Research Article

Heavy Sterile Neutrino in Dark Matter Searches

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1. Introduction

There exists evidence for existence of dark matter in almost all scales, from the dwarf galaxies, galaxies, and cluster of galaxies, with the most important ones being the observed rotational curves in the galactic halos; see, for example, the review [1]. Furthermore cosmological observations have provided plenty of additional evidence, especially the recent WMAP [2] and Planck [3] data.

In spite of this plethora of evidence, it is clearly essential to directly detect such matter in the laboratory in order to unravel its nature. At present there exist many such candidates, called Weakly Interacting Massive Particles (WIMPs). Some examples are the LSP (Lightest Supersymmetric Particle) [4–11], technibaryon [12, 13], mirror matter [14, 15], and Kaluza-Klein models with universal extra dimensions [16, 17]. Among other things these models predict an interaction of dark matter with ordinary matter via the exchange of a scalar particle, which leads to a spin independent interaction (SI) or vector boson interaction and therefore to a spin dependent (SD) nucleon cross section.

Since the WIMPs are expected to be extremely nonrelativistic, with average kinetic energy \( \langle T \rangle \approx 50 \text{ keV}(m_{\text{WIMP}}/100 \text{ GeV}) \), they are not likely to excite the nucleus, even if they are quite massive, \( m_{\text{WIMP}} > 100 \text{ GeV} \). Therefore they can be directly detected mainly via the recoiling of a nucleus, first proposed more than 30 years ago [18]. There exists a plethora of direct dark matter experiments with the task of detecting WIMP event rates for a variety of targets such as those employed in XENON10 [19], XENON100 [20], XMASS [21], ZEPLIN [22], PANDA-X [23], LUX [24], CDMS [25], CoGENT [26], EDELWEISS [27], DAMA [28, 29], KIMS [30], and PICASSO [31, 32]. These consider dark matter candidates in the multi-GeV region.

Recently, however, an important dark matter particle candidate of the Fermion variety in the mass range of \( 10–100 \text{ keV} \), obtained from galactic observables, has arisen [33–35]. This scenario produces basically the same behavior in the power spectrum (down to Mpc scales) with that of standard ΛCDM cosmologies, by providing the expected large-scale structure [36]. In addition, it is not too warm; that is, the masses involved are larger than \( m = 1–3 \text{ keV} \) to be in conflict with the current Lyα forest constraints [37] and the number of Milky Way satellites [38], as in standard AWDM cosmologies. In fact an interesting viable candidate has been suggested, namely, a sterile neutrino in the mass region of \( 48–300 \text{ keV} \) [33–35, 39–43], but most likely around \( 50 \text{ keV} \). For a recent
The existence of light sterile neutrinos had already been introduced to explain some experimental anomalies like those claimed in the short baseline LSND and MiniBooNE experiments [45–47], the reactor neutrino deficit [48], and the Gallium anomaly [49, 50], with possible interpretations discussed, for example, in [51, 52] as well as in [53, 54] for sterile neutrinos in the keV region. The existence of light neutrinos can be expected in an extended see-saw mechanism involving a suitable neutrino mass matrix containing a number of neutrino singlets not all of which being very heavy. In such models it is not difficult to generate more than one sterile neutrino, which can couple to the standard neutrinos [55].

As it has already been mentioned, however, the explanation of cosmological observations requires sterile neutrinos in the 50 keV region, which can be achieved in various models [33, 56].

In the present paper we will examine possible direct detection possibilities for the direct detection of these sterile neutrinos. Even though these neutrinos are quite heavy, their detection is not easy. Since like all dark matters candidates move in our galaxies with non-relativistic velocities, with average value about $10^{-3}$ c, and with energies about 0.05 eV, not all of them can be deposited in the detectors. Therefore the standard detection techniques employed in the standard dark matter experiments like those mentioned above are not applicable in this case. Furthermore, the size of the mixing parameter of sterile neutrinos with ordinary neutrinos is crucial for detecting sterile neutrinos. Thus our results concerning the expected event rates will be given in terms of this parameter.

