

## Research Article

# On a Neutral Particle with a Magnetic Quadrupole Moment in a Uniform Effective Magnetic Field

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Quantum effects on a Landau-type system associated with a moving atom with a magnetic quadrupole moment subject to confining potentials are analysed. It is shown that the spectrum of energy of the Landau-type system can be modified, where the degeneracy of the energy levels can be broken. In three particular cases, it is shown that the analogue of the cyclotron frequency is modified, and the possible values of this angular frequency of the system are determined by the quantum numbers associated with the radial modes and the angular momentum and by the parameters associated with confining potentials in order that bound states solutions can be achieved.

## 1. Introduction

It is well-known in the literature that the Landau quantization [1] takes place when the motion of a charged particle in a plane perpendicular to a uniform magnetic field acquires distinct orbits and the energy spectrum of this system becomes discrete and infinitely degenerate. It is important in studies of two-dimensional surfaces [2–4], the quantum Hall effect [5], and Bose-Einstein condensation [6, 7]. With the aim of building a quantum system where the quantum Hall effect for neutral particles could be observed, a model has been proposed in [8] where the electric field that interacts with the permanent magnetic dipole moment of the neutral particle must satisfy specific conditions: there is absence of torque on the magnetic dipole moment of the neutral particle, the electric field must satisfy the electrostatic conditions, and there exists the presence of a uniform effective magnetic field given by  $\vec{B}_{\text{eff}} = \vec{\nabla} \times \vec{A}_{\text{eff}}$ , where  $\vec{A}_{\text{eff}} = \vec{\sigma} \times \vec{E}$  corresponds to an effective vector potential,  $\vec{E}$  is the electric field, and  $\vec{\sigma}$  are the Pauli matrices. Therefore, by choosing an electric field that satisfies the above conditions, it has been shown in [8] that the motion of the neutral particle acquires discrete orbits, where the energy levels correspond to the analogue of the Landau levels. This analogue of the Landau levels corresponds

to the Landau-Aharonov-Casher quantization [8]. Another model of the Landau quantization for neutral particles has been proposed in [9] based on the dual effect of the Aharonov-Casher effect [10–12], where the neutral particle has a permanent electric dipole moment. In this model, there exists an effective vector potential given by  $\vec{A}_{\text{eff}} = \vec{\sigma} \times \vec{B}$ , where  $\vec{B}$  is a magnetic field that interacts with the permanent electric dipole moment of the neutral particle. This Landau-like quantization is called the Landau-He-McKellar-Wilkins quantization [9]. Recently, the Landau quantization for an atom with electric quadrupole moment has been proposed in [13] by imposing the notion that the electric quadrupole tensor must be symmetric and traceless and there exists the presence of a uniform effective magnetic field given by an effective vector potential defined as  $\vec{A}_{\text{eff}} = \vec{Q} \times \vec{B}$ , where  $\vec{Q}$  is a vector associated with the electric quadrupole tensor [13, 14] and  $\vec{B}$  is the magnetic field in the laboratory frame.

The aim of this work is to analyse quantum effects on a Landau-type system associated with an atom with a magnetic quadrupole moment subject to some confining potentials. A great deal of work can be found in the literature with respect to studies of quadrupole moments of atoms and molecules, for instance, in single crystals [15], refractive index

[16], nuclear quadrupole interactions [17–21], molecules [22–26], atoms [27, 28], superposition of chiral states [29], geometric quantum phases [14], and noncommutative quantum mechanics [30]. In particular, quantum particles with a magnetic quadrupole moment have attracted interests in atomic systems [31, 32], molecules [33–35], chiral anomaly [36], with  $P$ - and  $T$ -odds effects in atoms [37–39], Coulomb-type interactions [40, 41], Landau-type quantization [42, 43], and analogue of the quantum Hall effect [44]. Other interesting studies have been made in [45–49]. In this work, we consider the single particle approximation used in [40, 41, 47, 48] and then we deal with a system that consists in a moving atom with a magnetic quadrupole moment that interacts with external fields. Then, we introduce the Landau quantization associated with a moving atom that possesses a magnetic quadrupole moment, and thus we analyse the confinement of the Landau-type system to a hard-wall confining potential, a Coulomb-type potential, a linear confining potential, and a Coulomb-type plus a linear confining potential.

