Research Article

Relativistic Energy Analysis of Five-Dimensional q -Deformed Radial Rosen-Morse Potential Combined with q -Deformed Trigonometric Scarf Noncentral Potential Using Asymptotic Iteration Method

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We study the exact solution of Dirac equation in the hyperspherical coordinate under influence of separable q -deformed quantum potentials. The q -deformed hyperbolic Rosen-Morse potential is perturbed by q -deformed noncentral trigonometric Scarf potentials, where all of them can be solved by using Asymptotic Iteration Method (AIM). This work is limited to spin symmetry case. The relativistic energy equation and orbital quantum number equation l_{D-1} have been obtained using Asymptotic Iteration Method. The upper radial wave function equations and angular wave function equations are also obtained by using this method. The relativistic energy levels are numerically calculated using Matlab, and the increase of radial quantum number n causes the increase of bound state relativistic energy level in both dimensions $D = 5$ and $D = 3$. The bound state relativistic energy level decreases with increasing of both deformation parameter q and orbital quantum number n_l .

1. Introduction

Dirac equation as relativistic wave equation was formulated by P. A. M Dirac in 1928; the exact solution of Dirac equation for some quantum potentials plays a fundamental role in relativistic quantum mechanics [1]. In order to investigate nuclear shell model, spin symmetry and pseudospin symmetry solutions of Dirac equations have been an important field of study in nuclear physics. The concept of spin symmetry and pseudospin symmetry limit with nuclear shell model has been used widely in explaining a number of phenomena in nuclear physics and related field [2]. In nuclear physics, spin symmetry and pseudospin symmetry concepts have been used to study the aspect of deformed and super deformation nuclei. The concept of spin symmetry has been applied to the level of meson and antinucleon [3]. Pseudospin symmetry has

been observed in deformed nuclei and can be enhanced in heavy proton-rich nuclei [4].

Solutions of Dirac equation for some potentials under limit case of spin symmetry and pseudospin symmetry have been investigated intensively whether in three- [5, 6], two-, or one- [7–13] dimensional space some D -dimensional spherical symmetric spacetimes [14, 15]. However, The D -dimensional Dirac equation with $(D - 1)$ -dimensional separable noncentral potential has not been investigated yet; therefore, it may be worthy to investigate Dirac equation in 5 dimensions with separable 4-dimensional noncentral potential in this study.

In recent years, some researchers have studied solution of Dirac equation with quantum potentials with different application and methods. These investigations include Eckart potential and trigonometric Manning-Rosen potential

using Asymptotic Iteration Method (AIM) [5], q -deformed hyperbolic Pöschl-Teller potential and trigonometric Scarf II noncentral potential using Nikiforov-Uvarov method [6], q -deformed trigonometric Scarf potential with q -deformed Trigonometric Tensor Coupling Potential for Spin and Pseudospin Symmetries Using Romanovski Polynomial [7], generalized nuclear Wood-Saxon potential under relativistic spin symmetry limit [8], relativistic bound states of particle in Yukawa field with Coulomb tensor interaction [9], Rosen-Morse potential including the spin-orbit centrifugal term using Nikiforov-Uvarov (NU) method [3], pseudospin symmetric solution of the Morse potential for any κ state using AIM [11], Scalar, Vector, and Tensor Cornell Interaction using Ansatz method [12], Scalar and Vector Generalized Isotonic Oscillators and Cornell Tensor Interaction using Ansatz method [13], Mie-type potentials for energy dependent pseudoharmonic potential via SUSYQM [16], trigonometric Scarf potential in D -dimension for spin and pseudospin symmetry using Nikiforov-Uvarov (NU) method [15], Coulombic potential and its thermodynamics properties in D -dimensional space using NU method [17], and hyperbolic tangent potential and its application in material properties in D -dimensional space [18].

Asymptotic Iteration Methods (AIM) have small deviation for determination of eigenenergies and eigenfunctions of Dirac equation. The separable D -dimensional quantum potentials are not studied yet by some researchers. In this paper, we use Asymptotic Iteration Method (AIM) to solve the Dirac equation under influence of separable D -dimensional quantum potentials. The relativistic energy levels can be obtained from calculation of relativistic energy equation using Matlab R2013a. In Section 2, we present basic theory of Dirac equation in hyperspherical coordinate with D -dimensional separable quantum potential. In this section, deformed quantum potential is also included which is proposed by Dutra in 2005 [19]. In Section 3, we present Asymptotic Iteration Method. Result and discussion are included in Section 4, and in Section 5 we present the special case in 3-dimensional space. In the last section, we present conclusion.

2. Dirac Equation with Separable q -Deformed Quantum Potential in the Hyperspherical Coordinates

For single particle, Dirac equation with vector potential $V(r)$ and scalar potential $S(r)$ in the hyperspherical coordinate can be expressed as follows (in the unit $\hbar = c = 1$) [13, 14]:

$$\begin{aligned} \{\hat{\alpha} \cdot \vec{p} + \hat{\beta}(M + S(\vec{r}))\} \psi(\vec{r}) \\ = \{E - V(\vec{r})\} \psi(\vec{r}), \end{aligned} \quad (1)$$

where $\vec{p}$, E , and M are D -dimensional momentum operator, total relativistic energy, and relativistic mass of the particle, respectively:

$$\hat{\alpha}_i = \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix}, \quad (2)$$

$$\hat{\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (3)$$

where $\hat{\sigma}_i$ are Pauli's matrices and $\mathbf{1}$ is the 2×2 unit matrix. Here we use relations between Pauli's matrices as

$$\hat{\sigma}_i \hat{\sigma}_j + \hat{\sigma}_j \hat{\sigma}_i = 2\delta_{ij} \mathbf{1}. \quad (4)$$

The wave function of Dirac spinor can be classified in two forms, upper spinor $\chi(\vec{r})$ and lower spinor $\varphi(\vec{r})$ as follows [15, 16]:

$$\begin{aligned} \psi(\vec{r}) &= \begin{pmatrix} \chi(\vec{r}) \\ \varphi(\vec{r}) \end{pmatrix} \\ &= \begin{pmatrix} \frac{F_{nk}(r)}{r^{(D-1)/2}} Y_{\ell_1, \dots, \ell_{D-1}}^{\ell}(\hat{x} = \theta_1, \theta_2, \dots, \theta_{D-1}) \\ i \frac{G_{nk}(r)}{r^{(D-1)/2}} Y_{\tilde{\ell}_1, \dots, \tilde{\ell}_{D-1}}^{\tilde{\ell}}(\hat{x} = \theta_1, \theta_2, \dots, \theta_{D-1}) \end{pmatrix}. \end{aligned} \quad (5)$$

By substituting (2), (3), and (5) into (1), we get

$$\begin{aligned} \vec{\sigma} \cdot \vec{p} \varphi(\vec{r}) &= -(M - E + (V(\vec{r}) + S(\vec{r}))) \chi(\vec{r}), \\ \vec{\sigma} \cdot \vec{p} \chi(\vec{r}) &= (M + E - (V(\vec{r}) - S(\vec{r}))) \varphi(\vec{r}). \end{aligned} \quad (6)$$