The paper is organized as follows. In Section 2 we study the option on neutrino-electron scattering. In Section 3 we consider the case of low temperature bolometers. In Section 4 the possibility of neutrino induced atomic excitations is explored. In Section 5 we will consider the antineutrino absorption on nuclei, which normally undergo electron capture, and finally in Section 6 the modification of the end point electron energy in beta decay, for example, in the KATRIN experiment [57], is discussed. In Section 7, we summarize our conclusions.

## 2. The Neutrino-Electron Scattering

The sterile neutrino as dark matter candidate can be treated in the framework of the usual dark matter searches for light WIMPs except that its mass is very small. Its velocity follows a Maxwell-Boltzmann (MB) distribution with a characteristic velocity about $10^{-3}$ c. Since the sterile neutrino couples to the ordinary electron neutrino it can be detected in neutrino-electron scattering experiments with the advantage that the neutrino-electron cross section is very well known. Both the neutrino and the electron can be treated as nonrelativistic particles. Furthermore, we will assume that the electrons are free, since the WIMP energy is not adequate to ionize an atom. Thus the differential cross section is given by

$$
d\sigma = \frac{1}{v} C^2_v \left( g^2_v + g^2_\lambda \right) \left( \frac{G_F}{2\sqrt{2}} \right)^2 \frac{d^3 p'_e \cdot d^3 p_e}{(2\pi)^3} \delta \left( p'_e - p'_\nu - p_e \right) \delta \left( \frac{p^2_e}{2m_e} - \frac{p^2'_e}{2m_e} - \frac{p^2}{2m_\nu} \right),$$

(1)

where $C^2_v$ is the square of the mixing of the sterile neutrino with the standard electron neutrino $\nu_e$ and $G_F = G \cos\theta_c$, where $G = 1.1664 \times 10^{-5}$ GeV$^{-2}$ denotes the Fermi weak coupling constant and $\theta_c = 13^\circ$ is the Cabibbo angle [58].

The integration over the outgoing neutrino momentum is trivial due to the momentum $\delta$ function yielding

$$
d\sigma = \frac{1}{v} C^2_v \left( g^2_v + g^2_\lambda \right) \left( \frac{G_F}{2\sqrt{2}} \right)^2 \frac{1}{(2\pi)^2} \cdot d^3 p'_e \delta \left( p'_e - \frac{p^2}{2m_\nu} \right),$$

(2)

where $\xi = \frac{p'_e \cdot \hat{p}_e}{\mu_e}, 0 \leq \xi \leq 1, v$ is the WIMP velocity, and $\mu_e$ is the WIMP-electron reduced mass, $\mu_e \approx m_e$. The electron energy $T$ is given by

$$
T = \frac{p^2_e}{2m_e} = 2 \frac{m^2_e}{m_e} (v\xi)^2 \Rightarrow 0 \leq T \leq 2 \frac{m^2_e}{m_e} v^2_{\text{esc}},
$$

(3)

where $v_{\text{esc}}$ is the maximum WIMP velocity (escape velocity). Integrating (2) over the angles, using the $\delta$ function for the $\xi$ integration we obtain

$$
d\sigma = \frac{1}{v} C^2_v \left( g^2_v + g^2_\lambda \right) \left( \frac{G_F}{2\sqrt{2}} \right)^2 \frac{1}{2\pi} \frac{p^2}{2m_\nu} d\xi \frac{1}{|p_e v|} \Rightarrow
$$

$$
d\sigma = \frac{1}{v} C^2_v \left( g^2_v + g^2_\lambda \right) \left( \frac{G_F}{2\sqrt{2}} \right)^2 \frac{1}{2\pi} \frac{m_\nu}{m_e} dT.
$$

(4)

We are now in a position to fold the velocity distribution assuming it to be MB with respect to the galactic center:

$$
f (v') = \frac{1}{(\sqrt{2\pi} v_0)} e^{-v^2/v_0^2}.
$$

(5)

In the local frame, assuming that the sun moves around the center of the galaxy with velocity $v_0 = 220$ km/s, $v' = v + v_0 \hat{z}$, we obtain

$$
f_\xi (y, \xi) = \frac{1}{(\sqrt{2\pi} v_0)} e^{-(1+y^2+2y\xi)}, \quad y = \frac{v}{v_0},
$$

(6)
Figure 1: The shape of the spectrum of the emitted electrons in sterile neutrino-electron scattering.