The structure of this paper is as follows: in Section 2, we introduce the Landau-type system associated with an atom with a magnetic quadrupole moment by using the single particle approximation of [47, 48]; in Section 3, we confine the Landau-type system to a hard-wall confining potential and analyse the bound states solutions; in Section 4, we discuss the Landau-type system subject to a Coulomb-type confining potential; in Section 5, we discuss the Landau-type system subject to a linear confining potential; in Section 6, we discuss the Landau-type system subject to a Coulomb-type and a linear confining potential; in Section 7, we present our conclusions.

## 2. Landau-Type System

In this section, we make a brief introduction of the Landau quantization associated with neutral particle (atom or molecule) with a magnetic quadrupole moment. First of all, by following [47, 48], then, the single particle approximation that describes an atom with a magnetic quadrupole moment interacting with a magnetic field is given by a potential energy  $U_m = -\sum_{i,j} M_{ij} \partial_i B_j$  in the rest frame of the particle, where  $\vec{B}$  is the magnetic field and  $M_{ij}$  is the magnetic quadrupole moment tensor, whose characteristic is that it is a symmetric and a traceless tensor. Recently, by considering a moving atom, it has been shown in [40] that the magnetic quadrupole moment of the atom interacts with a magnetic field given by  $\vec{B}' = \vec{B} - (1/c^2) \vec{v} \times \vec{E}$  for  $v \ll c$  [50], where  $c$  is the velocity of light. Thereby, the quantum dynamics of a moving atom with magnetic quadrupole moment can be described by the Schrödinger equation (with SI units):

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m} \left[ \hat{p} - \frac{1}{c^2} (\vec{M} \times \vec{E}) \right]^2 \psi - \vec{M} \cdot \vec{B} \psi, \quad (1)$$

where vector  $\vec{M}$  has the components determined by  $M_i = \sum_j M_{ij} \partial_j$  and the fields  $\vec{E}$  and  $\vec{B}$  given in (1) are the electric and magnetic fields in the laboratory frame, respectively [40, 50].

An analogue of the Landau quantization for a moving atom that possesses a magnetic quadrupole moment was

proposed in [42, 44] based on the properties of the magnetic quadrupole tensor, and the field configuration in the laboratory frame that interacts with the magnetic quadrupole moment of the atom must produce a uniform effective magnetic field perpendicular to the plane of motion of the particle which is given by

$$\vec{B}_{\text{eff}} = \vec{\nabla} \times [\vec{M} \times \vec{E}], \quad (2)$$

where the vector  $\vec{E}$  is the electric field in the laboratory frame and satisfies the electrostatic conditions. From this perspective, it has been shown in [42] that an analogue of the Landau quantization can be obtained by considering the magnetic quadrupole moment tensor to be defined by the components:

$$M_{\rho z} = M_{z\rho} = M, \quad (3)$$

where  $M$  is a constant ( $M > 0$ ) and with all other components of  $M_{ij}$  being zero, and thus there exists an electric field given by

$$\vec{E} = \frac{\lambda \rho^2}{2} \hat{\rho}, \quad (4)$$

where  $\lambda$  is a constant associated with a nonuniform distribution of electric charges inside a nonconductor cylinder. In this particular case, we have that the magnetic quadrupole moment defined in (3) is a symmetric and traceless matrix, and we also have an effective vector potential given by  $\vec{A}_{\text{eff}} = \vec{M} \times \vec{E} = \lambda M \rho \hat{\phi}$  and, consequently, effective magnetic field (2) is uniform in the  $z$ -direction:

$$\vec{B}_{\text{eff}} = \lambda M \hat{z}, \quad (5)$$

that is, it is perpendicular to the plane of motion of the quantum particle. Therefore, the conditions for achieving the Landau quantization associated with a moving atom that possesses a magnetic quadrupole moment are satisfied. Henceforth, we are able to analyse the quantum effects of confining potentials on the Landau-type system described above.