Exact spin symmetry limit is characterized with $V(\vec{r}) = S(\vec{r})$, but for spin symmetry limit $V(r) - S(r) = \Delta$ which is constant, and for pseudospin symmetry limit $V(r) + S(r) = \Sigma$ which is also constant. For spin symmetry limit we have $d\Delta(\vec{r})/dr = 0$ since $\Delta(\vec{r}) = C_s = \text{constant}$; therefore, (6) can be rewritten as

$$\vec{\sigma} \cdot \vec{p} \varphi(\vec{r}) = -\{(M - E) + \Sigma(\vec{r})\} \chi(\vec{r}), \quad (7)$$

$$\varphi(\vec{r}) = \frac{\vec{\sigma} \cdot \vec{p}}{(M + E - C_s)} \chi(\vec{r}). \quad (8)$$

If (8) is substituted into (7) we get

$$\begin{aligned} \vec{p}^2 \chi(\vec{r}) \\ = \{(M + E) - C_s\} [-\{(M - E) + (\Sigma(\vec{r}))\}] \chi(\vec{r}), \end{aligned} \quad (9)$$

where

$$(\vec{\sigma} \cdot \vec{p})(\vec{\sigma} \cdot \vec{p}) = \vec{p}^2. \quad (10)$$

Using momentum operator definition in quantum mechanics, where $\vec{p} = -i\vec{\nabla}_D$, the hyperspherical Laplacian $\vec{\nabla}_D^2$ is given as [15]

$$\nabla_D^2 = \frac{1}{r^{D-1}} \frac{\partial}{\partial r} \left(r^{D-1} \frac{\partial}{\partial r} \right) + \left(\frac{L_{D-1}^2}{r^2} \right). \quad (11)$$

The eigenvalue of L_{D-1}^2 is $\ell'_{D-1}(\ell'_{D-1} + D - 2)$ and the angular momentum operator is expressed as [15]

$$\begin{aligned} L_{D-1}^2 = - \left\{ \frac{1}{\sin^{D-2}\theta_{D-1}} \frac{\partial}{\partial \theta_{D-1}} \left(\sin^{D-2}\theta_{D-1} \frac{\partial}{\partial \theta_{D-1}} \right) \right. \\ \left. - \frac{L_{D-2}^2}{\sin^2\theta_{D-1}} \right\}. \end{aligned} \quad (12)$$

By inserting (11) into (9), we get

$$\begin{aligned} & \frac{1}{r^{D-1}} \frac{\partial}{\partial r} \left(r^{D-1} \frac{\partial}{\partial r} \chi(\vec{r}) \right) + \left(\frac{L_{D-1}^2}{r^2} \right) \chi(\vec{r}) \\ & - (M^2 - E^2) \chi(\vec{r}) - \{(M + E) - C_s\} \Sigma(\vec{r}) \chi(\vec{r}) \quad (13) \\ & = 0. \end{aligned}$$

The separable variable potential used in this study is q -deformed hyperbolic Rosen-Morse potential plus q -deformed noncentral Scarf trigonometric potential in hyperspherical coordinate space. The effective potential can be written as

$$\begin{aligned} \Sigma(\vec{r}) = & \left[\left\{ -\frac{V_0}{\cosh_q^2(\alpha r)} + V_1 \tanh_q(\alpha r) \right\} \right. \\ & + \frac{1}{r^2} \left\{ \frac{V(\theta_1)}{\sin^2 \theta_2 \sin^2 \theta_3 \sin^2 \theta_4} + \frac{V(\theta_2)}{\sin^2 \theta_3 \sin^2 \theta_4} \right. \\ & \left. \left. + \frac{V(\theta_3)}{\sin^2 \theta_4} + V(\theta_4) \right\} \right] \quad (14) \end{aligned}$$

with angular potentials $V(\theta_i)$ taken as

$$V(\theta_1) = \frac{b_1^2 + a_1(a_1 - 1)}{\sin_q^2 \theta_1} - \frac{2b_1(a_1 - 1/2) \cos_q \theta_1}{\sin_q^2 \theta_1}, \quad (15)$$

$$V(\theta_2) = \frac{b_2^2 + a_2(a_2 - 1)}{\sin_q^2 \theta_2} - \frac{2b_2(a_2 - 1/2) \cos_q \theta_2}{\sin_q^2 \theta_2}, \quad (16)$$

$$V(\theta_3) = \frac{b_3^2 + a_3(a_3 - 1)}{\sin_q^2 \theta_3} - \frac{2b_3(a_3 - 1/2) \cos_q \theta_3}{\sin_q^2 \theta_3}, \quad (17)$$

$$V(\theta_4) = \frac{b_4^2 + a_4(a_4 - 1)}{\sin_q^2 \theta_4} - \frac{2b_4(a_4 - 1/2) \cos_q \theta_4}{\sin_q^2 \theta_4}. \quad (18)$$

By substituting (14) into (13) and using variable separation method, we get the radial part and the angular part of Dirac equations in hyperspherical coordinate with $D = 5$.

2.1. The Radial Part. The D -dimensional Dirac equation with q -deformed hyperbolic Rosen-Morse potential plus q -deformed trigonometric Scarf noncentral potentials can be resolved into the form of radial part and angular part equations. The radial part of D -dimensional Dirac equation in this case can be expressed as

$$\begin{aligned} & \frac{d^2 F_{nk}(r)}{dr^2} - \frac{(\ell'_{D-1} + (D-1)/2)(\ell'_{D-1} + (D-3)/2)}{r^2} \\ & \cdot F_{nk}(r) \\ & + \left[\left\{ \frac{V_0}{\cosh_q^2(\alpha r)} - V_1 \tanh_q(\alpha r) \right\} \{M + E\} \right. \\ & \left. - \{M + E - C_s\} \{M - E\} \right] F_{nk}(r) = 0 \quad (19) \end{aligned}$$

with $\lambda_4 = \lambda_{D-1} = \ell'_{D-1}(\ell'_{D-1} + D - 2)$.

