^{2}\Lambda\right)\Big).\end{aligned} \quad (25)$$

Using (25) and (22) in (23) we get a solution to $f(T)$ gravitational theory provided that $f(0)=0$.

4. Singularities

Now we are going to study the singularities of the derived solution. For this purpose we search the value at which r makes g_{00} and g_{11} tend to zero or ∞. This procedure may reproduce singularities corresponding to coordinate singularities. Therefore to procedure correct singularities we are going to demonstrate the invariants. In orthodox

general relativity or its modifications the invariants are the Ricci scalar, the Kretschmann scalar, or other invariants constructed from Riemann tensor and its contractions. In the teleparallel geometry of gravity we have two approaches of finding invariants. In the first approach one uses the solution of the vierbein and the Weitzenböck connection to calculate torsion invariants such as the torsion scalar T. In the second approach one uses the solution to construct metric and then construct the Levi-Civita connection and finally calculate curvature invariants such as the Ricci and Kretschmann scalars. The comparison of the two approaches is able to show differences between curvature and torsion gravity. In teleparallel theories we mean by singularity of spacetime the singularity of the scalar concomitants of the torsion and curvature tensors.

The torsion invariant T, that is, the torsion scalar, that arises from the vierbein solution (22) with the use of Weitzenböck's connection (25) is vanishing. The curvature invariants that arise from the metric solution (19) through the calculation of the Levi-Civita connection are [61]

$$\begin{aligned}
&R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma}\\
&= \frac{1}{3\Sigma^6}\left[8a^{12}\Lambda^2\cos^{12}\theta + 48a^{10}r^2\Lambda^2\cos^{10}\theta\right.\\
&\quad + 120a^8r^4\Lambda^2\cos^8\theta + 16a^6\cos^6\theta\left[10r^6\Lambda^2 - 9M^2\right]\\
&\quad + 120a^4r^2\cos^4\theta\left[r^6\Lambda^2 + 18M^2\right]\\
&\quad + 48a^2r^4\cos^2\theta\left[r^6\Lambda^2 - 45M^2\right]\\
&\quad \left. + 144r^6M^2 + 8r^{12}\Lambda^2\right],\\
&R^{\mu\nu}R_{\mu\nu} = 4\Lambda^2, \qquad R = -4\Lambda^2 = -2\nabla_\alpha T^\alpha,\\
&T^{\mu\nu\lambda}T_{\mu\nu\lambda} = \frac{-f(r,\theta)}{27\sin^2\theta\delta(r)\Sigma^5 f(\theta)^{3/2}},\\
&T^\mu T_\mu = \frac{-f_1(r,\theta)}{27\sin^2\theta\delta(r)\Sigma^3 f(\theta)},\\
&t^{\mu\nu\lambda}t_{\mu\nu\lambda} = \frac{-f_2(r,\theta)}{27\sin^2\theta\delta(r)\Sigma^3 f(\theta)},\\
&a^\mu a_\mu = \frac{2a^2\left[r^2\delta(r)\sin^2\theta + \cos^2\theta f(\theta)\right]}{9\delta(r)\Sigma f(\theta)},
\end{aligned} \tag{26}$$

where $f(r,\theta)$, $f_1(r,\theta)$, and $f_2(r,\theta)$ are functions of r and θ. Here $t_{\mu\nu\lambda}$, T_μ, and a_μ are the irreducible representation of the torsion tensor:

$$\begin{aligned}
&t_{\lambda\mu\nu} = \frac{1}{2}\left(T_{\lambda\mu\nu} + T_{\mu\lambda\nu}\right) + \frac{1}{6}\left(g_{\nu\lambda}V_\mu + g_{\mu\nu}V_\lambda\right) - \frac{1}{3}g_{\lambda\mu}V_\nu,\\
&T_\mu = T^\lambda{}_{\lambda\nu},\\
&a_\mu = \frac{1}{6}\epsilon_{\mu\nu\rho\sigma}T^{\nu\rho\sigma},
\end{aligned} \tag{27}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is defined by

$$\epsilon_{\mu\nu\rho\sigma} \overset{\text{def.}}{=} \sqrt{-g}\delta_{\mu\nu\rho\sigma} \tag{28}$$

with $\delta_{\mu\nu\rho\sigma}$ being completely antisymmetric and normalized as $\delta_{0123} = -1$.

Observing the forms of Ricci and Kretschmann torsion and its irreducible representation scalars in (26) we deduce that in the Kerr-dS case there are no divergence points at either $r = 0$ or $r = \infty$.

5. Main Results and Discussion

$f(T)$ gravitational theories are modifications of the TEGR that try to deal with the recent problems appearing in cosmology. In these theories, it is not easy to find exact solutions. We have rewritten the field equations of these theories in a simple form. This form enables us to show the extra terms that are responsible for the deviation from GR theory. In the nonvacuum case, those terms can be regarded as the effective energy momentum tensor and generally they depend on the scalar torsion and its derivatives. For vacuum solution one must show that the effective energy momentum tensors are vanishing. So if the torsion scalar is vanishing then it turns out that the extra terms are vanishing provided some constraints on the form of the zero function, that is, $f(0)$ and its first derivative.