## 3. Landau-Type System Subject to a Hard-Wall Confining Potential

In this section, we analyse the effects of the confinement of the Landau-type system established in the previous section to a hard-wall confining potential. By substituting (3) and (4) into Schrödinger equation (1), we have (with the units  $\hbar = c = 1$ )

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2m} \left[ \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} \right] \psi - i \frac{M \lambda}{m} \frac{\partial \psi}{\partial \varphi} + \frac{M^2 \lambda^2}{2m} \rho^2 \psi. \quad (6)$$

A particular solution to (6) is given in terms of the eigenvalues of the operators  $\hat{p}_z = -i\partial_z$  and  $\hat{L}_z = -i\partial_\varphi$  as

$$\psi(t, \rho, \varphi, z) = e^{-i\mathcal{E}t} e^{i\ell\varphi} e^{ikz} R(\rho), \quad (7)$$

where  $l = 0, \pm 1, \pm 2, \dots$  and  $k$  is a constant, since these quantum operators commute with the Hamiltonian of the right-hand side of (6). By performing a change of variables  $\xi = M\lambda\rho^2$  and after some calculations, we have that the radial wave function is given by [42]

$$R(\xi) = \xi^{|l|/2} e^{-\xi/2} M \left( \frac{|l|}{2} + \frac{1}{2} - \frac{m\mathcal{E}}{2M\lambda} + \frac{l}{2}, |l| + 1, \xi \right), \quad (8)$$

where  $M(|l|/2 + 1/2 - m\mathcal{E}/2M\lambda, |l| + 1, \xi)$  is the Kummer function of first kind or the confluent hypergeometric function [51].

In condensed matter physics, a hard-wall confining potential is used with the purpose of describing a more realistic geometry of quantum dots and quantum rings as shown in [52–57]. For this purpose, let us now assume that the wave function of the atom is well behaved at the origin and thus vanishes at a fixed radius  $\xi_0$ ; therefore, we have

$$R(\xi_0 = M\lambda\rho_0^2) = 0. \quad (9)$$

Note that the parameter  $\xi$  is defined in the range  $0 < \xi < \infty$  since the radial coordinate is defined in this range. However, by confining the system to a hard-wall confining potential, then, the parameter  $\xi$  becomes defined in the range  $0 < \xi < \xi_0$ . Therefore, we cannot impose the notion that the confluent hypergeometric series becomes a polynomial of degree  $n$  as made in [42] in order that the Landau quantization could be achieved because we now need to obtain a normalized wave function in the range  $0 < \xi < \xi_0$ . A particular solution to (9) is obtained by considering a fixed value for the parameter  $b = |l| + 1$  of the confluent hypergeometric function and the intensity of the electric field (4) to be small; that is, we consider the parameter  $\lambda$  in (4) to be small. As a consequence, we can consider the product  $M\lambda$  to be quite small, and hence the parameter  $a = |l|/2 + 1/2 - m\mathcal{E}/2M\lambda + l/2$  of the confluent hypergeometric function can be considered to be large, without loss of generality. Thereby, for a fixed  $\xi_0$ , the confluent hypergeometric function can be written in the following form [51]:

$$M(a, b, \xi_0 = M\lambda\rho_0^2) \approx \frac{\Gamma(b)}{\sqrt{\pi}} e^{\xi_0/2} \left( \frac{b\xi_0}{2} - a\xi_0 \right)^{(1-b)/2} \cdot \cos \left( \sqrt{2b\xi_0 - 4a\xi_0} - \frac{b\pi}{2} + \frac{\pi}{4} \right), \quad (10)$$

where  $\Gamma(b)$  is the gamma function. Next, by applying the boundary condition established in (9), we obtain the following expression for the energy levels:

$$\mathcal{E}_{n,l} \approx \frac{1}{2m\rho_0^2} \left[ n\pi + \frac{|l|}{2}\pi + \frac{3\pi}{4} \right]^2 + \frac{M\lambda l}{m}. \quad (11)$$

Thereby, (11) is the spectrum of energy of the Landau-type system of an atom with a magnetic quadrupole moment subject to a hard-wall confining potential. Despite the presence of an effective uniform magnetic field characteristic of the Landau quantization, the influence of the hard-wall confining potential yields a spectrum of energy that differs from that

of the analogue of the Landau quantization [42], where we have that the energy levels (11) are parabolic with respect to the quantum number  $n$  associated with radial modes and the analogue of the Landau levels is nonparabolic with respect to  $n$ .

