Research Article

Generalized Solutions of the Dirac Equation, W Bosons, and Beta Decay

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We study the 7×7 Hagen-Hurley equations describing spin 1 particles. We split these equations, in the interacting case, into two Dirac equations with nonstandard solutions. It is argued that these solutions describe decay of a virtual W boson in beta decay.

1. Introduction

Recently, we have shown that in the free case covariant solutions of the $s = 0$ and $s = 1$ Duffin-Kemmer-Petiau (DKP) equations are generalized solutions of the Dirac equation [1]. These wavefunctions are nonstandard since they involve higher-order spinors. We have demonstrated recently that in the $s = 0$ case the generalized solutions describe decay of a pion [2]. The aim of this work is to interpret spin 1 solutions, possibly in the context of weakly decaying particles.

There are several relativistic equations describing spin 1 particles; see [3, 4] for the reviews. The most common approach to study properties of spin 1 bosons is based on the 10×10 DKP equations (the DKP particles are bosons [5]). Several classes of potentials were used in DKP equations to investigate interactions of spin 1 particles [6–15]. However, we will apply the 7×7 Hagen-Hurley equations [16–18] in spinor form [1, 19, 20]. Our motivation stems from the observation that these equations violate parity and thus should describe weakly interacting particles (see [21] for weak interactions and violation of parity).

In the next section we transform the Hagen-Hurley equations, in the interacting case, into two Dirac equations with nonstandard solutions involving higher-order spinors, extending our earlier results described in [1]. These generalized solutions bear some analogy to generalized solutions of the Dirac equation argued to describe a lepton and three quarks [22]. In Section 3 we describe transition from nonstandard solutions of two Dirac equations to the Dirac

equation for a lepton and the Weyl equation for a neutrino. In the last section we show that the transition is consistent with decay of a virtual W boson in beta decay. In what follows we are using definitions and conventions of [23].

2. Generalized Solutions of the Dirac Equation in the Interacting Case

We have shown recently that, in the noninteracting case, solutions of the $s = 0$ and $s = 1$ DKP equations are generalized solutions of the Dirac equation [1]. In our derivation we have split the 10×10 DKP equations for $s = 1$ into two 7×7 Hagen-Hurley equations [16–18]. Let us note here that in the case of interaction with external fields such splitting is not possible since the identities (27) of [24], enabling the splitting, are not valid in the interacting case. Therefore, we will base our theory on the 7×7 formulation; see (18) and (19) in [1] and Subsection 6(ii) in [19]. These equations violate parity P [19], where $P : x^0 \rightarrow x^0, x^i \rightarrow -x^i$ ($i = 1, 2, 3$). Since parity is violated in weak interactions [21], these equations seem to describe weakly interacting particles.

We write one of these 7×7 equations (Eq. (19) of [1]), in the interacting case, in the form

$$\pi_B^A \zeta_{A\dot{D}} = m \chi_{B\dot{D}}, \quad (1a)$$

$$\pi_A^{\dot{D}} \chi_{B\dot{D}} = -m \zeta_{A\dot{B}},$$

$$\chi_{B\dot{D}} = \chi_{\dot{D}B}, \quad (1b)$$

where (1b) is the spin 1 constraint [19]. In (1a) we have $\pi^{A\dot{B}} = (\sigma^0 \pi^0 + \vec{\sigma} \cdot \vec{\pi})^{A\dot{B}}$, $\pi^\mu = p^\mu - qA^\mu$, where σ^k ($k = 1, 2, 3$) are the Pauli matrices and σ^0 is the 2×2 unit matrix. Equations (1a) and (1b), which were first proposed by Dirac [20], can be written in the 7×7 Hagen-Hurley form:

$$\beta^\mu p_\mu \Psi = m\Psi, \quad (2)$$

with β^μ matrices given in [3], with $\Psi = (\chi_{1i}, \chi, \chi_{22}, \zeta_{1i}, \zeta_{12}, \zeta_{2i}, \zeta_{22})^T$, where $\chi \stackrel{\text{df}}{=} \chi_{12} = \chi_{21}$ and T stands for transposition of a matrix.

