

Research Article

CFT and Logarithmic Corrections to the Black Hole Entropy Product Formula

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We examine the logarithmic corrections to the black hole (BH) entropy product formula of outer horizon and inner horizon by taking into account the *effects of statistical quantum fluctuations around the thermal equilibrium* and *via conformal field theory (CFT)*. We argue that, in logarithmic corrections to the BH entropy product formula when calculated using CFT and taking into account the effects of quantum fluctuations around the thermal equilibrium, the formula should *not be universal* and it also should *not be quantized*. These results have been explicitly checked by giving several examples.

1. Introduction

There has been considerable ongoing excitement in physics of BH thermodynamic product formula [particularly area (or entropy) product formula] of inner horizon (\mathcal{H}^-) and outer horizon (\mathcal{H}^+) for a wide variety of BHs [1–9] which have been examined so far *without considering any logarithmic correction*. For several cases, the product is mass-independent (universal) and in some specific cases the product is not mass-independent. This investigation has been more sparked by Cvetič et al. [2] for supersymmetric BPS (Bogomol'ni-Prasad-Sommerfield) class of BHs which have inner and outer BH entropy of the form $\mathcal{S}_{\pm} = 2\pi(\sqrt{N_L} \pm \sqrt{N_R})$, where N_L and N_R are excitation numbers of the left and right moving sectors of a weakly coupled two-dimensional (2D) CFT. Therefore, their product $\mathcal{S}_+ \mathcal{S}_-$ must be quantized in nature.

It has been suggested by Larsen [10] that BH event horizon is quantized in Planck units so it is natural to be valid for Cauchy horizon also. This also indicates that the product of area (or entropy) is quantized in terms of quantized charges and quantized angular momentum and so on. But this has been discussed without considering any logarithmic correction to the BH entropy. Now if we take into account the logarithmic correction to the BH entropy, what happens in case of logarithmic correction to the BH entropy product formula? What are the quantization rules for logarithmic

corrected BH entropy product formula? These are the main issues that we will be discussed in this work.

It should be noted that this is the continuation of our earlier investigation [11]. In the previous work, we derived the general logarithmic correction to the entropy product formula of event horizon and Cauchy horizon for various spherically symmetric and axisymmetric BHs by taking into account the effects of quantum fluctuations around the thermal equilibrium. These corrections are evaluated in terms of some BH thermodynamic parameters, namely, the *specific heat* C_{\pm} of \mathcal{H}^{\pm} and *BH temperature* T_{\pm} of \mathcal{H}^{\pm} , respectively.

The logarithmic correction of BH entropy of \mathcal{H}^{\pm} is described by the key formula

$$\mathcal{S}_{\pm} = \ln \rho_{\pm} = \mathcal{S}_{0,\pm} - \frac{1}{2} \ln |C_{\pm} T_{\pm}^2| + \dots, \quad (1)$$

where ρ_{\pm} is density of states of \mathcal{H}^{\pm} and \mathcal{S}_{\pm} is entropy of \mathcal{H}^{\pm} and their product is derived to be

$$\begin{aligned} \mathcal{S}_+ \mathcal{S}_- &= \mathcal{S}_{0,+} \mathcal{S}_{0,-} \\ &- \frac{1}{2} [\mathcal{S}_{0,+} \ln |C_- T_-^2| + \mathcal{S}_{0,-} \ln |C_+ T_+^2|] \\ &+ \frac{1}{4} \ln |C_+ T_+^2| \ln |C_- T_-^2| + \dots, \end{aligned} \quad (2)$$

where $\mathcal{S}_{0,\pm}$ is the entropy of \mathcal{H}^\pm without logarithmic correction.

In the present work, we shall compute the general logarithmic correction to BH entropy of \mathcal{H}^\pm whenever we have taken *the effects of statistical quantum fluctuations around the thermal equilibrium* and by using exact entropy function $\mathcal{S}_\pm(\beta_\pm)$ according to the formalism borrowed from the quantum theory of gravity [12–25]. Whenever we incorporated the effects of quantum fluctuations around the thermal equilibrium, the Bekenstein-Hawking entropy formula must be corrected and the entropy product formula of \mathcal{H}^\pm must also be corrected. This is the main motivation behind this work.