#### 4. Landau-Type System Subject to a Coulomb-Type Confinement

In this section, our focus is on the effects of a Coulomb-type interaction on the Landau-type system discussed in Section 2. Coulomb-type potentials have been reported as being in the interests of condensed matter physics, for instance, studies that have worked with 1-dimensional systems [58–62], molecules [63–65], position-dependent mass systems [66–68], and the Kratzer potential [69–71]. Furthermore, it is worth mentioning studies that have dealt with Coulomb-type potential in the propagation of gravitational waves [72], quark models [73], and relativistic quantum mechanics [74–76]. For this purpose, let us consider the Landau-type system to be subject to the confining potential:

$$V(\rho) = \frac{\alpha}{\rho}, \quad (12)$$

where  $\alpha$  is a constant that characterizes the Coulomb-type interaction. Since molecules can have a magnetic quadrupole moment [33–35], then, a particular interest in the Coulomb-type potential (12) comes from the studies of molecular interactions described by the Kratzer potential [69–71].

From (6) and (7), the radial equation with the Coulomb-type scalar potential (12) is given by

$$[2m\mathcal{E} + 2M\lambda l] R = -R'' - \frac{1}{\rho} R' + \frac{l^2}{\rho^2} R + M^2 \lambda^2 \rho^2 R + \frac{2m\alpha}{\rho} R, \quad (13)$$

where we have also taken  $k = 0$ . Now, we perform a change of variable given by  $r = \sqrt{M\lambda}\rho$  and, then, rewrite (13) as

$$R'' + \frac{1}{r} R' - \frac{l^2}{\rho^2} R - r^2 R - \frac{\nu}{r} R + \beta R = 0, \quad (14)$$

where

$$\nu = \frac{2m\alpha}{\sqrt{M\lambda}}, \quad (15)$$

$$\beta = \frac{1}{M\lambda} [2m\mathcal{E} + 2M\lambda l].$$

Observe that the asymptotic behaviour is determined for  $r \rightarrow 0$  and  $r \rightarrow \infty$ ; then, we can write the function  $R(r)$  in terms of an unknown function  $G(r)$  as follows [40, 77–79]:

$$R(r) = e^{-r^2/2} r^{|l|} G(r). \quad (16)$$

By substituting function (16) into (14), we obtain the following equation for  $G(r)$ :

$$G'' + \left[ \frac{2|l|+1}{r} - 2r \right] G' + \left[ \beta - 2 - 2|l| - \frac{\nu}{r} \right] G = 0, \quad (17)$$

which is called the biconfluent Heun equation [79], and the function  $G(r)$  is the biconfluent Heun function [79]:  $G(r) = H_B(2|l, 0, \beta, 2\nu, r)$ .

We proceed with using the Frobenius method [80]; then, we can write the solution to (17) as a power series expansion around the origin:  $G(r) = \sum_{k=0}^{\infty} a_k r^k$ . By substituting this series into (17), we obtain the recurrence relation:

$$a_{k+2} = \frac{\nu}{(k+2)(k+2+2|l|)} a_{k+1} - \frac{(\beta-2-2|l|-2k)}{(k+2)(k+2+2|l|)} a_k. \quad (18)$$

Let us start with  $a_0 = 1$ ; then, from (18), we can obtain other coefficients of the power series expansion, for instance, the coefficients  $a_1$  and  $a_2$ :

$$a_1 = \frac{\nu}{(1+2|l|)}; \quad (19)$$

$$a_2 = \frac{\nu^2}{2(2+2|l|)(1+2|l|)} - \frac{(\beta-2-2|l|)}{2(2+2|l|)}.$$

By focusing on achieving bound states solutions, then, we need to impose the notion that the biconfluent Heun series becomes a polynomial of degree  $n$ . From recurrence relation (18), we have that the biconfluent Heun series becomes a polynomial of degree  $n$  by imposing the notion that [40, 75]

