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Research Article

Entropy Product Formula for Gravitational Instanton

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We investigate the entropy product formula for various gravitational instantons. We speculate that due to the mass-independent features of the said instatons they are *universal* as well as *quantized*. For isolated Euclidean Schwarzschild black hole, these properties simply *fail*.

1. Introduction

There has been a strong interest in microscopic interpretation of black hole (BH) entropy [1–8] in terms of D-branes due to the work by Strominger and Vafa [9]. In d-dimension Euclidean quantum gravity, this entropy is due to the $(d-2)$-dimensional fixed point sets of the imaginary time translation Killing vector. There are many fixed point sets which can also give rise to BH entropy.

Previously, Area (or Entropy) product formula was evaluated for different class of BHs [10–28]. In some cases, the product formula is not mass-independent (universal) and in some cases the product formula is indeed mass-independent, that is, universal. To the best of the author's knowledge, there has been no attempt to compute the entropy product formula for *gravitational instanton*.

Thus, in the present work, we wish to investigate the entropy product formula for various gravitational instantons. Instantons are nonsingular and having imaginary time. They arise in quantum field theory (QFT) for evaluating the functional integral, in which the functional integral is Wick rotated and expressed as an integral over Euclidean field configurations. They are the solutions of the Euclidean Einstein equations. They have the signature of the form (++++). There are two types of instantons discovered so far. One is an asymptotically locally flat (ALF), which was first discovered by Hawking in 1977 [29] and the other is an asymptotically locally Euclidean (ALE) instanton, discovered by Gibbons and Hawking in 1979 [30]. The examples of ALE classes are Flat space, Eguchi-Hanson [31], and multi-instanton [30].

The ALF class of solutions are asymptotically flat (AF) in 3D sense, the fourth, imaginary time, direction being periodic. The surfaces of large radii should be thought of as an $\mathcal{S}^1$ bundle over $\mathcal{S}^2$. The product bundle corresponds to the AF solutions which include the Euclidean Schwarzschild and Euclidean Kerr solutions [32, 33]. The twisted bundles correspond to the multi-Taub-NUT solution [29], and the Taub-Bolt solution was first discovered by Page [34].

In our previous work [18, 27, 28], we investigated the properties of inner and outer horizon thermodynamics of Taub-NUT (Newman-Unti-Tamburino) BH, Kerr-Taub-NUT BH, and Kerr-Newman-Taub-NUT BH in four-dimensional Lorentzian geometry. The failure of *first law* of BH thermodynamics and *Smarr-Gibbs-Duhem relation for Taub-NUT and Kerr-Taub-NUT BH in the Lorentzian regime gives the motivation behind this work*. What happens when one can go from *Lorentzian geometry to Riemannian geometry?* This is the prime aim in this work. By studying the properties of these instantons what should be the effects on the BH entropy product formula due to the nontrivial NUT parameter?

In general relativity, the nontrivial value for the BH entropy is due to the presence of the fixed point set of the periodic imaginary time Killing vector. The fixed point set we considered here actually is the BH horizons ($\mathcal{H}^{\pm}$). Here $\mathcal{H}^{+}$ is called event horizon and $\mathcal{H}^{-}$ is called the Cauchy horizon. In four-dimensional space-time, such fixed point sets are of two types, isolated points or zero-dimensional which we call NUTs and two surfaces or two-dimensional which we call Bolts. Thus one can think of Bolts as being the analogue of

electric type mass-monopoles and the NUTs as being gravitational dyons endowed with a real electric type mass-monopole and an imaginary magnetic type mass-monopole. The presence of magnetic type mass introduces a Dirac string like singularity in the space-time, the so called Misner string, which was first pointed out by Misner in his paper for the Lorentzian Taub-NUT space-time [35]. A Misner string is a coordinate singularity which can be considered as a manifestation of a "nontrivial topological twisting" [36] of the manifold (M, g_{ab}). This twist is parametrized by a topological term, the NUT charge.