2.2. The Angular Part. The angular part of 5-dimensional Dirac equation obtained from (13)-(14) can be resolved into four parts, and for $C_s = 0$, we get

$$\frac{1}{P_1(\theta_1)} \left(\frac{\partial^2 P_1(\theta_1)}{\partial \theta_1^2} \right) + \{\lambda_1 - (M + E) V(\theta_1)\} = 0, \quad (20)$$

$$\begin{aligned} & \frac{1}{P_2(\theta_2)} \left\{ \frac{1}{\sin \theta_2} \frac{\partial}{\partial \theta_2} \left(\sin \theta_2 \frac{\partial P_2(\theta_2)}{\partial \theta_2} \right) \right\} \\ & + \left\{ \lambda_2 - \frac{\lambda_1}{\sin^2 \theta_2} - (M + E) V(\theta_2) \right\} = 0, \quad (21) \end{aligned}$$

$$\begin{aligned} & \frac{1}{P_3(\theta_3)} \frac{1}{\sin^2 \theta_3} \frac{\partial}{\partial \theta_3} \left(\sin^2 \theta_3 \frac{\partial P_3(\theta_3)}{\partial \theta_3} \right) \\ & + \left\{ \lambda_3 - \frac{\lambda_2}{\sin^2 \theta_3} - (M + E) V(\theta_3) \right\} = 0, \quad (22) \end{aligned}$$

$$\begin{aligned} & \frac{1}{P_4(\theta_4)} \frac{1}{\sin^3 \theta_4} \frac{\partial}{\partial \theta_4} \left(\sin^3 \theta_4 \frac{\partial P_4(\theta_4)}{\partial \theta_4} \right) \\ & + \left\{ \lambda_4 - \frac{\lambda_3}{\sin^2 \theta_4} - \{M + E\} V(\theta_4) \right\} = 0. \quad (23) \end{aligned}$$

Equations (20)–(23) are the angular part of Dirac equation for θ_1 until θ_4 , respectively. The D -dimensional relativistic wave functions and orbital quantum numbers are obtained from those equations.

3. Review of Asymptotic Iteration Method (AIM)

Asymptotic Iteration Method (AIM) is an alternative method which has accuracy and high efficiency to determine eigenenergies and eigenfunctions for analytically solvable hyperbolic-like potential. Asymptotic Iteration Method is also giving solution for exactly solvable problem [17].

AIM is used to solve the second-order homogeneous linear equation as follows [5, 20–23]:

$$y_n''(x) = \lambda_0(x) y_n'(x) + S_0(x) y_n(x), \quad (24)$$

where $\lambda_0(x) \neq 0$ and prime symbol refers to derivation along x . The other parameter n is interpreted as radial quantum number. Variables $\lambda_0(x)$ and $S_0(x)$ are variables that can be differentiated along x . To get the solution of (24), we have to differentiate (24) along x , and then we get

$$y_n'''(x) = \lambda_1(x) y_n'(x) + S_1(x) y_n(x), \quad (25)$$

where

$$\begin{aligned} \lambda_1(x) &= \lambda_0'(x) + S_0(x) + \lambda_0^2(x), \\ S_1(x) &= S_0'(x) + S_0(x) \lambda_0(x). \end{aligned} \quad (26)$$

Asymptotic Iteration Method (AIM) can be applied exactly in the different problem if the wave function has been known and fulfills boundary conditions zero (0) and infinity (∞).

Equation (2) can be simply iterated until $(k+1)$ and $(k+2)$, $k = 1, 2, 3, \dots$ and then we get

$$y_n^{k+1}(x) = \lambda_{k-1}(x) y_n'(x) + S_{k-1}(x) y_n(x), \quad (27)$$

$$y_n^{k+2}(x) = \lambda_k(x) y_n'(x) + S_k(x) y_n(x), \quad (28)$$

$$\lambda_k(x) = \lambda_{k-1}'(x) + S_{k-1}(x) + \lambda_0(x) \lambda_{k-1}(x), \quad (29)$$

$$S_k(x) = S_{k-1}'(x) + S_0(x) \lambda_{k-1}(x) \quad (30)$$

which is called as recurrence relation. Eigenvalue can be found using equation given as

$$\Delta_k(z) = \lambda_k(z) s_{k-1}(z) - \lambda_{k-1}(z) s_k(z) = 0, \quad (31)$$

where $k = 1, 2, 3, \dots$ is the iteration number and n_r is representation of radial quantum number.

Equation (2) is the second-order homogeneous linear equation which can be solved by comparing it with the second-order linear equation as follows [20]:

$$y''(z) = 2 \left(\frac{az^{N+1}}{1-bz^{N+2}} - \frac{t+1}{z} \right) y'(z) - \frac{wz^N}{1-bz^{N+2}} y(z), \quad (32)$$

where

$$\begin{aligned} (\sigma)_n &= \frac{\Gamma(\sigma+n)}{\Gamma(\sigma)}, \\ \sigma &= \frac{2t+N+3}{N+2}, \\ \rho &= \frac{(2t+1)b+2a}{(N+2)b}. \end{aligned} \quad (33)$$

The wave function of (2) is the solution of (32) which is given as [20]

$$y_n(z) = (-1)^n C_2 (N+2)^n (\sigma)_n {}_2F_1(-n, \rho+n; \sigma; bz^{N+2}). \quad (34)$$

The q -deformed hyperbolic and trigonometric functions are used as one of the parameters in the modified Rosen-Morse potential and noncentral Scarf trigonometric potentials were defined by Arai [24] some years ago as follows:

$$\sinh_q \alpha r = \frac{e^{\alpha r} - qe^{-\alpha r}}{2}, \quad (35a)$$

$$\cosh_q \alpha r = \frac{e^{\alpha r} + qe^{-\alpha r}}{2}, \quad (35b)$$

$$\tanh_q \alpha r = \frac{\sinh_q \alpha r}{\cosh_q \alpha r}, \quad (35c)$$

$$\cosh_q^2 \alpha r - \sinh_q^2 \alpha r = q. \quad (35d)$$

Deformation with q -parameter in the hyperbolic function can be extended into trigonometric function. Definition of trigonometric function can be arranged by the same way as in the hyperbolic function introduced by Suparmi et al. [7] as follows:

$$\begin{aligned} \sin_q ar &= \frac{e^{iar} - qe^{-iar}}{2}, \\ \cos_q ar &= \frac{e^{iar} + qe^{-iar}}{2}, \\ \cos_q^2 ar + \sin_q^2 ar &= q, \\ \tan_q ar &= \frac{\sin_q ar}{\cos_q ar}, \\ \sec_q ar &= \frac{1}{\cos_q ar}, \\ \frac{d \sin_q ar}{dr} &= a \cos_q ar, \\ \frac{d \cos_q ar}{dr} &= -a \sin_q ar, \\ \frac{d \tan_q ar}{dr} &= qa \sec_q^2 ar. \end{aligned} \quad (36)$$

By a convenient translation of spatial variable, one can transform the deformed potentials into the form of nondeformed potentials or vice-versa. In analogy to the translation of spatial variable for hyperbolic function introduced by Dutra [19], we propose the translation of spatial variable for hyperbolic and trigonometric function as follows:

$$r \longrightarrow r + \frac{\ln \sqrt{q}}{\alpha}, \quad (37a)$$

$$r \longrightarrow r - \frac{\ln \sqrt{q}}{\alpha},$$

$$r \longrightarrow r + \frac{\ln \sqrt{q}}{i\alpha}, \quad (37b)$$

$$r \longrightarrow r - \frac{\ln \sqrt{q}}{i\alpha}.$$

And then by inserting (21) into (18) and (19), we have

$$\begin{aligned} \sinh_q \alpha r &\longrightarrow \sqrt{q} \sinh \alpha r; \\ \cosh_q \alpha r &\longrightarrow \sqrt{q} \cosh \alpha r; \\ \sin_q \alpha r &\longrightarrow \sqrt{q} \sin \alpha r; \\ \cos_q \alpha r &\longrightarrow \sqrt{q} \cos \alpha r. \end{aligned} \quad (38)$$

The translation of spatial variable in (38) can be used to map the energy and wave function of nondeformed potential toward deformed potential of Scarf potential.