$$\begin{aligned} \beta - 2 - 2|l| &= 2n; \\ a_{n+1} &= 0, \end{aligned} \quad (20)$$

where  $n = 1, 2, 3, \dots$ . By analysing the first condition given in (20), we obtain

$$\mathcal{E}_{n,l} = \omega [n + |l| - l + 1], \quad (21)$$

where  $n = 1, 2, 3, \dots$  is the quantum number associated with the radial modes,  $l = 0, \pm 1, \pm 2, \dots$  is the angular momentum quantum number, and the angular frequency of the system becomes

$$\omega = \frac{M\lambda}{m}. \quad (22)$$

Note that the angular frequency is modified in contrast to the cyclotron frequency  $\omega = 2M\lambda/m$  obtained in [42].

Next, we analyse the condition  $a_{n+1} = 0$  given in (20) for the ground state of the system ( $n = 1$ ). Let us assume that the angular frequency  $\omega$  can be adjusted in such a way that the condition  $a_{n+1} = 0$ ; therefore, we will label  $\omega = \omega_{n,l}$  from now on. As a result of this assumption, we have that both conditions imposed in (20) are satisfied and a polynomial solution to the function  $G(r)$  is obtained. Hence, for  $n = 1$ , we have that  $a_{n+1} = a_2 = 0$ , and thus the possible values of the angular frequency  $\omega$  associated with the ground state of the system are given by

$$\omega_{1,l} = \frac{2m\alpha^2}{(1+2|l|)}, \quad (23)$$

which shows us that only some specific values of the angular frequency of the harmonic oscillator are allowed in the system in order that bound state solutions can be obtained. Besides, from (23), we have that the possible values of the angular frequency are determined by the quantum numbers  $\{n, l\}$  of the system and the parameter associated with Coulomb-type interaction. By substituting (23) into (21), we have that the energy of the ground state of the system is given by

$$\mathcal{E}_{1,l} = \frac{2m\alpha^2}{(1+2|l|)} [|l| - l + 2]. \quad (24)$$

Hence, the general expression for the energy levels (21) can be written as

$$\mathcal{E}_{n,l} = \omega_{n,l} [n + |l| - l + 1]. \quad (25)$$

In contrast to [42], we have that the energy levels of the Landau-type system are modified due to the influence of the Coulomb-type interaction. Moreover, the angular frequency is also modified by the effects of the Coulomb-type interaction, where the possible values of the angular frequency  $\omega$  are determined by the quantum numbers of the system  $\{n, l\}$  and the parameter associated with Coulomb-type interaction. The ground state of the system becomes determined by the quantum number  $n = 1$  instead of the quantum number  $n = 0$  given in [42], and the degeneracy of the analogue of the Landau levels is broken.

## 5. Landau-Type System Subject to a Linear Confining Potential

Several works have dealt with a linear scalar potential in molecular and atomic physics through the perturbation theory [81–86]. In the context of the quantum field theory, the interest in the linear scalar potential comes from the studies of confinement of quarks [87–92]. Besides, relativistic quantum systems [75, 77, 78, 93–99] have shown a great interest in the linear scalar potential. Now, we focus on the effects of a linear confining potential on the Landau-type system for an atom/molecule with a magnetic quadrupole moment by searching for analytical solutions. In this way, we introduce the following scalar potential into the Schrödinger equation (6):

$$V(\rho) = \eta\rho, \quad (26)$$

where  $\eta$  is a constant that characterizes the linear interaction. Then, by following the steps from (6) to (7), then, the radial equation with the linear confining potential (26) becomes

$$\begin{aligned} [2m\mathcal{E} + 2M\lambda] R &= -R'' - \frac{1}{\rho} R' + \frac{l^2}{\rho^2} R + M^2 \lambda^2 \rho^2 R \\ &+ 2m\eta\rho R. \end{aligned} \quad (27)$$