In our previous investigation [18], we have taken the metric in a Lorentzian space-time in 3 + 1 split form as

$$ds^2 = -\mathcal{F}\left(dt + w_i dx^i\right)^2 + \frac{\gamma_{ij}}{\mathcal{F}} dx^i dx^j. \tag{1}$$

In this work, we are interested in studying the metric in a *Riemannian* space-time which can be written in 3 + 1 split form:

$$ds^2 = \mathcal{F}\left(d\tau + w_i dx^i\right)^2 + \frac{\gamma_{ij}}{\mathcal{F}} dx^i dx^j. \tag{2}$$

Here all quantities are independent of t or τ. The Wick rotation that transforms from one case to the other is by the transformation $t \mapsto i\tau$ and $w_i \mapsto i\omega_i$. $\mathcal{F}$ can be thought of as an electric type potential, and ω_i or w_i as a magnetic type vector potential. The associated magnetic field is $H_{ij} = \partial_i \omega_j - \partial_j \omega_i$ and it is gauge invariant.

This should be used to define a magnetic monopole moment called the NUT charge n [30]. If $n \neq 0$, the fibration should not be trivial. In the Lorentzian geometry, these fixed point sets are the two-dimensional Boyer bifurcation sets of event horizon [5, 37]. On the other hand, in Riemannian geometry, these fixed point sets are of two types: zero-dimensional point or NUTs and two-dimensional surfaces or Bolts [30]. A NUT possesses a pair of surface gravities κ_1 and κ_2. p and q are a pair of coprime integers such that $\kappa_1/\kappa_2 = p/q$. If κ_1/κ_2 is irrational, $p = q = 1$. A NUT of type (p, q) has a NUT charge of $n = \beta/8\pi pq$, where β is the period of the imaginary time coordinate. Moreover, $n = Y\beta/8\pi$ for a Bolt of self-intersection number Y. It should be noted that, in the Riemannian case, the number of NUTs and Bolts is related to the Euler number χ and the Hirzebruch signature τ of the manifold M by

$$\chi = \sum_{\text{Bolts}} \chi_i + \sum_{\text{NUTs}} 1, \tag{3}$$

where χ_i is the Euler number for the ith Bolt, and

$$\tau = \sum_{\text{Bolts}} Y_i \csc^2\theta - \sum_{\text{NUTs}} \cot p_i\theta \cot q_i\theta + \eta(0,\theta). \tag{4}$$

Equation (4) is valid for arbitrary θ. Y_i is the self-intersection number of the ith Bolt, the ith NUT is of type (p_i, q_i), and $\eta(0,\theta)$ is a correction term which depends solely on the boundary. It should be mentioned that for multi-Taub-NUT and multi-instanton solutions there are k NUTs of type (1, 1) and $\tau = k - 1$; therefore

$$\tau = \sum_{\text{Bolts}} Y_i \csc^2\theta - \sum_{\text{NUTs}} \cot p_i\theta \cot q_i\theta + k\csc^2\theta - 1, \tag{5}$$

where the AF boundary conditions should be $\eta(0,\theta) = 0$ [33].

The structure of the paper is as follows. In Section 2, we have considered the Euclidean Schwarzschild BH. In Section 3, we have investigated the properties of self-dual Taub-NUT instantons. In Section 4, we have studied the mass-independent properties of Taub-Bolt instantons. In Section 5, we have described the properties of Eguchi-Hanson instantons. In Section 6, we have examined the properties of Taub-NUT-AdS space-time. In Section 7, we have studied the entropy product formula for Taub-Bolt-AdS space-time and finally in Section 8, we have examined the product rules for Dyonic Taub-NUT-AdS and Taub-Bolt-AdS space-time.

2. Euclidean Schwarzschild Metric

To give a warm up, let us first consider the Schwarzschild BH (where we have used units in $G = c = 1$) in Euclidean form as

$$\begin{aligned} ds^2 = &\left(1 - \frac{2M}{r}\right) d\tau^2 + \frac{dr^2}{(1 - 2M/r)} \\ &+ r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \end{aligned} \tag{6}$$

The apparent singularity at the event horizon $r_+ = 2M$ can be removed by identifying τ with a period $\beta = 8\pi M$ [29, 33]. The radial coordinate has the range $2M \leq r \leq \infty$. Then the topology of the manifold is $R^2 \times \mathcal{S}^2$. The isometry group is $O(2) \otimes O(3)$, where $O(2)$ corresponds to translations in the periodically identified imaginary time τ and $O(3)$ corresponds to rotations of the θ and ϕ coordinates.