4. Result and Discussion

4.1. Radial Part. The radial part of Dirac equation in hyper-spherical space when $C_s = 0$ can be expressed as

$$\begin{aligned} & \frac{d^2 F_{nk}(r)}{dr^2} - \frac{(\ell'_{D-1} + (D-1)/2)(\ell'_{D-1} + (D-3)/2)}{r^2} \\ & \cdot F_{nk}(r) \\ & + \left[\left\{ \frac{V_0}{\cosh_q^2(\alpha r)} - V_1 \tanh_q(\alpha r) \right\} \{M + E\} \right. \\ & \left. - \{M^2 - E^2\} \right] F_{nk}(r) = 0. \end{aligned} \quad (39)$$

Equation (39) cannot be solved directly when $\ell'_{D-1} \neq 0$; in this condition, we use approximation to solve centrifugal term with Pekeris approximation. Because we have used q -deformed quantum potential, the Pekeris approximation in this condition can be expressed as [3]

$$\frac{1}{r^2} = \frac{1}{r_e^2} \left(c_0 + c_1 \frac{-e^{-2\alpha r}}{1 + qe^{-2\alpha r}} + c_2 \left(\frac{-e^{-2\alpha r}}{1 + qe^{-2\alpha r}} \right)^2 \right) \quad (40)$$

with

$$c_0 = 1 - \left(\frac{1 + q \exp(-2\alpha r_e)}{2\alpha r_e} \right)^2 \left(\frac{8\alpha r_e}{1 + q \exp(-2\alpha r_e)} - (3 + 2\alpha r_e) \right), \quad (41a)$$

$$c_1 = -2(\exp(2\alpha r_e) + 1) \left(3 \left(\frac{1 + q \exp(-2\alpha r_e)}{2\alpha r_e} \right) - (3 + 2\alpha r_e) \left(\frac{1 + q \exp(-2\alpha r_e)}{2\alpha r_e} \right) \right), \quad (41b)$$

$$c_2 = (\exp(2\alpha r_e) + 1)^2 \left(\frac{1 + q \exp(-2\alpha r_e)}{2\alpha r_e} \right)^2 \left(3 + 2\alpha r_e - \frac{4\alpha r_e}{1 + q \exp(-2\alpha r_e)} \right). \quad (41c)$$

By inserting (38) into (37a) and (37b) and changing from exponential form to hyperbolic form, we get

$$\begin{aligned} & \frac{d^2 F_{nk}(r)}{dr^2} \\ & - \left(\omega \left(c_0 - \frac{c_1}{2\sqrt{q}} + \frac{c_2}{2q} \right) + \{M^2 - E^2\} \right) F_{nk}(r) \\ & - \left\{ \omega \left(\frac{c_1}{2\sqrt{q}} - \frac{c_2}{2q} \right) + V_1 \{M + E\} \right\} \tanh_q \alpha r F_{nk}(r) \\ & + \left[\left\{ \frac{\omega c_2}{4} + V_0 \{M + E\} \right\} \frac{1}{\cosh_q^2(\alpha r)} \right] F_{nk}(r) = 0 \end{aligned} \quad (42)$$

with

$$\frac{(\ell'_{D-1} + (D-1)/2)(\ell'_{D-1} + (D-3)/2)}{r_e^2} = \omega, \quad (43)$$

where r_e is equilibrium distance that can be derived by potential parameter.

Let us assume

$$\begin{aligned} \alpha^2 E' &= - \left(\omega \left(c_0 - \frac{c_1}{2\sqrt{q}} + \frac{c_2}{2q} \right) + \{M^2 - E^2\} \right), \\ \alpha^2 \rho &= \left\{ \omega \left(\frac{c_1}{2\sqrt{q}} + \frac{c_2}{2q} \right) + V_1 \{M + E\} \right\}, \\ \alpha^2 \nu(\nu + 1) &= \left\{ \frac{\omega c_2}{4} + V_0 \{M + E\} \right\}. \end{aligned} \quad (44)$$

Equations (44) are inserted into (40) so we obtain

$$\begin{aligned} & \frac{d^2 F_{nk}(r)}{dr^2} \\ & - \alpha^2 \left\{ \rho \tanh_q \alpha r - \nu(\nu + 1) \operatorname{sech}_q^2(\alpha r) \right\} F_{nk}(r) \\ & = -\alpha^2 E' F_{nk}(r). \end{aligned} \quad (45)$$

Let

$$\tanh_q \alpha r = 1 - 2z. \quad (46)$$

By using (45)-(46), we get

$$\begin{aligned} & z(1-z) \frac{\partial^2 F_{nk}(z)}{\partial z^2} + (1-2z) \frac{\partial F_{nk}(z)}{\partial z} \\ & - \frac{\left\{ \rho \tanh_q \alpha r - \nu(\nu + 1) \operatorname{sech}_q^2 \alpha r \right\}}{4z(1-z)} F_{nk}(z) \\ & = -\frac{E'}{4z(1-z)} F_{nk}(z). \end{aligned} \quad (47)$$

By substituting (46) and $\operatorname{sech}_q^2 \alpha r = 4z(1-z)/q$ into (47), we get

$$\begin{aligned} & z(1-z) \frac{\partial^2 F_{nk}(z)}{\partial z^2} + (1-2z) \frac{\partial F_{nk}(z)}{\partial z} \\ & + \left\{ \frac{\nu(\nu + 1)}{q} - \left(\frac{\rho - E'}{4z} \right) - \left(\frac{-\rho - E'}{4(1-z)} \right) \right\} F_{nk}(z) \\ & = 0. \end{aligned} \quad (48)$$

Equation (48) has two regular singular points for $z = 0$ and $z = 1$ so the general solution from (48) is $F_{nk}(z) = z^\delta (1-z)^\gamma f_n(z)$.

Let $4\delta^2 = \rho - E'$ and $4\gamma^2 = -\rho - E'$ and then substitute them into (48) to get

$$\begin{aligned} & z(1-z) f_n''(z) + \{(2\delta + 1) - z(2\delta + 2\gamma + 2)\} f_n'(z) \\ & + \left\{ \frac{\nu(\nu + 1)}{q} - (\delta + \gamma)(\delta + \gamma + 1) \right\} f_n(z) = 0. \end{aligned} \quad (49)$$

Equation (49) is the second differential equation that can be manipulated into the form as in (22):

$$f_n''(z) = \frac{\{z(2\delta + 2\gamma + 2)\} - (2\delta + 1)}{z(1-z)} f_n'(z) + \frac{\{(\delta + \gamma)(\delta + \gamma + 1) - \nu(\nu + 1)/q\}}{z(1-z)} f_n(z), \quad (50)$$

$$\lambda_0(z) = \frac{z(2\delta + 2\gamma + 2) - (2\delta + 1)}{z(1-z)}, \quad (51)$$

$$S_0(z) = \frac{\{(\delta + \gamma)(\delta + \gamma + 1) - \nu(\nu + 1)/q\}}{z(1-z)}. \quad (52)$$