Let us also take  $k = 0$  and perform the same change of variable made in the previous section; that is,  $r = \sqrt{M\lambda}\rho$ ; thus, we have

$$R'' + \frac{1}{r} R' - \frac{l^2}{\rho^2} R - r^2 R - \theta r R + \beta R = 0, \quad (28)$$

where the parameter  $\beta$  has been defined in (15) and the parameter  $\theta$  is defined as

$$\theta = \frac{2m\eta}{(M\lambda)^{3/2}}. \quad (29)$$

By analysing the asymptotic behaviour as in the previous section, we can write the function  $R(r)$  in terms of an unknown function  $H(r)$  as follows [40, 77–79]:

$$R(r) = e^{-r^2/2} e^{-\theta r/2} r^{|l|} H(r). \quad (30)$$

By substituting the function (30) into (28), we obtain the following equation for  $H(r)$ :

$$H'' + \left[ \frac{2|l|+1}{r} - \theta - 2r \right] H' + \left[ \beta + \frac{\theta^2}{4} - 2 - 2|l| - \frac{\theta(2|l|+1)}{2r} \right] H = 0, \quad (31)$$

which is the biconfluent Heun equation [79], and the function  $H(r)$  is the biconfluent Heun function [79]:  $H(r) = H_B(2|l|, \theta, \beta + \theta^2/4, 0, r)$ . Further, by following the steps from (18) to (19), we obtain a recurrence relation given by

$$a_{k+2} = \frac{\theta(2k+3+2|l|)}{2(k+2)(k+2+2|l|)} a_{k+1} - \frac{(4\beta + \theta^2 - 8 - 8|l| - 8k)}{4(k+2)(k+2+2|l|)} a_k, \quad (32)$$

and with  $a_0 = 1$ , we have that the coefficients  $a_1$  and  $a_2$  are given by

$$a_1 = \frac{\theta}{2}; \quad (33)$$

$$a_2 = \frac{\theta^2(2|l|+3)}{8(2+2|l|)} - \frac{(4\beta + \theta^2 - 8 - 8|l|)}{8(2+2|l|)}.$$

By imposing the notion that the biconfluent Heun series becomes a polynomial of degree  $n$ , then, from (32) we have that the biconfluent Heun series becomes a polynomial of degree  $n$  when

$$4\beta + \theta^2 - 8 - 8|l| = 8n; \quad (34)$$

$$a_{n+1} = 0,$$

where  $n = 1, 2, 3, \dots$ . The first condition given in (34) yields

$$\mathcal{E}_{n,l} = \omega[n + |l| - l + 1] - \frac{\eta^2}{2m\bar{\omega}^2}, \quad (35)$$

where  $n = 1, 2, 3, \dots$  is the quantum number associated with the radial modes,  $l = 0, \pm 1, \pm 2, \dots$  is the angular momentum quantum number, and  $\bar{\omega}$  is the angular frequency of the system given in (22).

Further, let us analyse the second condition  $a_{n+1} = 0$  given in (34) by considering the ground state of the system

as in the previous section. Again, we assume that the angular frequency  $\bar{\omega}$  is a parameter that can be adjusted in order to satisfy the condition  $a_{n+1} = 0$ . For  $n = 1$ , we obtain

$$\bar{\omega}_{1,l} = \left[ \frac{\eta^2}{2m} (2|l|+3) \right]^{1/3}, \quad (36)$$

where we also have that the possible values of the angular frequency are determined by the quantum numbers of the system  $\{n, l\}$  and, in this case, by the parameter associated with the linear interaction. Besides, from (35) and (36), the energy level associated with the ground state is

$$\mathcal{E}_{1,l} = \left[ \frac{\eta^2}{2m} (2|l|+3) \right]^{1/3} [ |l| - l + 2 ] - \frac{\eta^2}{2m} \left( \frac{2m}{\eta^2 [2|l|+3]} \right)^{2/3}. \quad (37)$$

Hence, the general form of the energy levels of the Landau-type system under the influence of a linear confining potential can be written as

$$\mathcal{E}_{n,l,s} = \bar{\omega}_{n,l} [n + |l| - l + 1] - \frac{\eta^2}{2m\bar{\omega}_{n,l}^2}. \quad (38)$$

By comparing the spectrum of energy (38) with the analogue of the Landau levels, we have that the energy levels are modified by the linear confining potential. The effects of the linear interaction yield a new contribution to the energy levels given by the last term of (35) and (38). The angular frequency is also modified due to the linear interaction in contrast to the cyclotron frequency  $\omega = 2M\lambda/m$  obtained in [42], but it has the same form of that obtained in the previous section for the Coulomb-type interaction case given in (22). Furthermore, in this case, the possible values of the angular frequency  $\bar{\omega}$  are determined by the quantum numbers of the system  $\{n, l\}$  and by the parameter associated with the linear interaction. The ground state of the system also becomes determined by the quantum number  $n = 1$  instead of the quantum number  $n = 0$  of the Landau-like levels. Finally, from (35) to (38), we have that the degeneracy of the analogue of the Landau levels is broken.