The Killing vector ∂_τ has unit magnitude at large radius and has a Bolt on the horizon $r_+ = 2M$ which is a 2-sphere $\mathcal{S}^2$ of area

$$A_+ = 16\pi M^2. \tag{7}$$

The surface gravity is given by

$$\kappa_+ = \frac{2\pi}{\beta} = \frac{1}{4M}, \tag{8}$$

and the BH temperature is

$$T_+ = \frac{\kappa_+}{2\pi} = \frac{1}{8\pi M}. \tag{9}$$

Thus for an isolated Euclidean Schwarzschild BH the area product becomes

$$A_+ = 16\pi M^2, \tag{10}$$

which tells us that the product is dependent on mass parameter and thus it is not universal. Also it is not quantized.

The Euclidean action derived in [29] is

$$I = -\ln Z = 4\pi M^2. \tag{11}$$

From that one can derive the entropy as in [29]

$$S_+ = -\left(\beta\frac{\partial}{\partial\beta} - 1\right)\ln Z = 4\pi M^2. \tag{12}$$

Thus the entropy product for isolated Euclidean Schwarzschild BH should be

$$S_+ = 4\pi M^2. \tag{13}$$

Indeed, it is not universal and it is not quantized.

3. Self-Dual Taub-NUT Instantons

In this section we shall calculate the entropy product and area product of four-dimensional Taub-NUT space-time. It is of ALF. ALF metrics have a NUT charge, or magnetic type mass, n, as well as the ordinary electric type mass, M. The NUT charge is $\beta c_1/8\pi$, where c_1 is the first Chern number of the $U(1)$ bundle over the sphere at infinity, in the orbit space Ξ. If $c_1 = 0$, then the boundary at infinity is $\mathcal{S}^1 \times \mathcal{S}^2$ and the space-time is AF. The BH metrics are saddle points in the path integral for the partition function. Thus, if $c_1 \neq 0$, the boundary at infinity is a squashed $\mathcal{S}^3$, and the metric should not be analytically continued to a Lorentzian metric. The squashed $\mathcal{S}^3$ is the three-dimensional space on which the boundary conformal field theory (CFT) will be compactified, with β identified with the inverse temperature; that is, $T = 1/\beta$.

Hawking [29] first given example of gravitational instanton was the self-dual Taub-NUT metric described by

$$\begin{aligned} ds^2 &= \mathcal{F}(r)\left(d\tau + 2n\cos\theta d\phi\right)^2 + \frac{dr^2}{\mathcal{F}(r)} \\ &\quad + \left(r^2 - n^2\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \\ \mathcal{F}(r) &= \frac{r-n}{r+n}. \end{aligned} \tag{14}$$

It is ALF with a central NUT. The self-dual Taub-NUT instanton has $M = n$ and the anti-self-dual instanton has $M = -n$. The value $r = n$ is now a zero in $\mathcal{F}(r)$. The (θ, ϕ) two-sphere has a zero area at $r = n$, so the zero in $\mathcal{F}(r)$ is a zero-dimensional fixed point of ∂_ϕ, a NUT.

In order to make the solution regular, we take the region $r \geq n$ and let the period of τ be $8\pi n$. The metric has a NUT at $r = n$, with a Misner string running along the z-axis from the NUT out to infinity; that is, $n \leq r \leq \infty$.

We know from the idea of path integral formulation of quantum gravity that the Euclidean action is derived in [29]

$$I = -\ln Z = 4\pi n^2, \tag{15}$$

where Z is the partition function of an ensemble

$$Z = \int [Dg]\,[D\phi]\, e^{-I(g,\phi)}, \tag{16}$$

with the path integral taken over all metrics g and matter field ϕ that are appropriately identified with the period β of τ. Therefore the entropy should be derived as

$$S = -\left(\beta\frac{\partial}{\partial\beta} - 1\right)\ln Z = 4\pi n^2. \tag{17}$$

It is indeed mass-independent and thus it is universal.

The surface gravity is calculated to be

$$\kappa = \frac{2\pi}{\beta} = \frac{1}{4n}. \tag{18}$$

Thus the BH temperature should be read off

$$T = \frac{\kappa}{2\pi} = \frac{1}{8\pi n}. \tag{19}$$

Now we see what happens in the above results for other instantons that are Taub-Bolt.