Let $A = (\delta + \gamma)(\delta + \gamma + 1) - \nu(\nu + 1)/q$ and using (29)-(30), together with (51) and (52), we get

$$\lambda_1(z) = \left\{ \frac{(2\delta + 1)}{z^2} + \frac{(2\gamma + 1)}{(1-z)^2} \right\} + \left\{ \frac{A}{z} + \frac{A}{(1-z)} \right\} + \left\{ -\frac{(2\delta + 1)}{z} + \frac{(2\gamma + 1)}{(1-z)} \right\}^2, \quad (53)$$

$$S_1(z) = \left\{ -\frac{A}{z^2} + \frac{A}{(1-z)^2} \right\} + \left\{ \frac{A}{z} + \frac{A}{(1-z)} \right\} \left\{ -\frac{(2\delta + 1)}{z} + \frac{(2\gamma + 1)}{(1-z)} \right\}. \quad (54)$$

The eigenvalue of (50) can be found by using (31):

$$\begin{aligned} \Delta_1 &= s_0 \lambda_1 - s_1 \lambda_0 = 0 \longrightarrow \\ \varepsilon_0 &= (\delta + \gamma)(\delta + \gamma + 1) \\ \Delta_2 &= s_1 \lambda_2 - s_2 \lambda_2 = 0 \longrightarrow \\ \varepsilon_1 &= (\delta + \gamma + 1)(\delta + \gamma + 2) \\ &\vdots \\ \varepsilon_n &= (\delta + \gamma + n)(\delta + \gamma + n + 1), \end{aligned} \quad (55)$$

where ε_n is n th eigenvalue when $n = 0, 1, 2, \dots$ and if n is radial quantum number. By using (29)-(30) and the last equation in (55), we have

$$\varepsilon_n = \frac{\nu(\nu + 1)}{q} = \left\{ \frac{(\ell'_4 + 3/2)(\ell'_4 + 1/2)}{r_e^2 4\alpha^2 q} c_2 + \frac{V_0(M + E)}{\alpha^2 q} \right\}. \quad (56)$$

The last equation in (55) can be rewritten as

$$\varepsilon_n = (\delta + \gamma + (n + 1))^2 - (\delta + \gamma + (n + 1)). \quad (57)$$

From (57), we get that the relativistic energy equation of this system is

$$(M^2 - E^2) = \alpha^2 \left[\frac{\rho^2}{4(\sqrt{\varepsilon_n + 1/4} - n - 1/2)^2} + \left(\sqrt{\varepsilon_n + \frac{1}{4}} - n - \frac{1}{2} \right)^2 \right] - \omega \left(c_0 - \frac{c_1}{2\sqrt{q}} + \frac{c_2}{2q} \right). \quad (58)$$

By comparing (32) and (50) and using (34), we obtain $f_n(z)$ as

$$f_n(z) = (-1)^n C' (1)^n (2\delta + 1)_n \cdot {}_2F_1(-n, 2\delta + 2\gamma + n + 1, 2\delta + 1, z). \quad (59)$$

From (59), we determine unnormalized radial wave function with $D = 5$ for various n as shown in Table 6, here C' is normalization constant.

4.2. Solution of Angular Part. In this study, the four angular parts of Dirac equations are presented in (20)–(23), so we have to solve each equation of angular Dirac equation using AIM.

4.2.1. Equation of Angular Part for θ_1 . We can solve (20) by changing it into the form of the second-order hypergeometric-type differential equation that is similar to (22) after we insert (15) in (20),

$$\begin{aligned} \frac{\partial^2 P_1(\theta_1)}{\partial \theta_1^2} - \{M + E\} \\ \cdot \left\{ \frac{b_1^2 + a_1(a_1 - 1)}{\sin_q^2 \theta_1} + \frac{2b_1(a_1 - 1/2) \cos_q \theta_1}{\sin_q^2 \theta_1} \right\} \\ \cdot P_1(\theta_1) + \lambda_1 P_1(\theta_1) = 0, \end{aligned} \quad (60)$$

and substitute $\cos_q \theta_1 = \sqrt{q}(1 - 2z_1)$ into (60) so we get

$$\begin{aligned} z_1(1 - z_1) \frac{d^2 P_1(z_1)}{dz_1^2} + \frac{1}{2}(1 - 2z_1) \frac{dP_1(z_1)}{dz_1} + \left\{ \lambda_1 \right. \\ \left. - \frac{\{M + E\} [(b_1^2 + a_1(a_1 - 1))/q + (2b_1(a_1 - 1/2) \sqrt{q})/q]}{4z_1} \right. \\ \left. - \frac{\{M + E\} [(b_1^2 + a_1(a_1 - 1))/q - (2b_1(a_1 - 1/2) \sqrt{q})/q]}{4(1 - z_1)} \right\} \\ \cdot P_1(z_1) = 0. \end{aligned} \quad (61)$$

Equation (62) has two regular singular points for $z_1 = 0$ and $z_1 = 1$, and then the solution of $P_1(z_1)$ is set as

$$P_1(z_1) = z_1^{\delta_{s1}} (1 - z_1)^{\gamma_{s1}} f_{n_1}(z_1). \quad (62)$$

TABLE 1: Relativistic energies with variation of q and n_l .

$D = 5, V_0 = 6, V_1 = -1, a_1 = a_2 = a_3 = a_4 = b_1 = b_2 = b_3 = b_4 = 2$									
q	r_e (fm)	n	$n_{l1} = n_{l2} = n_{l3} = n_{l4}$	E_q (fm $^{-1}$)	$n_{l1} = n_{l2} = n_{l3} = n_{l4}$	n	q	r_e (fm)	E_{n_l} (fm $^{-1}$)
0.2	0.0333	1	1	-5.2311	0	1	1	0.1671	-5.0977
0.4	0.0667	1	1	-5.4766	1	1	1	0.1671	-6.4721
0.6	0.1001	1	1	-5.7371	2	1	1	0.1671	-9.1544
0.8	0.1335	1	1	-6.0240	3	1	1	0.1671	-13.1643
1	0.1671	1	1	-6.4721	4	1	1	0.1671	-18.5283
1.2	0.2007	1	1	-6.9767	5	1	1	0.1671	-25.2837

If we replace $P_1(z_1)$ in (61) with (62) and simplify it by using appropriate variable substitution as follows:

$$O_{s1}^2 = \lambda_1, \quad (63a)$$

$$2\delta_{s1}(2\delta_{s1} - 1) = (M + E) \cdot \left(\frac{b_1^2 + a_1(a_1 - 1)}{q} - \frac{2b_1(a_1 - 1/2)\sqrt{q}}{q} \right), \quad (63b)$$

$$2\gamma_{s1}(2\gamma_{s1} - 1) = (M + E) \cdot \left(\frac{b_1^2 + a_1(a_1 - 1)}{q} + \frac{2b_1(a_1 - 1/2)\sqrt{q}}{q} \right) \quad (63c)$$

then (61) reduces to

$$f''_{n_l}(z_1) = \frac{\{(2\delta_{s1} + 2\gamma_{s1} + 1)z_1 - (2\delta_{s1} + 1/2)\}}{z_1(1 - z_1)} f'_{n_l}(z_1) + \frac{\{(\delta_{s1} + \gamma_{s1})^2 - O_{s1}^2\}}{z_1(1 - z_1)} f_{n_l}(z_1). \quad (64)$$