The equivalence between GR and TEGR emerges from the property that Einstein-Hilbert Lagrangian differs from TEGR Lagrangian by a four-divergence term. By using the Levi-Civita scalar curvature R for the metric (2) one gets the result of (13). Therefore, TEGR and Einstein-Hilbert actions are equivalent up to the equations of motion. This means that GR and TEGR possess the same number of degrees of freedom. In spite of the fact that the tetrad field contains 16 components (6 more than the metric field), TEGR is invariant under local Lorentz transformations of the tetrad due to the existence of the divergence term. In the TEGR the tetrad field changes under local Lorentz transformation as

$$e^i{}_\mu = \Lambda^i{}_j e^j{}_\mu, \qquad e_i{}^\mu = \Lambda_i{}^j e_j{}^\mu. \tag{29}$$

This transformation adds a four-divergence to the TEGR Lagrangian eT. This behavior is evident in (13) because eR is invariant under local Lorentz transformations. Instead $f(T)$ gravity, like other theories of amended gravity, possesses extra degrees of freedom. In fact the dynamical equations (7) are sensitive to local Lorentz transformations of the tetrad except for the case $f(T) = T$ (i.e., TEGR). This implies that the dynamical equations of $f(T)$ gravitational theories contain information not only about the evolution of the metric but also about some extra degrees of freedom exclusively associated with the tetrad that are not present in the undeformed theory [24, 25, 33–51]. For $f(T)$ theories, the Lagrangian changes under a local Lorentz transformation as

$$ef(T) = ef\left(T + \text{divergence term}\right). \tag{30}$$

In this case the divergence term remains free inside the function f damaging the principle of invariance under

local Lorentz transformation. The loss of the local Lorentz invariance leads to a preferred global reference frame defined by the autoparallel curves of the manifold that consistently solve the dynamical equations. This means that (7) not only determines the metric but also chooses some other properties of the tetrad field. The tetrads connected by local Lorentz transformations lead to the same metric; however they are different with respect to the parallel framework. Because of this fundamental property of $f(T)$ theories, when one is searching for solutions of a given symmetry it is quite difficult to do an ansatz for the tetrad field. For the metric case symmetry helps us to choose suitable coordinates to write the metric. However, this does not say much about the ansatz for the tetrad due to local Lorentz transformation [66, 67]. Certainly, in the context of $f(T)$ theories, the proper frame which parallelizes the spacetime for a given symmetry of the geometry must be independent of the function f [68]. This work is focused on how to find the parallelization for axially symmetric solutions in $f(T)$ theories. In particular, we want to know whether Kerr-dS geometry survives or not in $f(T)$ gravity. To answer this question we should find the correct ansatz to solve (7). This search is greatly facilitated by invoking the following argument concerning the survival of certain TEGR solutions [67]: if a vacuum solution of $f(T)$ gravity has $T = 0$, then it will be a solution of TEGR as well (a cosmological constant might be necessary). In fact, the replacement of $T = 0$ in (7), after neglecting the cosmological constant, leads to

$$e^{-1}e^{a}{}_{\mu}\partial_{\rho}\left(ee_{a}{}^{\alpha}S_{\alpha}{}^{\nu\rho}\right) - T^{\alpha}{}_{\mu\lambda}S_{\alpha}{}^{\nu\lambda} - \frac{1}{4}\delta^{\nu}_{\mu}\frac{f(0)}{f'(0)} = 0, \qquad (31)$$

which is a TEGR vacuum equation with cosmological constant $2\Lambda = f(0)/f'(0)$. We can neglect the cosmological constant term by restricting the family of functions f to have $f(0) = 0$ and $f'(0) \neq 0$. In other words, we can utilize the freedom to do local Lorentz transformations in TEGR to search a tetrad having $T = 0$; if we succeed, then we will state that such solution survives in $f(T)$ gravity. Notice that TEGR vacuum solutions do not force T to vanish; R must vanish. Thus (13) says that T is a four-divergence. So, the former argument is based on the sensitivity of T to local Lorentz transformations. The above argument means that TEGR vacuum solutions having $T = 0$ (or $T =$ constant) cannot be deformed by $f(T)$ gravity. We have shown that this is the case for Kerr-dS geometry; what is meant that $f(T)$ gravity is unable to smooth the singularity of a black hole [67, 69, 70].

In this study, we have used a diagonal tetrad field with six unknown functions of redial coordinate r and azimuthal angle θ. This type of tetrad does not depend on local Lorentz transformation. This tetrad field has been applied to the field equations of GR with a cosmological constant and got a system of nonlinear partial differential equations. This system have been solved and a solution with two constants of integration is derived. The associated metric of this solution is similar to Kerr-dS solution provided that the two constants of integration are the gravitational mass of the system and the rotating parameter [65]. The tetrad field of this solution has been multiplied by $so(3)$ which contains a generalization of the angle ϕ, that is, $N(\phi)$. The new tetrad field has been used to calculate the scalar torsion and differential equation with one unknown; $N(\phi)$ has been derived. This differential equation has been solved, assuming the vanishing of the scalar torsion, and the form of the unknown function has been derived. This form of the unknown function has been used in the new tetrad field which has been shown that the new tetrad is a special solution to $f(T)$ provided that $f(0) = 0$.

The singularities of the derived solution have been discussed and has been shown that the derived solution has no singularities at either $r = 0$ or $r = \infty$. This is done through the calculations of the scalars of Ricci, Kretschman, torsion tensor, and its irreducible representation. The situation of the singularities is expected to change for higher order of $f(T)$.

This is a first step in deriving a special solution within $f(T)$ gravitational theories, which proves that any GR solution can be regarded as a solution in $f(T)$ under some conditions [58]. This procedure can be generalized by trying to derive solutions for quadratic $f(T)$ by trying to solve the effective energy momentum tensor which will be responsible for such quadratic theory. This will be done elsewhere.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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