4. Taub-Bolt Instantons

The Taub-Bolt instanton is described by the metric [34]

$$\begin{aligned} ds^2 &= \mathcal{G}(r)\left(d\tau + 2n\cos\theta d\phi\right)^2 + \frac{dr^2}{\mathcal{G}(r)} \\ &\quad + \left(r^2 - n^2\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \\ \mathcal{G}(r) &= \frac{(r-2n)(r-n/2)}{(r^2-n^2)} = \frac{(r-r_+)(r-r_-)}{(r+n)(r-n)}. \end{aligned} \tag{20}$$

It is a non-self-dual, noncompact solution of the vacuum Euclidean Einstein equations. In order to make the solution regular we are restricted to the region $r = r_+ \geq 2n$, and the Euclidean time has period $\beta = 8\pi n$. Asymptotically, the Taub-Bolt instanton behaves in a similar manner to the Taub-NUT, so it is ALF. Since we are setting the fixed point at $r = r_+ = 2n$, therefore the area of the $\mathcal{S}^2$ does not vanish there and the fixed point set is 2-dimensional; thus it is a Bolt of area

$$A_+ = 12\pi n^2. \tag{21}$$

Thus the area product for Taub-Bolt instanton will be

$$A_+ = 12\pi n^2. \tag{22}$$

Thus the area product is independent of mass and also quantized. Similarly, the action was calculated in [38]

$$I = -\ln Z = \pi n^2. \tag{23}$$

Thus the entropy was derived by the universal formula

$$S_+ = -\left(\beta\frac{\partial}{\partial\beta} - 1\right)\ln Z = \pi n^2. \tag{24}$$

It indicates that the entropy product should be universal and quantized.

5. Eguchi-Hanson Instantons

A noncompact instanton which is a limiting case of the Taub-NUT solution is the Eguchi-Hanson metric [31],

$$\begin{aligned} ds^2 &= \left(1 - \frac{n^4}{r^4}\right)\left(\frac{r}{8n}\right)^2\left(d\tau + 4n\cos\theta d\phi\right)^2 \\ &\quad + \frac{dr^2}{\left(1 - n^4/r^4\right)} + \frac{r^2}{4}\left(d\theta^2 + \sin^2\theta d\phi^2\right). \end{aligned} \tag{25}$$

The instanton is regular if we consider the region $r \geq n$, and let τ has period $8\pi n$. The metric is ALE type. There is a Bolt of area at $r = n$ given by

$$A = \pi n^2, \tag{26}$$

which gives rise to a Misner string along the z-axis. Thus the product is universal and should be quantized. The Euclidean action is derived in [38]

$$I = 0. \tag{27}$$

Thus entropy corresponds to

$$S = \left(\beta \frac{\partial}{\partial \beta} - 1\right) I = 0. \tag{28}$$

We now turn our attention to the Taub-NUT and Taub-Bolt geometries in four-dimensional locally AdS space-time. The spacetimes have a global nontrivial topology due to the fact that one of the Killing vectors has a zero-dimensional fixed point set called NUT or a two-dimensional fixed point set called Bolt. Moreover, these four-dimensional spacetimes have Euclidean sections which can not be exactly matched to AdS space-time at infinity.

6. Taub-NUT-AdS Space-Time

In this section we shall consider the space-time which is only locally asymptotically AdS and with nontrivial topology. The metric on the Euclidean section of this family of solutions could be written as [39, 40]

$$\begin{aligned} ds^2 = \mathscr{H}(r)\left(d\tau + 2n\cos\theta d\phi\right)^2 + \frac{dr^2}{\mathscr{H}(r)} \\ + \left(r^2 - n^2\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \end{aligned} \tag{29}$$

where

$$\mathscr{H}(r) = \frac{\left(r^2 + n^2\right) - 2Mr + \ell^{-2}\left(r^4 - 6n^2r^2 - 3n^4\right)}{r^2 - n^2} \tag{30}$$

and $\ell^2 = -3/\Lambda$, with $\Lambda < 0$ being the cosmological constant. Here M is a (generalized) mass parameter and r is a radial coordinate. Also, τ, the analytically continued time, that is, Euclidean time, parametrizes a circle $\mathcal{S}^1$, which is fibered over the two-sphere $\mathcal{S}^2$, with coordinates θ, ϕ. The nontrivial fibration is a consequence of a nonvanishing NUT parameter n.