It has been known that BHs in Einstein's gravity as well as other theories of gravity are much larger than the Planck scale length where the Bekenstein-Hawking entropy is precisely proportional to the horizon area [12, 13, 18–20]. Thus, it is quite natural to investigate the *leading order corrections in Bekenstein-Hawking entropy* as well as in *Bekenstein-Hawking entropy product formula* of \mathcal{H}^\pm and when one can reduce the size of BH. For large BHs, it has been proved that the logarithm of the density of states is exactly the Bekenstein-Hawking entropy plus the corrections term $-\chi \ln A$, where $\chi = 3/2$ and A is the area of the event horizon [22, 23]. Thus, it appears that the logarithmic corrections to the Bekenstein-Hawking entropy as well as *Bekenstein-Hawking entropy product formula* of \mathcal{H}^\pm are a generic feature of BHs. It has been verified earlier by Das et al. [24] for \mathcal{H}^+ and here we have tried to examine for \mathcal{H}^- followed by our earlier investigation [11].

It should be emphasized that logarithmic corrections to the Bekenstein-Hawking formula are very interesting and a great deal about such corrections is known in string theory and beyond. Logarithmic corrections arise from various sources, the simplest of which are the statistical fluctuations around thermal equilibrium. These are always present because they arise from saddle point Gaussian corrections to the integral that computes the density of states from the partition function. In some cases, such as the BTZ BH in pure 3D gravity, these are the only logarithmic corrections to the Bekenstein-Hawking entropy. However, more generally the logarithm of the partition function, $\ln Z$, itself receives corrections from the massless spectrum of particles in the theory whose solution contains the BH. These corrections therefore cannot be determined from the BH solution only. They are universal only in the sense that they are independent from the UV completion of the theory (see review article [26]).

Moreover, it must be noted that a given theory of quantum gravity will assign a Hilbert space to \mathcal{H}^+ counting the number of microstates of the Hilbert space which gives us the entropy of \mathcal{H}^+ of the BH by Boltzmann's entropy formula

$$\mathcal{S}_+ = k_B \ln \Omega_+, \quad (3)$$

where k_B is the Boltzmann constant and Ω_+ is the microstates of \mathcal{H}^+ only. Analogously, there must exist *inner Hilbert space* for \mathcal{H}^- . Therefore, Boltzmann's entropy formula for \mathcal{H}^- becomes

$$\mathcal{S}_- = k_B \ln \Omega_-, \quad (4)$$

where Ω_- is the microstates of \mathcal{H}^- . Finally, their product should be

$$\mathcal{S}_+ \mathcal{S}_- = k_B^2 \ln \Omega_+ \ln \Omega_-. \quad (5)$$

Now we turn to compute the logarithmic corrections to BH entropy product formula by using the CFT formalism.

2. Logarithmic Corrections to the BH Entropy Product Formula of \mathcal{H}^\pm via CFT

We have started with the partition function [11, 24] of any thermodynamic system consisting of \mathcal{H}^\pm which should read

$$\mathcal{Z}_\pm(\beta_\pm) = \int_0^\infty \rho_\pm(E) e^{-\beta_\pm E} dE, \quad (6)$$

where $T_\pm = 1/\beta_\pm$ can be defined as the temperature of \mathcal{H}^\pm . We have to set Boltzmann constant $k_B = 1$.

The density of states of the said thermodynamic system may be expressed by taking an inverse Laplace transformation (keeping E fixed) of the partition function defined at \mathcal{H}^\pm [24, 27, 28]

$$\begin{aligned} \rho_\pm(E) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{Z}_\pm(\beta_\pm) e^{\beta_\pm E} d\beta_\pm \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{S_\pm(\beta_\pm)} d\beta_\pm, \end{aligned} \quad (7)$$

where c is a real constant and defining

$$S_\pm = \ln \mathcal{Z}_\pm + \beta_\pm E \quad (8)$$

is the exact entropy of \mathcal{H}^\pm as a function of temperature.

Now if we consider the system to be in equilibrium, then the inverse temperature is defined to be $\beta_\pm = \beta_{0,\pm}$; therefore, we can expand the entropy function of \mathcal{H}^\pm as

$$S_\pm(\beta_\pm) = S_{0,\pm} + \frac{1}{2}(\beta_\pm - \beta_{0,\pm})^2 S''_{0,\pm} + \dots, \quad (9)$$

where $S_{0,\pm} := S_\pm(\beta_{0,\pm})$ and $S''_{0,\pm} = \partial^2 S_\pm / \partial \beta_\pm^2$ at $\beta_\pm = \beta_{0,\pm}$.