From (64), we get

$$\lambda_0(z_1) = \frac{\{(2\delta_{s1} + 2\gamma_{s1} + 1)z_1 - (2\delta_{s1} + 1/2)\}}{z_1(1 - z_1)}; \quad (65)$$

$$S_0(z_1) = \frac{\{(\delta_{s1} + \gamma_{s1})^2 - O_{s1}^2\}}{z_1(1 - z_1)}, \quad (66)$$

$$\lambda_1(z_1) = \left\{ \frac{(2\delta_{s1} + 1/2)}{z_1^2} + \frac{(2\gamma_{s1} + 1/2)}{(1 - z_1)^2} \right\} + \left\{ \frac{I_{s1}}{z_1} + \frac{I_{s1}}{(1 - z_1)} \right\} + \left\{ -\frac{(2\delta_{s1} + 1/2)}{z_1} + \frac{(2\gamma_{s1} + 1/2)}{(1 - z_1)} \right\}^2, \quad (67)$$

$$S_1(z_1) = \left\{ -\frac{I_{s1}}{z_1^2} + \frac{I_{s1}}{(1 - z_1)^2} \right\} + \left\{ \frac{I_{s1}}{z_1} + \frac{I_{s1}}{(1 - z_1)} \right\} \cdot \left\{ -\frac{(2\delta_{s1} + 1/2)}{z_1} + \frac{(2\gamma_{s1} + 1/2)}{(1 - z_1)} \right\}. \quad (68)$$

TABLE 2: Relativistic energies with variation of radial quantum number n .

$D = 5, V_0 = 6, V_1 = -1, \alpha = 0.5$ (fm $^{-1}$), and $r_e = 0.1671$ (fm)			
$a_1 = a_2 = a_3 = a_4 = b_1 = b_2 = b_3 = b_4 = 2$			
n	q	$n_{l1} = n_{l2} = n_{l3} = n_{l4}$	E_n (fm $^{-1}$)
0	1	1	-6.4725
1	1	1	-6.4721
2	1	1	-6.4717
3	1	1	-6.4713
4	1	1	-6.4709

TABLE 3: The relativistic energies with variation of potential parameters $a_1; a_2; a_3; a_4$ and $b_1; b_2; b_3; b_4$ with $M = 5; C_s = 0; n_r = 0; n_{l1} = n_{l2} = n_{l3} = n_{l4} = 0; V_0 = 6; \alpha = 0.5; q = 1; V_1 = -1$; and $r_e = 0.1671$.

a_1	b_1	a_2	b_2	a_3	b_3	a_4	b_4	E_{ab} (fm)
2	2	2	2	2	2	2	2	-5.0980
4	4	2	2	2	2	2	2	-5.0821
6	6	2	2	2	2	2	2	-5.0561
8	8	2	2	2	2	2	2	-5.0382
10	10	2	2	2	2	2	2	-5.0269
2	2	4	4	2	2	2	2	-5.0528
2	2	6	6	2	2	2	2	-5.0319
2	2	8	8	2	2	2	2	-5.0211
2	2	10	10	2	2	2	2	-5.0171
2	2	2	2	4	4	2	2	-5.0383
2	2	2	2	6	6	2	2	-5.0136
2	2	2	2	8	8	2	2	-5.0084
2	2	2	2	10	10	2	2	-5.0055
2	2	2	2	2	2	4	4	-5.0453
2	2	2	2	2	2	6	6	-5.0266
2	2	2	2	2	2	8	8	-5.0200
2	2	2	2	2	2	10	10	-5.0156

By using (65)–(68) and using relation of (31), we get

$$(\ell_1'^2)_{n_l} = (\delta_{s1} + \gamma_{s1} + n_l)^2, \quad (69)$$

where n_l is orbital quantum number. The angular wave function of (64) is determined by using (34) and then we get

$$f_{n_l} = (-1)^{n_l} C' (1)^{n_l} \left(2\delta_{s1} + \frac{1}{2} \right)_{n_l} \cdot {}_2F_1 \left(-n_l, 2\delta_{s1} + 2\gamma_{s1} + n_l, 2\delta_{s1} + \frac{1}{2}, z \right), \quad (70)$$

where C' is normalization constant and ${}_2F_1(a, b; c; z)$ is hypergeometry function. From (70) we solve unnormalized angular wave function as function of z completely for variation of n_l as shown in Table 7.

4.2.2. Equation of Angular Part for θ_2 , θ_3 , and θ_4 . Solutions for θ_2 , θ_3 , and θ_4 are determined by the same way as the

solution for θ_1 , by setting subscript i with 2, 3, and 4. So we get the solution for orbital quantum number for θ_2 , θ_3 , and θ_4 , respectively, as follows:

$$(\ell'_2)_{n_l} = \delta_{s2} + \gamma_{s2} + \left(n_l - \frac{1}{2} \right), \quad \text{for } \theta_2 \ n_l = 0, 1, \dots, \quad (71)$$

$$(\ell'_3)_{n_l} = \delta_{s3} + \gamma_{s3} + (n_l - 1), \quad \text{for } \theta_3 \ n_l = 0, 1, \dots, \quad (72)$$

$$(\ell'_4)_{n_l} = \delta_{s4} + \gamma_{s4} + \left(n_l - \frac{3}{2} \right), \quad \text{for } \theta_4 \ n_l = 0, 1, \dots, \quad (73)$$

where

$$\delta_{s2} = \frac{\sqrt{\lambda_1 + (M + E)((b_2^2 + a_2(a_2 - 1))/q - 2b_2(a_2 - 1/2)/\sqrt{q})} + 1/2}{2}, \quad (74)$$

$$\delta_{s3} = \frac{\sqrt{\lambda_2 + 1/4 + (M + E)((b_3^2 + a_3(a_3 - 1))/q - 2b_3(a_3 - 1/2)/\sqrt{q})} + 1/2}{2}, \quad (75)$$

$$\delta_{s4} = \frac{\sqrt{\lambda_3 - 1/2 + (M + E)((b_4^2 + a_4(a_4 - 1))/q - 2b_4(a_4 - 1/2)/\sqrt{q})} + 1/2}{2}, \quad (76)$$

$$\gamma_{s2} = \frac{\sqrt{\lambda_1 + (M + E)((b_2^2 + a_2(a_2 - 1))/q + 2b_2(a_2 - 1/2)/\sqrt{q})} + 1/2}{2}, \quad (77)$$

$$\gamma_{s3} = \frac{\sqrt{\lambda_2 + (M + E)((b_3^2 + a_3(a_3 - 1))/q + 2b_3(a_3 - 1/2)/\sqrt{q})} + 1/4 + 1/2}{2}, \quad (78)$$

$$\gamma_{s4} = \frac{\sqrt{\lambda_3 - 1/2 + (M + E)((b_4^2 + a_4(a_4 - 1))/q + 2b_4(a_4 - 1/2)/\sqrt{q})} + 1/2}{2}. \quad (79)$$

The unnormalized angular wave functions for θ_2 , θ_3 , and θ_4 are in the same pattern as the angular wave function for θ_1 so we obtain the last three angular wave functions by simple change of parameters in (70) with parameters in (74)–(79).