There are some restrictions [41] for existence of a regular NUT parameter. Firstly, in order to ensure that the fixed point set is zero-dimensional, it is necessary that the Killing vector ∂_τ has a fixed point which occurs precisely when the area of the two-sphere is zero in size. Secondly, in order for the Dirac-Misner string [35] to be unobservable, it is necessary that the period of τ be $\beta = 8\pi n$. To avoid the conical singularity, we must check $\mathscr{H}'(r_+ = n) = 1/2n$. Thirdly, the mass parameter M must be $M = n - 4n^3/\ell^2$. After simplifying the metric coefficients, we obtain

$$\mathscr{H}(r) = \left(\frac{r-n}{r+n}\right)\left[1 + \frac{(r-n)(r+3n)}{\ell^2}\right] \tag{31}$$

and the range of the radial coordinate becomes $n \leq r \leq \infty$. For our requirement, the Euclidean action for this space-time was calculated in [42, 43]

$$I = -\ln Z = 4\pi n^2\left(1 - \frac{2n^2}{\ell^2}\right), \tag{32}$$

and the entropy will be

$$S_+ = \left(\beta \frac{\partial}{\partial \beta} - 1\right) I = 4\pi n^2\left(1 - \frac{6n^2}{\ell^2}\right). \tag{33}$$

Thus the entropy product should be

$$S_+ = 4\pi n^2\left(1 - \frac{6n^2}{\ell^2}\right). \tag{34}$$

It is independent of mass parameter and does depend on NUT parameter and cosmological constant. Thus the entropy product is universal for Taub-NUT-AdS space-time. The Hawking temperature $T_+ = 1/8\pi n$ is the same as Taub-NUT BH. The first law of thermodynamics is also satisfied as $dM = T_+ dS$.

If we consider the *extended phase space* following our previous work [24] and in this framework, the cosmological constant (Λ) should be treated as thermodynamical pressure, that is, $P = -\Lambda/8\pi = 3/8\pi\ell^2$, and its conjugate variable should be treated as thermodynamic volume, that is, $V_+ = (4/3)\pi r_+^3$, where r_+ is the horizon radius. Then one should interpret the ADM mass M parameter not to be the energy; rather it should be interpreted as enthalpy $H = M = U + PV_+$ of the gravitational thermodynamical system. Therefore the thermodynamic volume has been calculated in [44] for Taub-NUT-AdS space-time:

$$V_+ = \left(\frac{\partial H}{\partial P}\right)_S = -\frac{8}{3}\pi n^3. \tag{35}$$

One aspect is that this is a peculiar result in a sense that the thermodynamic volume is negative and the other aspect is that the thermodynamic volume is *universal* because it is independent of the ADM mass parameter. It should be noted that the first law is fulfilled in this case and it yields

$$dH = T_+ d\mathcal{S} + V_+ dP. \tag{36}$$

Analogously, the Smarr-Gibbs-Duhem relation should be

$$H = 2T_+\mathcal{S} - 2PV_+, \tag{37}$$

and in another interesting result we first claimed that the internal energy for Taub-NUT-AdS BH is *universal*. It is given by

$$U = n\left(1 - 8\pi P n^2\right). \tag{38}$$

7. Taub-Bolt-AdS Space-Time

For Taub-Bolt-AdS, the metric has the same form as in (29) but the fixed point set here is two-dimensional or Bolt

and with additional restrictions the metric coefficients $\mathscr{H}(r)$ vanish at $r = r_b > n$. In order to have a regular Bolt at $r = r_b$, the following conditions must be satisfied: (i) $\mathscr{H}(r_b) = 0$, (ii) $\mathscr{H}'(r_b) = 1/2n$, and the numerator of $\mathscr{H}(r)$ at $r = r_b$ being a single one. From condition (i), we get the mass parameter at $r = r_b$:

$$M = M_b = \frac{r_b^2 + n^2}{2r_b} + \frac{1}{2\ell^2}\left(r_b^3 - 6n^2 r_b - 3\frac{n^4}{r_b}\right). \quad (39)$$