Substituting (9) in (7), we find

$$\rho_\pm(E) = \frac{e^{S_{0,\pm}}}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{(\beta_\pm - \beta_{0,\pm})^2 S''_{0,\pm} / 2} d\beta_\pm. \quad (10)$$

Let us put $\beta_\pm - \beta_{0,\pm} = ix_\pm$ and choosing $c = \beta_{0,\pm}$, x_\pm is a real variable and evaluating a contour integration, one obtains

$$\rho_\pm(E) = \frac{e^{S_{0,\pm}}}{\sqrt{2\pi S''_{0,\pm}}}. \quad (11)$$

Thus, the logarithm of the density of states gives the corrected entropy of \mathcal{H}^\pm :

$$\mathcal{S}_\pm := \ln \rho_\pm = \mathcal{S}_{0,\pm} - \frac{1}{2} \ln S''_{0,\pm} + \dots \quad (12)$$

The main aim of this work is to compute the term $S''_{0,\pm}$ by using the exact entropy function $\mathcal{S}_{\pm}(\beta_{\pm})$, evaluated at the equilibrium temperature $\beta_{\pm} = \beta_{0,\pm}$.

Assume that the exact entropy function $\mathcal{S}_{\pm}(\beta_{\pm})$ of \mathcal{H}^{\pm} (the entropy function is defined in [20] for \mathcal{H}^+ only. We here prescribed that this entropy function is valid for \mathcal{H}^- as well) which is followed from the CFT [24] is

$$\mathcal{S}_{\pm}(\beta_{\pm}) = a\beta_{\pm} + \frac{b}{\beta_{\pm}}. \quad (13)$$

We choose the entropy function [24] in more general form (because this form admits a saddle point) as

$$\mathcal{S}_{\pm}(\beta_{\pm}) = a\beta_{\pm}^m + \frac{b}{\beta_{\pm}^n}, \quad (14)$$

where $m, n, a, b > 0$. When we have considered the special case which is dictated by CFT, then $m = n = 1$.

Now the above function has an extremum value at

$$\beta_{0,\pm} = \left(\frac{nb}{ma}\right)^{1/(m+n)} \equiv \frac{1}{T_{\pm}}. \quad (15)$$

Expanding around $\beta_{0,\pm}$ and by evaluating second-order derivative, one obtains

$$\begin{aligned} \mathcal{S}_{\pm}(\beta_{\pm}) &= \alpha (a^n b^m)^{1/(m+n)} \\ &+ \frac{\gamma}{2} (a^{n+2} b^{m-2})^{1/(m+n)} (\beta_{\pm} - \beta_{0,\pm})^2 + \dots, \end{aligned} \quad (16)$$

where

$$\begin{aligned} \alpha &= \left(\frac{n}{m}\right)^{m/(m+n)} + \left(\frac{m}{n}\right)^{n/(m+n)}, \\ \gamma &= (m+n)m^{((n+2)/(m+n))} + n^{((m-2)/(m+n))}. \end{aligned} \quad (17)$$

We also derived in [11] close to the equilibrium and at the inverse temperature $\beta_{\pm} = \beta_{0,\pm}$ the entropy function of \mathcal{H}^{\pm} as

$$S_{\pm}(\beta_{\pm}) = S_{0,\pm} + \frac{1}{2} (\beta_{\pm} - \beta_{0,\pm})^2 S''_{0,\pm} + \dots, \quad (18)$$

where $S_{0,\pm} := S_{\pm}(\beta_{0,\pm})$ and $S''_{0,\pm} = \partial^2 S_{\pm}(\beta_{\pm})/\partial\beta_{\pm}^2$ at $\beta_{\pm} = \beta_{0,\pm}$. Comparing (16) and (18), we find

$$\begin{aligned} S_{0,\pm} &= \alpha (a^n b^m)^{1/(m+n)}, \\ S''_{0,\pm} &= \gamma (a^{n+2} b^{m-2})^{1/(m+n)}. \end{aligned} \quad (19)$$

Inverting these equations, one can find a and b in terms of $S_{0,\pm}$ and $S''_{0,\pm}$

$$\begin{aligned} a &= \left(\frac{\alpha^{(m-2)/2}}{\gamma^{m/2}}\right) \left[\frac{(S''_{0,\pm})^{m/2}}{(S_{0,\pm})^{m/2}}\right], \\ b &= \left[\frac{\gamma}{\alpha^{(n+2)/2}}\right]^{n/2} \left[\frac{(S''_{0,\pm})^{(n+2)/2}}{(S_{0,\pm})^{n/2}}\right]. \end{aligned} \quad (20)$$

Putting these values in (15), we get

$$\beta_{0,\pm} = \frac{1}{T_{\pm}} = \left(\frac{n}{m}\right)^{1/(m+n)} \sqrt{\frac{\gamma S_{0,\pm}}{\alpha S''_{0,\pm}}}. \quad (21)$$