where $\xi$ is now the cosine of the angle between the WIMP velocity $v$ and the direction of the sun’s motion. Eventually we will need the flux so we multiply with the velocity before we integrate over the velocity. The limits of integration are between $v_{\text{min}}$ and $v_{\text{esc}}$. The velocity is given via (3); namely,

$$v = \frac{\sqrt{2m_e T}}{2m_\xi} \Rightarrow v_{\text{min}} = \frac{\sqrt{2m_e T}}{2m_\gamma}$$

(7)

We find it convenient to express the kinetic energy $T$ in units of $T_0 = 2(m_e^2/m_\gamma)v_0^2$. Then

$$y_{\text{min}} = \sqrt{x}, \quad x = \frac{T}{T_0}.$$  

(8)

Thus

$$\left\langle \frac{d\sigma}{dT} \right\rangle = \frac{1}{v_0} \int_0^{y_{\text{esc}}} \int \frac{2}{\sqrt{\pi}} e^{-(1+y^2)} \int_1^{y_{\text{esc}}} d\xi e^{-2y\xi}.$$  

(9)

These integrals can be done analytically to yield

$$\left\langle \frac{d\sigma}{dT} \right\rangle = \frac{1}{v_0} \int_0^{y_{\text{esc}}} \int C_\sigma^2 G_F^2 (g_\gamma^2 + g_A^2) \cdot g(x),$$  

(10)

where $g(x)$ characterizes the spectrum of the emitted electrons and is exhibited in Figure 1 and it is without any particular structure, which is the case in most WIMP searches. For a 50 keV sterile neutrino we find that

$$T_0 = 2 \left( \frac{m_e}{m_\nu} \right)^2 \left( \frac{2.2}{3} \right)^2 10^{-6} m_e c^2 \approx 5.0 \times 10^{-3} \text{ eV}$$

$$T_{\text{max}} = T_0 y_{\text{esc}}^2 = 5 \times 10^{-3} \times 2.84^2 = 0.04 \text{ eV}$$

$$\langle T \rangle = 1.6 T_0 = 8.0 \times 10^{-3} \text{ eV}.$$  

(11)

Now $dT = T_0 dt$. Thus

$$\frac{\langle v \sigma \rangle}{v_0} = \frac{1}{v_0} \int_0^{y_{\text{esc}}} \int C_\sigma^2 G_F^2 (g_\gamma^2 + g_A^2) \cdot g(x)$$

$$= 1.43 \frac{m_e^2}{8\pi} C_\sigma^2 G_F^2 (g_\gamma^2 + g_A^2),$$  

(12)

where

$$\int_0^{y_{\text{esc}}} \int C_\sigma^2 G_F^2 (g_\gamma^2 + g_A^2) = 1.43.$$  

(13)

It is clear that with this amount of energy transferred to the electron it is not possible to eject an electron out of the atom. One therefore must use special materials such that the electrons are loosely bound. It has recently been suggested that it is possible to detect even very light WIMPs, much lighter than the electron, utilizing Fermi-degenerate materials like superconductors [59]. In this case the energy required is essentially the gap energy of about 1.5$kT_e$, which is in the meV region; that is, the electrons are essentially free. In what follows, we assume the values

$$g_A = 1,$$

$$g_\gamma = 1 + 4 \sin^2 \theta_W = 1.92,$$

$$G_F^2 = 5.02 \times 10^{-44} \text{ cm}^2/\text{MeV}^2$$

(14)

while $C^2_\sigma$ is taken as a parameter and will be discussed in Section 7. Thus we obtain

$$\frac{\langle v \sigma \rangle}{v_0} = 3.47 \times 10^{-47} C_\sigma^2 \text{ cm}^2.$$  

(15)

The neutrino particle density is

$$N_\nu = \rho \frac{m_\nu}{m_\nu} = 0.3 \text{ GeV/cm}^3 = 6 \times 10^3 \text{ cm}^{-3}$$  

(16)

while the neutrino flux

$$\Phi_\nu = \rho v_\nu = 1.32 \times 10^{11} \text{ cm}^{-2} \text{ s}^{-1}.$$  

(17)
where \( \rho = 0.3 \text{GeV/cm}^3 \), being the dark matter density. Assuming that the number of electron pairs in the target is 2 \( \times N_A = 2 \times 10^{24} \) we find that the number of events per year is

\[
\Phi_{\nu} \frac{\nu_{\sigma}}{\nu_0} 2 \times N_A = 2.89 \times 10^{-4} \text{cm}^{-1} \text{y}^{-1}.
\]

The authors of [59] are perhaps aware of the fact that the average energy for very light WIMPs is small and as we have seen above a small portion of it is transferred to their system. With their bolometer detector these authors probably have a way to circumvent the fact that a small amount of energy will be deposited, about 0.4 eV in a year for \( N_A \approx 10^{24} \). Perhaps they may manage to accumulate a larger number of loosely bound electrons in their target.