Then we find [41]

$$\mathscr{H}'(r_b) = \frac{3}{\ell^2}\left(\frac{r_b^2 - n^2 + \ell^2/3}{r_b}\right). \quad (40)$$

To satisfy condition (ii) we must have the quadratic equation for r_b:

$$6nr_b^2 - \ell^2 r_b - 6n^3 + 2n\ell^2 = 0, \quad (41)$$

which gives the solution for r_b in two branches:

$$r_{b\pm} = \frac{\ell^2}{12n}\left(1 \pm \sqrt{1 - 48\frac{n^2}{\ell^2} + 144\frac{n^4}{\ell^4}}\right). \quad (42)$$

The discriminant of the above equation must be negative for r_b to be real and for $r_b > n$ we obtain the following inequality for n:

$$n \le n_{\max} = \sqrt{\frac{1}{6} - \frac{\sqrt{3}}{12}}\,\ell. \quad (43)$$

The Euclidean action was computed in [43]

$$I = \frac{4\pi n}{\ell^2}\left(M_b \ell^2 + 3n^2 r_b - r_b^3\right). \quad (44)$$

Now the entropy via the universal entropy formula can be easily derived:

$$S_+ = \left(\beta\frac{\partial}{\partial\beta} - 1\right) I = 4\pi n\left(M_b - \frac{3n^2 r_b}{\ell^2} + \frac{r_b^3}{\ell^2}\right). \quad (45)$$

Now substituting the values of M_b, we find the value of entropy

$$S_+ = 4\pi n\left[\frac{r_b^2 + n^2}{2r_b} + \frac{1}{2\ell^2}\left(3r_b^3 - 12n^2 r_b - 3\frac{n^4}{r_b}\right)\right]. \quad (46)$$

Again, putting the values of r_b, we see that the entropy is universal as well as quantized.

8. Dyonic Taub-NUT-AdS and Taub-Bolt-AdS Space-Time

The general form of the metric for dyonic Taub-NUT-AdS space-time [45–47] is given by

$$\begin{aligned} ds^2 = {} & \mathcal{N}(r)\left(d\tau + 2n\cos\theta d\phi\right)^2 + \frac{dr^2}{\mathcal{N}(r)} \\ & + \left(r^2 - n^2\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \end{aligned} \quad (47)$$

where

$$\begin{aligned} & \mathcal{N}(r) \\ & = \frac{\left(r^2 + n^2 + 4n^2\nu^2 - q^2\right) - 2Mr + \ell^{-2}\left(r^4 - 6n^2 r^2 - 3n^4\right)}{r^2 - n^2}. \end{aligned} \quad (48)$$

The gauge field reads off

$$\begin{aligned} A & \equiv A_\mu dx^\mu \\ & = \left(\frac{qr}{r^2 - n^2} + \nu\frac{r^2 + n^2}{r^2 - n^2}\right)\left(d\tau - 2n\cos\theta d\phi\right). \end{aligned} \quad (49)$$

The conditions of smoothness of the Euclidean section implies that the parameter q is related to the parameter ν which gives a deformation from the uncharged system. When these parameters go to zero value, we obtain simply Taub-NUT-AdS space-time.

In order to have a regular position of NUT or Bolt at $r = r_\pm$, we set $\mathcal{N}(r) = 0$ and also the gauge field A must be regular at that point. Thus we obtain the mass parameter as

$$\begin{aligned} M & = \frac{r_\pm^2 + n^2 + 4n^2\nu^2 - \nu^2}{2r_\pm} \\ & \quad + \frac{1}{2\ell^2}\left(r_\pm^3 - 6n^2 r_\pm - 3\frac{n^4}{r_\pm}\right), \\ q & = -\frac{r_\pm^2 + n^2}{r_\pm}\nu. \end{aligned} \quad (50)$$

The electric charge and potential at infinity correspond to

$$\begin{aligned} Q & = q, \\ \phi_\pm & = -q\frac{r_\pm}{r_\pm^2 + n^2} = \nu. \end{aligned} \quad (51)$$