The relativistic energy levels are calculated numerically using Matlab program R2013a. Table 1 shows the relativistic energy levels as a function of deformation parameter q ; the relativistic energy E decreases when deformation parameter q increases. Here we apply the value of q from 0.2 until 1.2 with step 0.2. The relativistic energy levels as a function of orbital quantum numbers n_l are also shown in Table 1. The magnitude of energies E decreases when the orbital quantum numbers n_l increase. The relativistic energy levels as a function of radial quantum number n are shown in Table 2. The relativistic energies E increase when radial quantum number n increases. The relativistic energies numerically also change as a function of potential parameters a_i and b_i where $i = 1, 2, 3, 4$ are the components of i th noncentral potential (Table 3). The relativistic energies E increase when

both potential parameters in each potential component increase. This suggests that the bounded energies become less bounded with increasing of potential parameters. The unnormalized angular wave functions are listed in Table 4. The unnormalized radial wave functions are plotted by using (46) and (59) as shown in Figure 1. From Figures 1(a)–1(c), it is seen that the amplitude of the wave function increases when the orbital quantum number increases. This suggests that the probability of finding particles is larger for higher orbital quantum number n_l .

5. Special Case for Three-Dimensional System

In three-dimensional case, $D = 3$, the bound state relativistic energy levels are calculated numerically from (58), (43), (44), (56), (69), (74), and (77) using Matlab software R2013a and are presented in Tables 5, 6, and 7. By using (44) and (62), the unnormalized radial wave functions are listed in Table 8. In Table 5, the relativistic energy levels decrease

TABLE 4: The unnormalized upper angular wave function for higher dimension $D = 5$.

n_i	Unnormalized angular wave function
0	$Q(\theta_1) = C' \left(\frac{1 - \cos \theta_1}{2} \right)^{\delta_{s1}} \left(\frac{1 + \cos \theta_1}{2} \right)^{\gamma_{s1}}$
	$Q(\theta_2) = C' \left(\frac{1 - \cos \theta_2}{2} \right)^{\delta_{s2}} \left(\frac{1 + \cos \theta_2}{2} \right)^{\gamma_{s2}}$
	$Q(\theta_3) = C' \left(\frac{1 - \cos \theta_3}{2} \right)^{\delta_{s3}} \left(\frac{1 + \cos \theta_3}{2} \right)^{\gamma_{s3}}$
	$Q(\theta_4) = C' \left(\frac{1 - \cos \theta_4}{2} \right)^{\delta_{s4}} \left(\frac{1 + \cos \theta_4}{2} \right)^{\gamma_{s4}}$
1	$Q(\theta_1) = C' \left(\frac{1 - \cos \theta_1}{2} \right)^{\delta_{s1}} \left(\frac{1 + \cos \theta_1}{2} \right)^{\gamma_{s1}} (-1) \left(2\delta_{s1} + \frac{1}{2} \right) \left[1 + \frac{(-1)(2\delta_{s1} + 2\gamma_{s1} + 1)((1 - \cos \theta_1)/2)}{(2\delta_{s1} + 1/2)} \right]$
	$Q(\theta_2) = C' \left(\frac{1 - \cos \theta_2}{2} \right)^{\delta_{s2}} \left(\frac{1 + \cos \theta_2}{2} \right)^{\gamma_{s2}} (-1) \left(2\delta_{s2} + \frac{1}{2} \right) \left[1 + \frac{(-1)(2\delta_{s2} + 2\gamma_{s2} + 1)((1 - \cos \theta_2)/2)}{(2\delta_{s2} + 1/2)} \right]$
	$Q(\theta_3) = C' \left(\frac{1 - \cos \theta_3}{2} \right)^{\delta_{s3}} \left(\frac{1 + \cos \theta_3}{2} \right)^{\gamma_{s3}} (-1) \left(2\delta_{s3} + \frac{1}{2} \right) \left[1 + \frac{(-1)(2\delta_{s3} + 2\gamma_{s3} + 1)((1 - \cos \theta_3)/2)}{(2\delta_{s3} + 1/2)} \right]$
	$Q(\theta_4) = C' \left(\frac{1 - \cos \theta_4}{2} \right)^{\delta_{s4}} \left(\frac{1 + \cos \theta_4}{2} \right)^{\gamma_{s4}} (-1) \left(2\delta_{s4} + \frac{1}{2} \right) \left[1 + \frac{(-1)(2\delta_{s4} + 2\gamma_{s4} + 1)((1 - \cos \theta_4)/2)}{(2\delta_{s4} + 1/2)} \right]$

(a)

(b)

(c)

FIGURE 1: Unnormalized radial wave functions versus hyperspherical radius r with variation of radial quantum number, where $M = 5 \text{ fm}^{-1}$, $C_s = 0$, $V_0 = 6$, $V_1 = -1$, $r_e = 0.1671 \text{ fm}$, and $n_{ii} = 1$, $i = 1, 2, 3, 4$. (a) For $n = 0$, (b) for $n = 1$, and (c) for $n = 2$.

TABLE 5: Relativistic energy with variation of q and n_l .

$V_0 = 6, V_1 = -1, \alpha = 0.5 \text{ (fm}^{-1}) \text{ } a_1 = a_2 = b_1 = b_2 = 2$									
q	$r_e \text{ (fm)}$	n	$n_{l1} = n_{l2}$	$E_q \text{ (fm}^{-1})$	$n_{l1} = n_{l2}$	n_r	q	$r_e \text{ (fm)}$	$E_{n_l} \text{ (fm}^{-1})$
0.2	0.0333	1	1	-5.1360	0	1	1	0.1671	-5.1394
0.4	0.0667	1	1	-5.2867	1	1	1	0.1671	-6.1034
0.6	0.1001	1	1	-5.4552	2	1	1	0.1671	-7.9374
0.8	0.1335	1	1	-5.6616	3	1	1	0.1671	-10.6614
1	0.1671	1	1	-6.1034	4	1	1	0.1671	-14.3053
1.2	0.2007	1	1	-6.5579					

TABLE 6: Relativistic energies with variation of n_r and potential parameter.