### 3. Sterile Neutrino Detection via Low Temperature Bolometers

Another possibility is to use bolometers, like the CUORE detector exploiting Low Temperature Specific Heat of Crystalline \( ^{130}\text{TeO}_2 \) at low temperatures. The energy of the WIMP will now be deposited in the crystal, after its interaction with the nuclei via Z-exchange. In this case the Fermi component of interaction with neutrons is coherent, while that of the protons is negligible. Thus the matrix element becomes

\[
\begin{align*}
\text{ME} &= \frac{G_F}{2\sqrt{2}} N g_{\nu}, \\
N &= \text{number of neutrons in the nucleus}. 
\end{align*}
\]

A detailed analysis of the frequencies of \( ^{130}\text{TeO}_2 \) can be found [60]. The analysis involved crystalline phases of tellurium dioxide: paratellurite \( \alpha\text{-TeO}_2 \), tellurite \( \beta\text{-TeO}_2 \), and the new phase \( \text{-TeO}_2 \), recently identified experimentally. Calculated Raman and IR spectra are in good agreement with available experimental data. The vibrational spectra of \( \alpha \) and \( \beta\text{-TeO}_2 \) can be interpreted in terms of vibrations of \( \text{TeO}_2 \) molecular units. The \( \alpha\text{-TeO}_2 \) modes are associated with the symmetry \( D_4h \) or 422, which has 5 irreducible representations, two 1-dimensional representations of the antisymmetric type indicated by \( A_1 \) and \( A_2 \), two 1-dimensional representations of the symmetric types \( B_1 \) and \( B_2 \), and one 2-dimensional representation, usually indicated by \( E \). They all have been tabulated in [60]. Those that can be excited must be below the Debye frequency which has been determined [61] and found to be quite low:

\[
T_D = (232 \pm 7) \text{K} \implies \omega_D = 0.024 \text{eV}. \tag{20}
\]

This frequency is smaller than the maximum sterile neutrino energy estimated to be \( T_{\text{max}} = 0.11 \text{eV} \). Those frequency modes of interest to us are given in Table 1. The differential cross section is, therefore, given by

\[
\begin{align*}
\frac{d\sigma}{d\nu} &= \frac{1}{\nu} C_s^2 N^2 \left( g_{\nu} \right)^2 \left( \frac{G_F}{2\sqrt{2}} \right)^2 \sum_{k=1}^{8} \sum_{n_k=0}^{N_k} \frac{d^3p'_k}{2m_{\nu}} \cdot \frac{d^3\mathbf{q}}{(2\pi)^3} \left( 2\nu \right)^4 \\
&\cdot F^2 \left( \mathbf{q}^2 \right) \delta \left( \mathbf{p}_\nu - \mathbf{p}'_k - \mathbf{q} \right) \cdot \delta \left( \frac{p^2_\nu}{2m_{\nu}} - \frac{(p')^2}{2m_{\nu}} - (n_k + \frac{1}{2}) \omega_k \right),
\end{align*}
\]

where \( N_k \) will be specified below and \( \mathbf{q} \) is the momentum transferred to the nucleus. The momentum transfer is small and the form factor \( F^2(\mathbf{q}^2) \) can be neglected.

In deriving this formula we tacitly assumed a coherent interaction between the WIMP and several nuclei, thus creating a collective excitation of the crystal, that is, a phonon or few phonons. This of course is a good approximation provided that the energy transferred is small, of a few tens of meV. We see from Table 1 that the maximum allowed energy is small, around 100 meV. We find that, if we restrict the maximum allowed energy by a factor of 2, the obtained results are reduced only by a factor of about 10%. We may thus assume that this approximation is good.

