Research Article

Q-Deformed Morse and Oscillator Potential

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We studied the q -deformed Morse and harmonic oscillator systems with appropriate canonical commutation algebra. The analytic solutions for eigenfunctions and energy eigenvalues are worked out using time-independent Schrödinger equation and it is also noted that these wave functions are sensitive to variation in the parameters involved.

1. Introduction

Quantum groups and q -deformed algebras have been the subject of intense study and investigation in the last decade. In the past few years, a q -deformed harmonic oscillator was introduced [1–3] and, then inspired from such deformation, quantum groups and q -deformations have found applications in various branches of physics and chemistry; specially, they have been utilized to express electronic conductance in disordered metals and doped semiconductors [4], to analyze the phonon spectrum in ^4He [5], to specify the oscillatory-rotational spectra of diatomic [6] and multiatomic molecules [7]. As the main application of quantum groups, q -deformed quantum mechanics [8, 9] was developed by generalizing the standard quantum mechanics which was based on the Heisenberg commutation relation (the Heisenberg algebra). Furthermore, quantum q -analogues of several fundamental notions and models in quantum mechanics have been mentioned such as phase space [10], uncertainty relation [11, 12], density matrix [13], harmonic oscillator [1, 2, 10], hydrogen atom [14], creation and annihilation operators, and coherent states [1, 2, 15–18] so that they reduce to their standard counterparts as $q \rightarrow 1$. It can also be interpreted that some properties of generalized q -variables

are notably different from the properties of the standard quantum mechanics because of imposing the deformation.

Recently, some efforts have been made to solve various problems of quantum mechanics by the Lie algebraic methods. These methods have been the subject of interest in many fields of physics and chemistry. For example, these methods provide a way to obtain the wave functions of potentials in nuclear and polyatomic molecules [19, 20]. On the other hand, the deformed algebras are deformed versions of the usual Lie algebras which are obtained by introducing a deformation parameter q . The deformed algebras provide appropriate tools for describing systems which cannot be described by the ordinary Lie algebras. With this motivation, we studied the q -deformed Morse and harmonic potential in the nonrelativistic time-independent Schrödinger equation.

The present paper is organized as follows. In Section 2, a kind of deformation of quantum mechanics is introduced. Then as first study in such deformation, q -deformed Morse system has been investigated in Section 3. Furthermore, q -deformed harmonic oscillator is studied in Section 4 and, finally, concluding remarks are given in Section 5.

2. The Classical Oscillator Algebra

The classical oscillator algebra is defined by the canonical commutation relations [18–20]:

$$\begin{aligned} [a, a^\dagger] &= 1, \\ [N, a] &= -a, \\ [N, a^\dagger] &= a^\dagger, \end{aligned} \quad (1)$$

where N is called a number operator and it is assumed to be Hermitian. The first deformation was accomplished by Arik and Coon [21] as follows:

$$\begin{aligned} aa^\dagger - qa^\dagger a &= 1, \\ [N, a^\dagger] &= a^\dagger, \\ [N, a] &= -a, \end{aligned} \quad (2)$$

where the relation between the number operator and step operators becomes

$$a^\dagger a = [N]_q, \quad (3)$$

where a q -number is defined as

$$[X]_q = \frac{1 - q^X}{1 - q}. \quad (4)$$

The Jackson derivative is defined as follows [22]:

$$\partial_x^q f(x) = \frac{f(x) - f(qx)}{(1 - q)x} \quad (5)$$

which reduces to the ordinary derivative when $q \rightarrow 1$. If we introduce the coordinate realization of the deformed momentum $\partial_q = (\hbar/i)\partial_x^q$, we can obtain the q -deformed Schrödinger equation of the following form:

$$\left[-\frac{\hbar^2}{2m} (\partial_x^q)^2 + U(x) \right] \psi(x) = E\psi(x). \quad (6)$$

and the energy eigenvalue is

$$E_n = -D_e - \frac{\beta^2 \hbar^2}{8m} (2n+1)^2 + \frac{\beta^2 \hbar^2}{2m} (2n+1) \sqrt{\frac{2mD_e}{\beta^2 \hbar^2}}. \quad (13)$$

4. The q -Deformed Harmonic Oscillator

The q -deformed harmonic oscillator [16, 27] is given by

$$V(X) = \frac{1}{2} m \omega^2 X^2, \quad (14)$$

where m and ω are the mass and frequency of oscillator.

But, this equation is not easily solved for some potentials which has analytic solutions in the limit $q \rightarrow 1$. Recently, another type of q -deformed theory appeared in statistical physics, which was first proposed by Tsallis [23, 24]. He used the q -deformed logarithm instead of an ordinary logarithm in defining the entropy called a Tsallis entropy.

3. The q -Deformed Morse Potential

In this section, we demonstrate the viability of the canonical algebra as discussed in Section 2 for obtaining energy eigenvalue and eigenfunctions for q -deformed Morse potential.