$M = 5, V_0 = 6, V_1 = -1, q = 1, \alpha = 0.5 \text{ (fm}^{-1}) \text{, and } r_e = 0.1671 \text{ (fm)}$							
n	$n_{l1} = n_{l2}$	$a_1 = a_2 = b_1 = b_2$	$E_{mn_l n_{l2}} \text{ (fm}^{-1})$	n	$n_{l1} = n_{l2}$	$a_1 = a_2 = b_1 = b_2$	$E_{mn_l n_{l2}} \text{ (fm}^{-1})$
0	1	2	-6.1042	0	1	0	-0.0616
1	1	2	-6.1034	1	1	0	-0.0564
2	1	2	-6.1024	2	1	0	-0.0509
3	1	2	-6.1014	3	1	0	-0.0454
4	1	2	-6.1004	4	1	0	-0.0398

TABLE 7: Relativistic energies for $M = 1 \text{ fm}^{-1}$, $C_s = 0$, $V_0 = 6$, and $V_1 = -1$ without presence of q -deformed noncentral potentials.

ℓ	n	K	(ℓ, j)	$E_{nK}^{ss} \text{ (fm}^{-1}) \text{, } q = 1, \text{ } r_e = 0.1671$	$E_{nK}^{ss} \text{ (fm}^{-1}) \text{, } q = 1.2, \text{ } r_e = 0.2007$	$E_{nK}^{ss} \text{ (fm}^{-1}) \text{, } q = 1.4, \text{ } r_e = 0.2334$
1	0	-2	0p _{3/2}	-0.5028	-0.5014	-0.5006
2	0	-3	0d _{5/2}	-0.5008	-0.5002	-0.4999
3	0	-4	0f _{7/2}	-0.5002	-0.4999	-0.4996
4	0	-5	0g _{9/2}	-0.5000	-0.4998	-0.4995
1	1	-2	1p _{3/2}	-0.5042	-0.5028	-0.5020
2	1	-3	1d _{5/2}	-0.5013	-0.5007	-0.5003
3	1	-4	1f _{7/2}	-0.5005	-0.5002	-0.4999
4	1	-5	1g _{9/2}	-0.5002	-0.4999	-0.4997
1	0	1	0p _{1/2}	-0.5028	-0.5014	-0.5006
2	0	2	0d _{3/2}	-0.5008	-0.5002	-0.4999
3	0	3	0f _{5/2}	-0.5002	-0.4999	-0.4996
4	0	4	0g _{7/2}	-0.5000	-0.4998	-0.4995
1	1	1	1p _{1/2}	-0.5042	-0.5028	-0.5020
2	1	2	1d _{3/2}	-0.5013	-0.5007	-0.5003
3	1	3	1f _{5/2}	-0.5005	-0.5002	-0.4999
4	1	4	1g _{7/2}	-0.5002	-0.4999	-0.4997

TABLE 8: Unnormalized upper radial wave function for dimension $D = 3$.

n	Radial wave function
0	$F_0(r) = C' \left(\frac{1 - \tanh_q \alpha r}{2} \right)^\delta \left(\frac{1 + \tanh_q \alpha r}{2} \right)^\gamma$
1	$F_1(r) = C' \left(\frac{1 - \tanh_q \alpha r}{2} \right)^\delta \left(\frac{1 + \tanh_q \alpha r}{2} \right)^\gamma (-1) C' (2\delta + 1) \left(1 + \frac{(-1)_1 (2\delta + 2\gamma + 2)_1 ((1 - \tanh_q \alpha r)/2)^1}{(2\delta + 1)_1 1!} \right)$

TABLE 9: The unnormalized upper angular wave function for dimension $D = 3$.

n_l	Unnormalized angular wave function
0	$Q(\theta_1) = C' \left(\frac{1 - \cos \theta_1}{2} \right)^{\delta_{s1}} \left(\frac{1 + \cos \theta_1}{2} \right)^{\gamma_{s1}}$
	$Q(\theta_2) = C' \left(\frac{1 - \cos \theta_2}{2} \right)^{\delta_{s2}} \left(\frac{1 + \cos \theta_2}{2} \right)^{\gamma_{s2}}$
1	$Q(\theta_1) = C' \left(\frac{1 - \cos \theta_1}{2} \right)^{\delta_{s1}} \left(\frac{1 + \cos \theta_1}{2} \right)^{\gamma_{s1}} (-1) \left(2\delta_{s1} + \frac{1}{2} \right) \left[1 + \frac{(-1)(2\delta_{s1} + 2\gamma_{s1} + 1)((1 - \cos \theta_1)/2)}{(2\delta_{s1} + 1/2)} \right]$
	$Q(\theta_2) = C' \left(\frac{1 - \cos \theta_2}{2} \right)^{\delta_{s2}} \left(\frac{1 + \cos \theta_2}{2} \right)^{\gamma_{s2}} (-1) \left(2\delta_{s2} + \frac{1}{2} \right) \left[1 + \frac{(-1)(2\delta_{s2} + 2\gamma_{s2} + 1)((1 - \cos \theta_2)/2)}{(2\delta_{s2} + 1/2)} \right]$

with the increase of both deformation parameter q and orbital quantum number n_l , where n_{l1} is orbital quantum number for θ_1 and n_{l2} is orbital quantum number for θ_2 . From Table 7, we can conclude that the value of relativistic energies increases with the increase of the radial quantum number n and with either the presence or absence of angular potential parameter. The angular potential parameters a and b influence the relativistic energy level, where the relativistic energy levels increase by increasing the angular potential parameter. The unnormalized upper radial wave functions are listed in Table 8 when the radial quantum numbers are $n = 0$ and $n = 1$ and C' is the normalization constant. The unnormalized angular wave functions as functions of θ_1 and θ_2 are listed in Table 9. Without the presence of the noncentral potential, degeneracy energy spectra occur in the spin doublets, with quantum numbers $(n, \ell, j = \ell + 1/2)$ and $(n, \ell, j = \ell - 1/2)$, where n , ℓ , and j are the radial, the orbital, and the total angular momentum quantum numbers, respectively, for example, $(np_{1/2}, np_{3/2})$ for $\ell = 1$ but for difference values of $j = 1/2$ and $3/2$, $(nd_{3/2}, nd_{5/2})$ for $\ell = 2$ with $(j = 3/2$ and $5/2)$, $(nf_{5/2}, nf_{7/2})$ for $\ell = 3$ with $(j = 5/2$ and $7/2)$, and $(ng_{7/2}, ng_{9/2})$ for $\ell = 4$ with $(j = 7/2$ and $9/2)$. The degeneracy energies can be removed with the presence of noncentral potential, by changing $\ell \rightarrow K$, where $K = -\ell - 1$ and $K = \ell$ for $K < 0$ and $K > 0$.

6. Conclusion

In this study, we have obtained the bound state solution of the 5-dimensional Dirac equation with separable q -deformed quantum potential under condition of spin symmetry. The upper component of Dirac spinors and relativistic energy have been obtained using Asymptotic Iteration Method (AIM). The numerical result shows that the bound state relativistic energy level for spin symmetry case with both dimensions $D = 5$ and $D = 3$ increases with increasing of radial quantum number n and decreases with increasing of both deformation parameter q and orbital quantum number n_l . The degeneracy energy states occur in the absence of q -deformed noncentral potential.

Competing Interests

The authors declare that they have no competing interests.

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