Now inverting $S''_{0,\pm}$ in terms of $S_{0,\pm}$ and T_{\pm} , one obtains

$$S''_{0,\pm} = \left[\left(\frac{\gamma}{\alpha}\right)\left(\frac{n}{m}\right)^{2/(m+n)}\right] S_{0,\pm} T_{\pm}^2. \quad (22)$$

Substituting these values in (12), we have

$$\mathcal{S}_{\pm} = \ln \rho_{\pm} = \mathcal{S}_{0,\pm} - \frac{1}{2} \ln |T_{\pm}^2 S_{0,\pm}| + \dots. \quad (23)$$

This is in fact the generic formula for leading order correction to Bekenstein-Hawking formula. It should be noted that the formula is indeed independent of a and b . *What is new in this formula is that one could calculate the logarithmic correction to the Bekenstein-Hawking entropy of \mathcal{H}^{\pm} without knowing the values of any specific heat of the BH but only knowing the values of T_{\pm} of \mathcal{H}^{\pm} and $S_{0,\pm}$ for the said BH.*

Therefore, the product becomes

$$\begin{aligned} \mathcal{S}_+ \mathcal{S}_- &= \mathcal{S}_{0,+} \mathcal{S}_{0,-} \\ &- \frac{1}{2} [\mathcal{S}_{0,+} \ln |T_-^2 \mathcal{S}_{0,-}| + \mathcal{S}_{0,-} \ln |T_+^2 \mathcal{S}_{0,+}|] \\ &+ \frac{1}{4} \ln |T_+^2 \mathcal{S}_{0,+}| \ln |T_-^2 \mathcal{S}_{0,-}| + \dots. \end{aligned} \quad (24)$$

We have already argued the implication of this formula in [11] as when we take the first-order correction, it indicates that the product is always dependent on mass parameter. Therefore, the theorem of Ansorg-Hennig [1], "the area product formula of \mathcal{H}^{\pm} being independent of mass," is no longer true when we have taken into consideration the leading order logarithmic correction.

For completeness, we further compute the logarithmic correction of *entropy sum*, *entropy minus*, and *entropy division* using (23). They are

$$\begin{aligned} \mathcal{S}_+ + \mathcal{S}_- &= \ln \rho_+ + \ln \rho_- \\ &= (\mathcal{S}_{0,+} + \mathcal{S}_{0,-}) \\ &- \frac{1}{2} [\ln |T_+^2 \mathcal{S}_{0,+}| + \ln |T_-^2 \mathcal{S}_{0,-}|] + \dots \\ \mathcal{S}_+ - \mathcal{S}_- &= \ln \rho_+ - \ln \rho_- \\ &= (\mathcal{S}_{0,+} - \mathcal{S}_{0,-}) \\ &- \frac{1}{2} [\ln |T_+^2 \mathcal{S}_{0,+}| - \ln |T_-^2 \mathcal{S}_{0,-}|] + \dots, \\ \frac{\mathcal{S}_+}{\mathcal{S}_-} &= \frac{\ln \rho_+}{\ln \rho_-} = \frac{\mathcal{S}_{0,+} - (1/2) \ln |T_+^2 \mathcal{S}_{0,+}| + \dots}{\mathcal{S}_{0,-} - (1/2) \ln |T_-^2 \mathcal{S}_{0,-}| + \dots}. \end{aligned} \quad (25)$$

These equations indicate that they are all mass-independent; hence, they are not *universal* in this sense.

3. Examples

Now we apply this formula for specific BHs in order to calculate the logarithm correction to the Bekenstein-Hawking entropy of \mathcal{H}^\pm . First we take the four-dimensional Reissner-Nordström (RN) BH.

Example 1 (RN BH). The BH entropy and BH temperature [15] become

$$\begin{aligned} \mathcal{S}_{0,\pm} &= \pi r_\pm^2, \\ T_\pm &= \frac{r_\pm - r_\mp}{4\pi r_\pm^2}, \end{aligned} \quad (26)$$

where $r_\pm = M \pm \sqrt{M^2 - Q^2}$ and M and Q are mass and charge of BH, respectively.

Therefore, the entropy correction is given by

$$\mathcal{S}_\pm = \ln \rho_\pm = \mathcal{S}_{0,\pm} - \frac{1}{2} \ln \left| \frac{(r_\pm - r_\mp)^2}{16\pi r_\pm^2} \right| + \dots \quad (27)$$

We can conclude that the product includes the mass term so it is not *universal*.

The second example we take is the Kehagias-Sfetsos (KS) BH [29] in Hořava-Lifshitz gravity [30–32].