Equations (1a) in explicit form read

$$\begin{aligned} -(\pi^1 + i\pi^2) \chi_{1i} - (\pi^0 - \pi^3) \chi_{2i} &= -m\zeta_{1i}, \\ (\pi^0 + \pi^3) \chi_{1i} + (\pi^1 - i\pi^2) \chi_{2i} &= -m\zeta_{2i}, \end{aligned} \quad (3a)$$

$$\begin{aligned} -(\pi^1 - i\pi^2) \zeta_{1i} - (\pi^0 - \pi^3) \zeta_{2i} &= m\chi_{1i}, \\ (\pi^0 + \pi^3) \zeta_{1i} + (\pi^1 + i\pi^2) \zeta_{2i} &= m\chi_{2i}, \\ -(\pi^1 + i\pi^2) \chi_{12} - (\pi^0 - \pi^3) \chi_{22} &= -m\zeta_{12}, \\ (\pi^0 + \pi^3) \chi_{12} + (\pi^1 - i\pi^2) \chi_{22} &= -m\zeta_{22}, \end{aligned} \quad (3b)$$

$$-(\pi^1 - i\pi^2) \zeta_{12} - (\pi^0 - \pi^3) \zeta_{22} = m\chi_{12},$$

$$(\pi^0 + \pi^3) \zeta_{12} + (\pi^1 + i\pi^2) \zeta_{22} = m\chi_{22},$$

where condition (1b), $\chi_{B\dot{D}} = \chi_{D\dot{B}}$, is not imposed. We thus get two Dirac equations or, alternatively, a single Dirac equation

with generalized solution $\mathbb{B} = \begin{pmatrix} \zeta_{1i} & \zeta_{12} \\ \zeta_{2i} & \zeta_{22} \\ \chi_{1i} & \chi_{12} \\ \chi_{2i} & \chi_{22} \end{pmatrix}$:

$$(\pi^0 \gamma^0 - \pi^1 \gamma^1 - \pi^2 \gamma^2 - \pi^3 \gamma^3) \mathbb{B} = m\mathbb{B}, \quad (4)$$

generalizing (24) of [1].

3. Decay of Spin 1 Bosons

We note that solutions of two Dirac equations (3a) and (3b) are nonstandard since they involve higher-order spinors rather than spinors ξ_A, η_B . To interpret (3a) and (3b) we put

$$\chi_{B\dot{D}}(x) = \eta_B(x) \alpha_{\dot{D}}(x), \quad (5a)$$

$$\zeta_{A\dot{B}}(x) = \xi_A(x) \alpha_{\dot{B}}(x), \quad (5b)$$

where $\alpha_{\dot{A}}(x)$ is the Weyl spinor, describing massless neutrinos, while $\eta_B(x)$ and $\xi_A(x)$ are the Dirac spinors. Our analysis below will not be exact since neutrinos are massive [21]. On the other hand, their masses are very small so this approximation should not lead to significant errors.

Note that now $\chi_{12} \neq \chi_{21}$ and, accordingly, the spin is not determined: more exactly, the spin is in the $0 \oplus 1$ space. It means that we consider not real but virtual (off-shell) bosons [25]. This substitution is in the spirit of the method of fusion of de Broglie [26, 27] (similar ansatz was used in the $s = 0$

case [2]). After the substitution of (5a) and (5b) into (3a) and (3b) we obtain two equations:

$$\begin{aligned} -(\pi^1 + i\pi^2) \eta_i \alpha_{\dot{A}} - (\pi^0 - \pi^3) \eta_2 \alpha_{\dot{A}} &= -m\xi_1 \alpha_{\dot{A}}, \\ (\pi^0 + \pi^3) \eta_i \alpha_{\dot{A}} + (\pi^1 - i\pi^2) \eta_2 \alpha_{\dot{A}} &= -m\xi_2 \alpha_{\dot{A}}, \\ -(\pi^1 - i\pi^2) \xi_1 \alpha_{\dot{A}} - (\pi^0 - \pi^3) \xi_2 \alpha_{\dot{A}} &= m\eta_i \alpha_{\dot{A}}, \\ (\pi^0 + \pi^3) \xi_1 \alpha_{\dot{A}} + (\pi^1 + i\pi^2) \xi_2 \alpha_{\dot{A}} &= m\eta_2 \alpha_{\dot{A}}, \end{aligned} \quad (6)$$

where $\dot{A} = \dot{1}, \dot{2}$ and, after substituting solution of the Weyl equation,