## 6. Landau-Type System Subject to a Coulomb-Type Plus Linear Confining Potential

Linear plus Coulomb-type potential has been studied in the context of the perturbation theory [100], with the WKB approximation [101] and the Bohr-Sommerfeld quantization [102]. Analytical solutions to the Schrödinger equation in the presence of linear and Coulomb-type potentials have been discussed in the context of the quark systems [103] and in mesoscopic systems. In this section, we consider the Landau-type system for an atom/molecule with a magnetic quadrupole moment to be subject to the Coulomb-type plus linear confining potential and search for analytical solutions.

Thereby, let us introduce a scalar potential into the Schrödinger equation (6) given by

$$V(\rho) = \frac{\alpha}{\rho} + \eta\rho, \quad (39)$$

where  $\alpha$  and  $\eta$  are constants as we have established previously. From (6) and (7), the radial equation becomes

$$R'' + \frac{1}{r}R' - \frac{l^2}{\rho^2}R - r^2R - \theta rR - \frac{\nu}{r}R + \beta R = 0, \quad (40)$$

where parameters  $\nu$ ,  $\beta$ , and  $\theta$  have been defined in (15) and (29), respectively. By analysing the asymptotic behaviour, we can write the radial wave function  $R(r)$  in the same way as that given in (30); thus, substituting radial wave function (30) into (40), we obtain the following second-order differential equation. By substituting function (30) into (28), we obtain the following equation for  $H(r)$ :

$$\begin{aligned} H'' + \left[ \frac{2|l|+1}{r} - \theta - 2r \right] H' \\ + \left[ \beta + \frac{\theta^2}{4} - 2 - 2|l| - \frac{\theta(2|l|+1)+2\nu}{2r} \right] H \\ = 0, \end{aligned} \quad (41)$$

which is also the biconfluent Heun equation [79], and the function  $H(r)$  is the biconfluent Heun function [79]:  $H(r) = H_B(2|l|, \theta, \beta + \theta^2/4, 2\nu, r)$ . Further, by following the steps from (18) to (19), we obtain a recurrence relation given by

$$\begin{aligned} a_{k+2} = \frac{\theta(2k+3+2|l|)+2\nu}{2(k+2)(k+2+2|l|)} a_{k+1} \\ - \frac{(4\beta + \theta^2 - 8 - 8|l| - 8k)}{4(k+2)(k+2+2|l|)} a_k, \end{aligned} \quad (42)$$

and with  $a_0 = 1$ , we have that the coefficients  $a_1$  and  $a_2$  are given by

$$\begin{aligned} a_1 &= \frac{\theta}{2} + \frac{\nu}{(1+2|l|)}; \\ a_2 &= \frac{\theta^2(3+2|l|)}{8(2+2|l|)} + \frac{\theta\nu(1+|l|)}{(2+2|l|)(1+2|l|)} \\ &\quad + \frac{\nu^2}{2(2+2|l|)(1+2|l|)} \\ &\quad - \frac{(4\beta + \theta^2 - 8 - 8|l|)}{8(2+2|l|)}. \end{aligned} \quad (43)$$

Again, we must impose the notion that the biconfluent Heun series becomes a polynomial of degree  $n$ ; then, from (42), we have that the biconfluent Heun series becomes a polynomial of degree  $n$  with the following conditions:

$$\begin{aligned} 4\beta + \theta^2 - 8 - 8|l| &= 8n; \\ a_{n+1} &= 0, \end{aligned} \quad (44)$$

where  $n = 1, 2, 3, \dots$ . The first condition of (44) yields a general expression for the energy levels:

$$\mathcal{E}_{n,l} = \omega [n + |l| - l + 1] - \frac{\eta^2}{2m\omega^2}, \quad (45)$$

where  $n = 1, 2, 3, \dots$  is the quantum number associated with the radial modes,  $l = 0, \pm 1, \pm 2, \dots$  is the angular momentum quantum number, and  $\omega$  is the angular frequency of the system given in (22).