Example 2 (KS BH). The entropy for KS BH [8] should read

$$\begin{aligned} \mathcal{S}_{0,\pm} &= \pi r_\pm^2, \\ r_\pm &= M \pm \sqrt{M^2 - \frac{1}{2\omega}}, \end{aligned} \quad (28)$$

where ω is coupling constant and r_+ and r_- are EH and CH, respectively. The Hawking temperature becomes

$$T_\pm = \frac{\omega(r_\pm - r_\mp)}{4\pi(1 + \omega r_\pm^2)}. \quad (29)$$

Therefore, the entropy correction for KS BH is given by

$$\mathcal{S}_\pm = \mathcal{S}_{0,\pm} - \frac{1}{2} \ln \left| \frac{\omega^2 r_\pm^2 (r_\pm - r_\mp)^2}{16\pi(1 + \omega r_\pm^2)} \right| + \dots \quad (30)$$

It implies that when the logarithmic correction is taken into consideration the entropy product is not mass-independent (universal).

Now we take the AdS space. First, it should be Schwarzschild-AdS space-time (in the limit $\ell \rightarrow \infty$, one gets the Schwarzschild BH. Here the horizon is at $r_+ = 2M$ and $\mathcal{S}_{0,+} = 4\pi M^2$. Thus, one obtains the logarithmic correction as $\mathcal{S}_+ = 4\pi M^2 + (1/2) \ln |16\pi|$. Therefore, it indicates that the logarithmic correction must be mass dependent. Therefore, it is not universal).

Example 3 (Schwarzschild-AdS BH). The only physical horizon [4] is at

$$r_+ = \frac{2\ell}{\sqrt{3}} \sinh \left[\frac{1}{3} \sinh^{-1} \left(3\sqrt{3} \frac{M}{\ell} \right) \right]. \quad (31)$$

Thus, the entropy of \mathcal{H}^+ should be

$$\mathcal{S}_{0,+} = \pi r_+^2, \quad (32)$$

where $\Lambda = -3/\ell^2$ is cosmological constant. The BH temperature reads

$$T_+ = \frac{1}{4\pi r_+} \left(1 + 3 \frac{r_+^2}{\ell^2} \right). \quad (33)$$

Now the entropy correction formula should read

$$\mathcal{S}_+ = \mathcal{S}_{0,+} - \frac{1}{2} \ln \left| \frac{(1 + 3(r_+^2/\ell^2))^2}{16\pi} \right| + \dots \quad (34)$$

In fact, in both cases, with logarithmic correction and without logarithmic correction, the entropy depends on the mass parameter; thus, it is not universal and therefore it is not quantized.

Now we take the RN-AdS case [4].

Example 4 (RN-AdS BH). The quartic Killing horizon equation becomes

$$r^4 + \ell^2 r^2 - 2M\ell^2 r + Q^2 \ell^2 = 0. \quad (35)$$

There are at least two real zeros which correspond to two physical horizons, namely, EH, r_+ , and CH, r_- .

The entropy should read

$$\mathcal{S}_{0,\pm} = \pi r_\pm^2. \quad (36)$$

The BH temperature of \mathcal{H}^\pm is given by

$$T_\pm = \frac{1}{4\pi r_\pm} \left(3 \frac{r_\pm^2}{\ell^2} - \frac{Q^2}{r_\pm^2} + 1 \right). \quad (37)$$

Therefore, the logarithmic correction becomes

$$\mathcal{S}_\pm = \mathcal{S}_{0,\pm} - \frac{1}{2} \ln \left| \frac{(3(r_\pm^2/\ell^2) - Q^2/r_\pm^2 + 1)^2}{16\pi} \right| + \dots \quad (38)$$

It implies that the product of logarithmic correction is not mass-independent.

Now we take the spinning BH.

Example 5 (Kerr BH). The BH entropy and BH temperature [15] are

$$\begin{aligned} \mathcal{S}_{0,\pm} &= \pi (r_\pm^2 + a^2), \\ T_\pm &= \frac{r_\pm - r_\mp}{4\pi (r_\pm^2 + a^2)}, \end{aligned} \quad (39)$$

where $r_\pm = M \pm \sqrt{M^2 - a^2}$, M , and $a = J/M$ are mass and spin parameter of the BH, respectively.

Now the logarithmic correction is computed to be

$$\mathcal{S}_\pm = \mathcal{S}_{0,\pm} - \frac{1}{2} \ln \left| \frac{(r_\pm - r_\mp)^2}{16\pi (r_\pm^2 + a^2)} \right| + \dots \quad (40)$$

It also indicates that whenever we take the logarithmic correction; the entropy product of \mathcal{H}^\pm is not universal.