The time-independent Schrödinger equation is given by

$$\left(-\frac{\hbar^2}{2m} D_x^2 + V(x) \right) U(x) = EU(x), \quad (7)$$

where

$$D_x = (1 + q|x|) \frac{\partial}{\partial x}. \quad (8)$$

By using changing variable $X = (1/q) \ln(1 + q|x|)$, (7) reduces to

$$\left(-\frac{\hbar^2}{2m} \frac{d^2}{dX^2} + V(X) \right) U(X) = EU(X). \quad (9)$$

Now considering the q -deformed Morse potential [25, 26] of the form

$$V(X) = D_e \left(e^{-2\beta(X-X_0)} - 2e^{-\beta(X-X_0)} \right) \quad (10)$$

and by putting $s = e^{-\beta(X-X_0)}$, (9) becomes

$$\begin{aligned} \frac{d^2 U(s)}{ds^2} + \frac{1}{s} \frac{dU(s)}{ds} \\ + \frac{1}{s^2} \left(-\frac{2mD_e}{\beta^2 \hbar^2} s^2 + \frac{4mD_e}{\beta^2 \hbar^2} s + \frac{2mE}{\beta^2 \hbar^2} \right) U(s) = 0. \end{aligned} \quad (11)$$

Now solving (11) eigenfunction can be written as follows:

$$U(X) = e^{-\beta(X-X_0)\sqrt{-2mE/\beta^2 \hbar^2}} e^{-\sqrt{2mD_e/\beta^2 \hbar^2} e^{-\beta(X-X_0)}} L_n^{2\sqrt{-2mE/\beta^2 \hbar^2}} \left(2\sqrt{\frac{2mD_e}{\beta^2 \hbar^2}} e^{-\beta(X-X_0)} \right) \quad (12)$$

Now substituting (14) into (9) and by using changing variable $s = (m\omega/\hbar)X^2$, we obtain

$$s \frac{d^2 U(s)}{ds^2} + \frac{1}{2} \frac{dU(s)}{ds} + \left(\frac{E}{2\hbar\omega} - \frac{s}{4} \right) U(s) = 0. \quad (15)$$

For further convenience, we apply the gauge transformation $U(s) = e^{-s/2} \chi(s)$ which leads to

$$s \frac{d^2 \chi(s)}{ds^2} + \left(\frac{1}{2} - s \right) \frac{d\chi(s)}{ds} + \left(\frac{E}{2\hbar\omega} - \frac{1}{4} \right) \chi(s) = 0. \quad (16)$$

Equation (16) is identified as the Kummer differential equation. In view of the above equations, the even and odd eigenfunctions may be, respectively, expressed as follows [28]:

$$U_{\text{even}} = N_n e^{-(m\omega/2\hbar)X^2} H_n \left(X \sqrt{\frac{m\omega}{\hbar}} \right) \quad (17a)$$

$$U_{\text{odd}} = N_n e^{-(m\omega/2\hbar)X^2} H_n \left(X \sqrt{\frac{m\omega}{\hbar}} \right), \quad (17b)$$

where N_n is the normalization constant. However, the even and odd eigenfunctions may be combined and the stationary states of the relativistic oscillator are

$$U_n(x) = N_n e^{-(m\omega/2\hbar)((1/q)\ln(1+q|x|))^2} H_n \left(\frac{1}{q} \cdot \ln(1+q|x|) \sqrt{\frac{m\omega}{\hbar}} \right). \quad (18)$$

The energy eigenvalues of spin-zero particles bound in this oscillator potential may be found using (15). Therefore, the energy for even and odd states can be written as follows:

$$E_n = \left(n + \frac{1}{2} \right) \hbar\omega, \quad (19)$$

where $n = 0, 1, 2, 3, \dots$ are the integers. Note that (19) is in agreement with the energy of the harmonic oscillator.

5. Conclusion

In this paper, we have introduced the q -deformed Morse and harmonic oscillator potential functions in the light of canonical commutation algebra. We have computed the energy eigenvalues and corresponding eigenfunctions for these potentials in one-dimensional nonrelativistic Schrödinger equation. The exact solution for the eigenfunction is obtained in terms of Laguerre polynomial for Morse potential. However, in case of harmonic oscillator, even and odd eigenfunctions are obtained in terms of Hermite polynomials. It is also noted that the function behavior depends on the variation of q -deformed parameter as well as the strength of the potential parameter. Moreover, it is worth mentioning that, in the limit case, results in ordinary quantum mechanics can be recovered.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