From the second condition  $a_{n+1} = 0$  given in (44), where we also assume that the angular frequency  $\omega$  is a parameter that can be adjusted in order to satisfy the condition  $a_{n+1} = 0$ , we have for the ground state of the system ( $n = 1$ ) that the possible values of the angular frequency associated with the ground state are determined by the following third-degree algebraic equation [41, 75]:

$$\begin{aligned} \omega_{1,l}^3 - \frac{2m\alpha^2}{(1+2|l|)} \omega_{1,l}^2 - \frac{4(1+|l|)}{(1+2|l|)} \omega_{1,l} - \frac{\eta^2}{2m} (3+2|l|) \\ = 0. \end{aligned} \quad (46)$$

Despite the fact that (46) has at least one real solution, we do not write it because its expression is very long. Besides, the general expression for energy levels (45) should be written in the same form of (38).

Hence, we have obtained analytical solutions to the Landau-type system for an atom/molecule with a magnetic quadrupole moment to be subject to the Coulomb-type and linear confining potentials. Note that the possible values of the angular frequency  $\omega$  are determined by the quantum numbers of the system  $\{n, l\}$  and by the parameters associated with the linear and Coulomb-type confining potentials in order that a polynomial solution to the function  $H(r)$  can be obtained. The ground state of the system is also determined by the quantum number  $n = 1$  instead of the quantum number  $n = 0$  of the Landau-like levels. Finally, from (45) to (46), we have that the degeneracy of the analogue of the Landau levels is broken.

## 7. Conclusions

We have investigated the behaviour of a neutral particle (atom or molecule) with a magnetic quadrupole moment in a region with a uniform effective magnetic field subject to confining potentials. We have analysed the confinement of the Landau-type system to a hard-wall confining potential, a Coulomb-type potential, a linear confining potential, and a linear plus Coulomb-type potential. In the confinement to a hard-wall confining potential, we have seen that the spectrum of energy is modified in contrast to the Landau-type levels, where the energy levels are parabolic with respect to the quantum number associated with the radial modes.

On the other hand, with respect to the confinement to a Coulomb-type potential, a linear confining potential, and a linear plus Coulomb-type potential, we have obtained different spectrum of energies. In these three cases, the ground state of the system becomes determined by the quantum number  $n = 1$  instead of the quantum number  $n = 0$

obtained in the Landau-like levels, and the degeneracy of the analogue of the Landau levels is broken. Moreover, the cyclotron frequency of the Landau-type system is modified by the influence of the confining potentials, where the possible values of the angular frequency of the system are determined by the quantum numbers  $\{n, l\}$  and by the parameters associated with confining potentials. In particular, in the case of the confinement to the linear plus Coulomb-type potential, the possible values of the angular frequency associated with the ground state are determined by a third-degree algebraic equation.

## Competing Interests

The authors declare that they have no competing interests.

## Acknowledgments

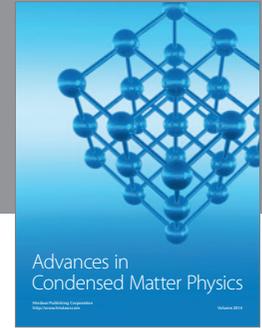
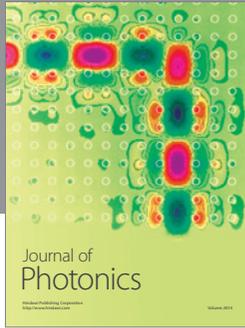
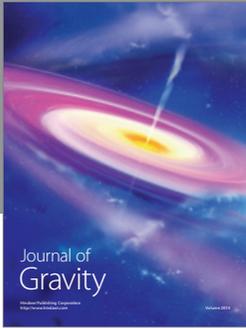
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