Next we take charged rotating BH.

Example 6 (Kerr-Newman BH). The BH entropy and BH temperature [15] should read

$$\begin{aligned} \mathcal{S}_{0,\pm} &= \pi (r_{\pm}^2 + a^2), \\ T_{\pm} &= \frac{r_{\pm} - r_{\mp}}{4\pi (r_{\pm}^2 + a^2)}, \end{aligned} \quad (41)$$

where $r_{\pm} = M \pm \sqrt{M^2 - a^2 - Q^2}$ and M , a , and Q correspond to the mass, the spin parameter, and the charge of BH, respectively.

The logarithmic correction is derived to be

$$\mathcal{S}_{\pm} = \mathcal{S}_{0,\pm} - \frac{1}{2} \ln \left| \frac{(r_{\pm} - r_{\mp})^2}{16\pi (r_{\pm}^2 + a^2)} \right| + \dots \quad (42)$$

Again we observe that when we take logarithmic correction the entropy product of \mathcal{H}^{\pm} for Kerr-Newman BH [1, 6] is not mass-independent.

Example 7 (Kerr-Newman AdS BH). The horizon equation [33] is given by

$$\Delta_r = (r^2 + a^2) \left(1 + \frac{r^2}{\ell^2} \right) - 2Mr + Q_e^2 + Q_m^2 = 0 \quad (43)$$

which implies that the quartic order horizon equation

$$r^4 + (\ell^2 + a^2)r^2 - 2M\ell^2 r + (a^2 + Q_e^2 + Q_m^2)\ell^2 = 0. \quad (44)$$

This equation has two real zeros which correspond to two physical horizons, namely, r_{\pm} , where Q_e and Q_m are electric and magnetic charge parameters, respectively. The BH entropy and BH temperature are

$$\begin{aligned} \mathcal{S}_{0,\pm} &= \frac{\pi (r_{\pm}^2 + a^2)}{(1 - a^2/\ell^2)} \\ T_{\pm} &= \frac{r_{\pm} (1 + a^2/\ell^2 + 3(r_{\pm}^2/\ell^2) - (a^2 + Q_e^2 + Q_m^2)/r_{\pm}^2)}{4\pi (r_{\pm}^2 + a^2)}. \end{aligned} \quad (45)$$

The logarithmic correction for KN-AdS BH should read

$$\begin{aligned} \mathcal{S}_{\pm} &= \mathcal{S}_{0,\pm} - \frac{1}{2} \\ &\cdot \ln \left| \frac{r_{\pm}^2 (1 + a^2/\ell^2 + 3(r_{\pm}^2/\ell^2) - (a^2 + Q_e^2 + Q_m^2)/r_{\pm}^2)}{16\pi (1 - a^2/\ell^2) (r_{\pm}^2 + a^2)} \right| \\ &+ \dots \end{aligned} \quad (46)$$

Example 8 (nonrotating BTZ BH). The BH horizon is at $r_+ = \sqrt{8G_3 M \ell}$. G_3 is 3D Newtonian constant. The BH entropy of \mathcal{H}^+ for BTZ BH is

$$\mathcal{S}_{0,+} = \frac{2\pi r_+}{4G_3}. \quad (47)$$

The BH temperature is

$$T_+ = \frac{r_+}{2\pi \ell^2}, \quad (48)$$

where $\Lambda = -3/\ell^2$ is cosmological constant. Thus, the BH entropy correction for BTZ BH is

$$\mathcal{S}_+ = \mathcal{S}_{0,+} - \frac{1}{2} \ln \left| \frac{r_+^3}{8\pi G_3 \ell^4} \right| + \dots \quad (49)$$

In fact, it is isolated case and there is only one horizon; therefore, both the logarithmic correction of entropy and without logarithmic corrections term are mass dependent.

Example 9 (rotating BTZ BH). The BH horizons for rotating BTZ BH [34, 35] are given by

$$r_{\pm} = \sqrt{4G_3 \mathcal{M} \ell^2 \left(1 \pm \sqrt{1 - \frac{J^2}{\mathcal{M}^2 \ell^2}} \right)}. \quad (50)$$

The BH entropy of \mathcal{H}^{\pm} is

$$\mathcal{S}_{0,\pm} = \frac{2\pi r_{\pm}}{4G_3}, \quad (51)$$

and the Hawking temperature of \mathcal{H}^{\pm} should read

$$T_{\pm} = \frac{r_{\pm}^2 - r_{\mp}^2}{2\pi \ell^2 r_{\pm}}. \quad (52)$$

Therefore, the BH entropy correction is calculated to be

$$\mathcal{S}_{\pm} = \mathcal{S}_{0,\pm} - \frac{1}{2} \ln \left| \frac{(r_{\pm}^2 - r_{\mp}^2)^2}{8\pi G_3 \ell^4 r_{\pm}} \right| + \dots \quad (53)$$

It is clear from the calculation that the product depends on the mass parameter.