Now the Euclidean action for the above space-time was calculated in [48] (in units where $G = c = 1$)

$$\begin{aligned} & I_\pm = -2\pi \\ & \cdot \frac{\left[r_\pm^4 - \ell^2 r_\pm^2 + n^2\left(3n^2 - \ell^2\right)\right] r_\pm^2 - \left(r_\pm^4 + 4n^2 r_\pm^2 - n^4\right)\ell^2\nu^2}{\left(3r_\pm^2 - 3n^2 + \ell^2\right) r_\pm^2 + \left(r_\pm^2 - n^2\right)\ell^2\nu^2}. \end{aligned} \quad (52)$$

The entropy was calculated as

$$S_\pm = 2\pi\frac{\left[3r_\pm^4 + \left(\ell^2 - 12n^2\right) r_\pm^2 + n^2\left(\ell^2 - 3n^2\right)\right] r_\pm^2 + \left(r_\pm^4 + 4n^2 r_\pm^2 - n^4\right)\ell^2\nu^2}{\left(3r_\pm^2 - 3n^2 + \ell^2\right) r_\pm^2 + \left(r_\pm^2 - n^2\right)\ell^2\nu^2}. \quad (53)$$

When we set $r_\pm = r_n = n$, we get a dyonic NUT spacetime. For this spacetime the above calculations are reduced to

$$\begin{aligned} M &= n - \frac{4n^3}{\ell^2}, \\ Q &= -2n\nu, \\ \phi_\pm &= \nu, \\ I_\pm &= 4\pi n^2 \left(1 - 2\frac{n^2}{\ell^2} + 2\nu^2\right). \end{aligned} \tag{54}$$

Finally, the entropy is given by

$$S_\pm = 4\pi n^2 \left(1 - 6\frac{n^2}{\ell^2} + 2\nu^2\right). \tag{55}$$

Thus the entropy product formula for dyonic Taub-NUT is

$$S_+ S_- = S_+^2 = S_-^2 = \left(4\pi n^2\right)^2 \left(1 - 6\frac{n^2}{\ell^2} + 2\nu^2\right)^2. \tag{56}$$

The product formula is indeed independent of mass and depends on n, ℓ, and ν.

For a dyonic Bolt, we set $r_\pm = r_b$ and satisfy the fourth-order equation for r_b:

$$6nr_b^4 - \ell^2 r_b^3 + 2n\left(\ell^2 - 3n^2 + \ell^2\nu^2\right) r_b^2 - 2\ell^2\nu^2 n^3 = 0. \tag{57}$$

Let $r_b = r_{b\pm}$ be the solution of the equation. Then we find the action as before:

$$\begin{aligned} I_\pm &= -2\pi \\ &\cdot \frac{\left[r_b^4 - \ell^2 r_b^2 + n^2\left(3n^2 - \ell^2\right)\right] r_b^2 - \left(r_b^4 + 4n^2 r_b^2 - n^4\right)\ell^2\nu^2}{\left(3r_b^2 - 3n^2 + \ell^2\right) r_b^2 + \left(r_b^2 - n^2\right)\ell^2\nu^2}. \end{aligned} \tag{58}$$

Similarly the entropy is given by

$$S_\pm = 2\pi \frac{\left[3r_b^4 + \left(\ell^2 - 12n^2\right) r_b^2 + n^2\left(\ell^2 - 3n^2\right)\right] r_b^2 + \left(r_b^4 + 4n^2 r_b^2 - n^4\right)\ell^2\nu^2}{\left(3r_b^2 - 3n^2 + \ell^2\right) r_b^2 + \left(r_b^2 - n^2\right)\ell^2\nu^2}. \tag{59}$$

After substituting the value of $r_{b\pm}$ in the entropy product formula, it seems that the product is independent of mass and depends on n, ℓ, ν for dyonic Taub-Bolt instanton.

9. Conclusion

We have studied the mass-independent feature for various gravitational instantons. This universal feature gives us strong indication towards understanding the microscopic properties of BH entropy. It would be interesting if one considered the entropy product formula for other instantons like multi-Taub-NUT, non-self-dual Taub-NUT, S^4, CP^2, $S^2 \times S^2$, and twisted $S^2 \times S^2$. We expect these instantons also to give us universal features.

Conflicts of Interest

The author declares that they have no conflicts of interest.

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