- [1] A. J. Macfarlane, "On q -analogues of the quantum harmonic oscillator and the quantum group $SU(2)_q$," *Journal of Physics. A. Mathematical and General*, vol. 22, no. 21, pp. 4581–4588, 1989.
- [2] L. C. Biedenharn, "The quantum group $SU_q(2)$ and a q -analogue of the boson operators," *Journal of Physics. A: Mathematical and General*, vol. 22, no. 18, pp. L873–L878, 1989.
- [3] P. P. Kulish and E. V. Damaskinsky, "On the q oscillator and the quantum algebra $textrmsu_q(1, 1)$," *Journal of Physics. A. Mathematical and General*, vol. 23, no. 9, pp. L415–L419, 1990.
- [4] S. A. Alavi and S. Rouhani, "Exact analytical expression for magnetoresistance using quantum groups," *Physics Letters. A*, vol. 320, no. 4, pp. 327–332, 2004.
- [5] M. R-Monteiro, L. M. C. S. Rodrigues, and S. Wulck, "Quantum algebraic nature of the phonon spectrum in ^4He ," *Physical Review Letters*, vol. 76, no. 7, 1996.
- [6] R. S. Johal and R. K. Gupta, "Two parameter quantum deformation of $U(2) \supset U(1)$ dynamical symmetry and the vibrational spectra of diatomic molecules," *International Journal of Modern Physics. E. Nuclear Physics*, vol. 7, no. 5, pp. 553–557, 1998.
- [7] D. Bonatsos, C. Daskaloyannis, and P. Kolokotronis, "Coupled Q -oscillators as a model for vibrations of polyatomic molecules," *Journal of Chemical Physics*, vol. 106, p. 605, 1997.
- [8] S. Chaturvedi and V. Srinivasan, "Aspects of q -oscillator quantum mechanics," *Physical Review. A. Third Series*, vol. 44, no. 12, pp. 8020–8023, 1991.
- [9] J. Zhang, "A q -deformed quantum mechanics," *Physics Letters B*, vol. 440, p. 66, 1998.
- [10] A. Lorek, A. Ruffing, and J. Wess, "A q -deformation of the harmonic oscillator," *Journal of Physics C*, vol. 74, p. 369, 1997.
- [11] J. Zhang, "A q -deformed uncertainty relation," *Physics Letters A*, vol. 262, p. 125, 1999.
- [12] R. J. Finkelstein, "q-uncertainty relations," *International Journal of Modern Physics A*, vol. 13, p. 1795, 1998.
- [13] R. Parthasarathy and R. Sridhar, "A diagonal representation of the quantum density matrix using q -boson oscillator coherent states," *Physics Letters. A*, vol. 305, no. 3-4, pp. 105–110, 2002.
- [14] R. J. Finkelstein, "Observable properties of q -deformed physical systems," *Letters in Mathematical Physics*, vol. 49, no. 2, pp. 105–114, 1999.
- [15] X.-M. Liu and C. Quesne, "Even and odd q -deformed charge coherent states and their nonclassical properties," *Physics Letters. A*, vol. 317, no. 3-4, pp. 210–222, 2003.
- [16] V. V. Eremin and A. A. Meldianov, "The q -deformed harmonic oscillator, coherent states, and the uncertainty relation," *Theoretical and Mathematical Physics*, vol. 147, no. 2, pp. 709–715, 2006.
- [17] A. Dobrogowska, "The q -deformation of the Morse potential," *Applied Mathematics Letters. An International Journal of Rapid Publication*, vol. 26, no. 7, pp. 769–773, 2013.
- [18] A. G. Dijkstra, C. Wang, J. Cao, and G. R. Fleming, "Coherent exciton dynamics in the presence of underdamped vibrations," *Journal of Physical Chemistry Letters*, vol. 6, no. 4, pp. 627–632, 2015.
- [19] M. R. Setare and E. Karimi, "Algebraic approach to the Kratzer potential," *Physica Scripta*, vol. 75, no. 1, pp. 90–93, 2007.
- [20] M. R. Setare and G. Olfati, "An algebraic approach for a charged particle in a certain magnetic field," *Physica Scripta. An International Journal for Experimental and Theoretical Physics*, vol. 75, no. 3, pp. 250–252, 2007.
- [21] M. Arik and D. D. Coon, "Hilbert spaces of analytic functions and generalized coherent states," *Journal of Mathematical Physics*, vol. 17, no. 4, pp. 524–527, 1976.
- [22] F. Jackson, *Math Mess*, vol. 38, 57 pages, 1909.
- [23] C. Tsallis, "Possible generalization of Boltzmann-Gibbs statistics," *Journal of Statistical Physics*, vol. 52, no. 1-2, pp. 479–487, 1988.

- [24] E. M. F. Curado and C. Tsallis, “Generalized statistical mechanics: connection with thermodynamics,” *Journal of Physics A: Mathematical and General*, vol. 24, no. 2, article no. 004, pp. L69–L72, 1991.
- [25] M. R. Setare and O. Hatami, “An algebraic approach to the q -deformed Morse potential,” *Modern Physics Letters A. Particles and Fields, Gravitation, Cosmology, Nuclear Physics*, vol. 24, no. 5, pp. 361–367, 2009.
- [26] M. S. Abdalla and H. Eleuch, “Exact analytic solutions of the Schrödinger equations for some modified q -deformed potentials,” *Journal of Applied Physics*, vol. 115, Article ID 234906, 2014.
- [27] S. Zare and H. Hassanabadi, “Properties of quasi-oscillator in position-dependent mass formalism,” *Advances in High Energy Physics*, vol. 2016, Article ID 4717012, 7 pages, 2016.
- [28] A. R. Nagalakshmi and B. A. Kagali, “Energy profile of the one-dimensional Klein–Gordon oscillator,” *Phys. Script*, vol. 77, Article ID 015003, 2008.