$$p^{A\dot{B}} \alpha_{\dot{B}} = 0, \quad (7)$$

$\alpha_{\dot{A}}(x) = \hat{\alpha}_{\dot{A}} e^{ik \cdot x}$, $k^\mu k_\mu = 0$, we get a single Dirac equation for spinors $\xi_A(x)$ and $\eta_B(x)$:

$$\begin{aligned} -(\tilde{\pi}^1 + i\tilde{\pi}^2) \eta_i - (\tilde{\pi}^0 - \tilde{\pi}^3) \eta_2 &= -m\xi_1, \\ (\tilde{\pi}^0 + \tilde{\pi}^3) \eta_i + (\tilde{\pi}^1 - i\tilde{\pi}^2) \eta_2 &= -m\xi_2, \\ -(\tilde{\pi}^1 - i\tilde{\pi}^2) \xi_1 - (\tilde{\pi}^0 - \tilde{\pi}^3) \xi_2 &= m\eta_i, \\ (\tilde{\pi}^0 + \tilde{\pi}^3) \xi_1 + (\tilde{\pi}^1 + i\tilde{\pi}^2) \xi_2 &= m\eta_2, \end{aligned} \quad (8)$$

where $\tilde{\pi}^\mu \equiv \pi^\mu + k^\mu$, since components $\alpha_1(x)$ and $\alpha_2(x)$ cancel out.

Equations (7) and (8) describe two spin 1/2 particles, whose spins can couple to $s = 0$ or $s = 1$: that is, $1/2 \otimes 1/2 = 0 \oplus 1$.

4. Conclusions

Results obtained in Sections 2 and 3 cast new light on the Hagen-Hurley equations as well as on weak decays of spin 1 bosons. We have shown that transition from (1a) and (1b), describing a spin $s = 1$ particle, to (7) and (8), via substitution of (5a) and (5b), which means that now $s \in 0 \oplus 1$, corresponds to decay of this particle into a Weyl antineutrino, cf. (7), and a Dirac lepton, cf. (8). Indeed, it should be a weak decay since (1a) and (1b) violate parity. The spin of this particle becomes undetermined in the process of decay; more exactly it belongs to the $0 \oplus 1$ space: this suggests that this is a virtual particle. Therefore, the products, a lepton and a Weyl antineutrino (this suggests maximal parity violation [25]), should have total spin 0 or 1 and there should be a third particle to secure spin conservation.

The above description fits a (three-body) beta decay with formation of a virtual W^- boson, decaying into a lepton and antineutrino. This is most conveniently explained in the case of a mixed beta decay [28]:

$$\begin{aligned} n(\uparrow) &\longrightarrow \begin{cases} p(\downarrow) + [e(\uparrow) \bar{\nu}_e(\uparrow)], & \text{Gamow-Teller transition,} \\ p(\uparrow) + [e(\uparrow) \bar{\nu}_e(\downarrow)], & \text{Fermi transition,} \end{cases} \end{aligned} \quad (9)$$

where products of the W^- boson decay (see [29]) are shown in square brackets and $(\uparrow)$ denotes spin $1/2$: this seems to correspond well to the proposed transition from (1a) and (1b) to (7) and (8). Since spin of the products of decay of the virtual W^- boson belongs to the $0 \oplus 1$ space, their spin can be $s = 0$ or $s = 1$. Moreover, in the case of the Gamow-Teller transition there must be a spin-flip in the decaying nucleon. Let us add here the fact that in reaction (9) some neutrons (82%) decay according to the Gamow-Teller mechanism while some (18%) undergo the Fermi transition [28]. This mixed mechanism is explained by decoupled spins of the just born products: indeed, the condition $\chi_{12} = \chi_{21}$ for the spinor χ_{AB} , due to substitution (5a), does not hold and spin of the products is in the $0 \oplus 1$ space. More about properties of $W^\pm$ bosons can be found in [21].

It is now obvious that another set of 7×7 equations, involving spinor η_{AB} rather than χ_{AB} (see (18) of [1]), describes a β^+ decay with intermediate W^+ boson. Let us note finally that kinematics of the neutrino appears in the Dirac equation for the electron with arbitrary neutrino four-momentum, suggesting a continuous distribution of neutrino energy.

Competing Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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