Example 10 (charged dilaton BH). The horizons of charged dilation BH [36] are at

$$\begin{aligned} r_+ &= M + \sqrt{M^2 - \left(\frac{2n}{1+n} \right) Q^2}, \\ r_- &= \frac{1}{n} \left[M + \sqrt{M^2 - \left(\frac{2n}{1+n} \right) Q^2} \right], \end{aligned} \quad (54)$$

where n is given by

$$n = \frac{1 - a^2}{1 + a^2}. \quad (55)$$

The BH temperature of \mathcal{H}^{\pm} reads

$$\begin{aligned} T_+ &= \frac{1}{4\pi r_+} \left(\frac{r_+ - r_-}{r_+} \right)^n, \\ T_- &= 0. \end{aligned} \quad (56)$$

The entropy for both the horizons is

$$\mathcal{S}_{0,+} = \pi r_+^2 \left(\frac{r_+ - r_-}{r_+} \right)^{1-n}, \quad (57)$$

$$\mathcal{S}_{0,-} = 0.$$

Thus, the BH entropy correction for \mathcal{H}^+ should be

$$\mathcal{S}_+ = \mathcal{S}_{0,+} - \frac{1}{2} \ln \left| \frac{((r_+ - r_-)/r_+)^{1+n}}{16\pi} \right| + \dots \quad (58)$$

And the entropy correction for \mathcal{H}^- is

$$\mathcal{S}_- = 0. \quad (59)$$

This is an interesting case because the entropy product of \mathcal{H}^\pm and the entropy product with logarithmic correction both go to zero value. The logarithmic correction survives for \mathcal{H}^+ and only when we have taken into account the logarithmic correction for \mathcal{H}^- it breaks down and therefore the product also breaks down.

Example 11 (Kerr-Sen BH). The horizon for Kerr-Sen [37, 38] BH is situated at

$$r_\pm = \left(M - \frac{Q^2}{2M} \right) \pm \sqrt{\left(M - \frac{Q^2}{2M} \right)^2 - a^2}. \quad (60)$$

The BH entropy and BH temperature for Sen BH are

$$\mathcal{S}_{0,\pm} = 2\pi M r_\pm \quad (61)$$

$$T_\pm = \frac{r_\pm - r_\mp}{8\pi M r_\pm}.$$

Therefore, the logarithmic correction is calculated to be

$$\mathcal{S}_\pm = \mathcal{S}_{0,\pm} - \frac{1}{2} \ln \left| \frac{(r_\pm - r_\mp)^2}{32\pi M r_\pm} \right| + \dots \quad (62)$$

Example 12 (Sultana-Dyer BH). This is an example of a dynamical cosmological BH [39]. The horizon is located at $r_+ = 2m$, where m is the mass of BH.

The entropy (the surface area at $t = 0$ indicates that the cosmological BH is formed initially from Big-Bang singularity. t^4 is the conformal factor) and temperature of this BH are

$$\mathcal{S}_{0,+} = \frac{\mathcal{A}_+}{4} = \pi t^4 r_+ \quad (63)$$

$$T_+ = \frac{1}{4\pi t^2 r_+}.$$

The logarithmic correction is found to be

$$\mathcal{S}_+ = \mathcal{S}_{0,+} - \frac{1}{2} \ln \left| \frac{1}{16\pi} \right| + \dots \quad (64)$$

The interesting fact in this case is that we have found that the entropy correction term in the logarithmic correction term is mass-independent, whereas without logarithmic term it is mass-dependent.

Example 13 (charged BHs in $f(R)$ gravity). The $f(R)$ gravity [40, 41] is interesting because it is equivalent to Einstein gravity coupled to matter, where $f(R)$ is an arbitrary function of the scalar curvature. The horizon function at the constant scalar curvature $R = R_0$ is given by

$$\mathcal{N}(r) = 1 - \frac{2\mu}{r} + \frac{q^2}{\alpha r^2} - \frac{R_0 r^2}{12} = 0, \quad (65)$$

where $\alpha = 1 + f'(R_0)$. The quantities μ and q are related to M (ADM mass) and Q (electric charge) which are

$$M = \mu\alpha, \quad (66)$$

$$Q = \frac{q}{\sqrt{2\alpha}}.$$

The entropy for all the horizons is

$$\mathcal{S}_{0,i} = \pi \alpha r_i^2, \quad (67)$$

and the BH temperature should read

$$T_i = \frac{1}{4\pi r_i} \left(1 - \frac{q^2}{\alpha r_i^2} - \frac{R_0 r_i^2}{4} \right). \quad (68)$$

The logarithmic correction of entropy becomes

$$\mathcal{S}_i = \mathcal{S}_{0,i} - \frac{1}{2} \ln \left| \frac{\alpha \left(1 - \frac{q^2}{\alpha r_i^2} - \frac{R_0}{4} r_i^2 \right)^2}{16\pi r_i^2} \right| + \dots \quad (69)$$

Example 14 (5D Gauss-Bonnet BH). The horizon radii for 5D Gauss-Bonnet BH [41] are located at

$$r_\pm = \frac{1}{\sqrt{2}} \sqrt{(2\mu - \alpha_d) \pm \sqrt{(2\mu - \alpha_d)^2 - 4q^2}}, \quad (70)$$

where $\mu = 4M/3\pi$ and $q = (4/\pi)^{2/3} Q$.

The entropy of \mathcal{H}^\pm is

$$\mathcal{S}_{0,\pm} = \frac{\pi^2 r_\pm^3}{2} \left(1 + \frac{6\alpha_d}{r_\pm^2} \right). \quad (71)$$

The BH temperature of \mathcal{H}^\pm reads

$$T_\pm = \frac{r_\pm^2 - r_\mp^2}{2\pi r_\pm (r_\pm^2 + 2\alpha_d)}, \quad (72)$$

where $\alpha_d = (d-3)(d-4)\alpha = 2\alpha$ and α_d is Gauss-Bonnet coupling constant.

The logarithmic correction of entropy \mathcal{H}^\pm becomes

$$\mathcal{S}_\pm = \mathcal{S}_{0,\pm} - \frac{1}{2} \ln \left| \frac{r_\pm (1 + 6\alpha_d/r_\pm^2) (r_\pm^2 - r_\mp^2)^2}{8 (r_\pm^2 + 2\alpha_d)^2} \right| + \dots \quad (73)$$

It follows from the several examples that when the logarithmic correction is considered the entropy product formula is *not mass-independent (universal)* and therefore it is *not quantized*.

To sum up, we computed the general logarithmic corrections to the BH entropy product formula of inner horizon and outer horizon by taking into consideration the effects of statistical quantum fluctuations around the thermal equilibrium and also via CFT. We showed, followed by our earlier work [11], that whenever we take the first-order logarithmic correction to the entropy product formula, it is not universal and also it cannot be quantized. What is new in this work is that when we have chosen the exact entropy function followed by CFT and by taking the effects of quantum fluctuations, the logarithmic correction formula of \mathcal{H}^\pm should depend solely on the value of BH temperature of \mathcal{H}^\pm and BH entropy of \mathcal{H}^\pm at the thermal equilibrium.

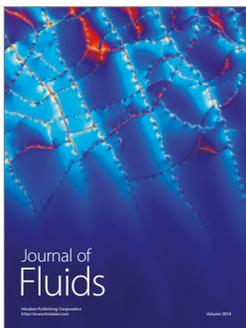
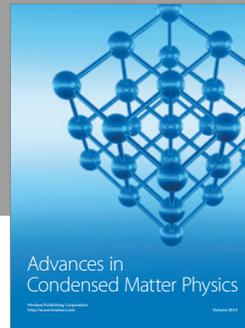
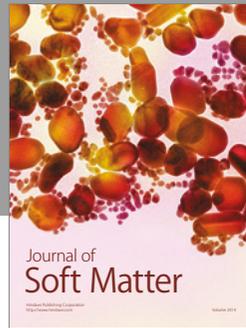
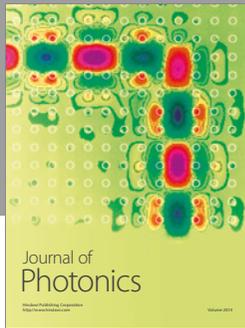
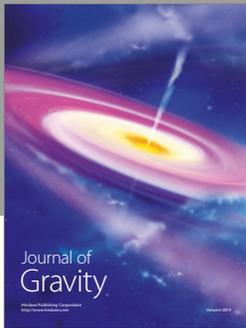
Conflicts of Interest

The author declares that they have no conflicts of interest.

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