Integrating over the nuclear momentum we get

\[
\begin{align*}
\frac{d\sigma}{d\nu} &= \frac{1}{\nu} C_s^2 N^2 \left( g_{\nu} \right)^2 \left( \frac{G_F}{2\sqrt{2}} \right)^2 \frac{1}{(2\pi)^3} \\
&\cdot \sum_{k=1}^{8} \sum_{n_k=0}^{N_k} \frac{d^3p'_k}{2m_{\nu}} \cdot \frac{d^3p'}{2m_{\nu}} (n_k + \frac{1}{2}) \omega_k, \tag{22}
\end{align*}
\]

### Table 1: The frequency modes below the Debye temperature for \( \alpha\text{-TeO}_2 \) obtained from Table VIII of [60] (for notation see text).

<table>
<thead>
<tr>
<th>( \nu_i = \frac{\omega_i}{2\pi} ) \text{ (cm}^{-1} \text{)}</th>
<th>52</th>
<th>124</th>
<th>128</th>
<th>152</th>
<th>157</th>
<th>176</th>
<th>177</th>
<th>179</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symmetry</td>
<td>( B_1 )</td>
<td>( E )</td>
<td>( B_1 )</td>
<td>( A_1 )</td>
<td>( B_1 )</td>
<td>( A_1 )</td>
<td>( E )</td>
<td>( B_1 )</td>
</tr>
<tr>
<td>( \omega_i ) \text{ (eV)}</td>
<td>0.006</td>
<td>0.015</td>
<td>0.016</td>
<td>0.019</td>
<td>0.019</td>
<td>0.022</td>
<td>0.022</td>
<td>0.022</td>
</tr>
<tr>
<td>( N_i )</td>
<td>16</td>
<td>6</td>
<td>6</td>
<td>5</td>
<td>5</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>( E_{\text{max}}(i) ) \text{ (meV)}</td>
<td>106</td>
<td>100</td>
<td>102</td>
<td>103</td>
<td>107</td>
<td>98</td>
<td>99</td>
<td>100</td>
</tr>
</tbody>
</table>

Another possibility is to use bolometers, like the CUORE detector exploiting Low Temperature Specific Heat of Crystalline \( ^{130}\text{TeO}_2 \) at low temperatures. The energy of the WIMP will now be deposited in the crystal, after its interaction with the nuclei via Z-exchange. In this case the Fermi component of interaction with neutrons is coherent, while that of the protons is negligible. Thus the matrix element becomes

\[
\begin{align*}
\text{ME} &= \frac{G_F}{2\sqrt{2}} N g_{\nu}, \\
N &= \text{number of neutrons in the nucleus}. 
\end{align*}
\]
The functions exhibited as a function of energy; namely, $W$ etc. have imposed by the available energy, namely, $\delta$ performing the integration using the $\delta$ function we get
\[ \sigma = \frac{1}{v} C_N^2 N^2 \left( \frac{G_F}{2 \sqrt{2}} \right)^2 \frac{1}{\pi} \]
\[ \cdot m_\nu \sqrt{2m_\nu} \sum_{k=1}^{8} \sum_{n_1=0}^{N_k} \sqrt{E_k - \left( n_1 + \frac{1}{2} \right) \omega_k} \]
where $T_1 = (1/2)m_\nu u_0^2$, $y = v/u_0$

Folding with the velocity distribution we obtain
\[ \langle \nu \sigma \rangle = \nu_0 C_N^2 N^2 \left( \frac{G_F}{2 \sqrt{2}} \right)^2 \frac{1}{\pi} m_\nu^2 \sum_{k=1}^{8} \sum_{n_1=0}^{N_k} I_{n_1, \omega_k} \]
\[ I_{n_1, \omega_k} = \int_{y_{\min}}^{y_{\max}} dy f_{n_1, \omega_k}(y), \]
\[ f_{n_1, \omega_k}(y) = \sqrt{y^2 - \left( n_1 + 1 \right) / 2} - \omega_k \frac{y e^{-y} - y^2 \sinh y}{T_1} \]
\[ y_{\min} = \sqrt{\left( n_1 + 1 \right) / 2} \omega_k / T_1 \]

We see that we have the constraint imposed by the available energy; namely,
\[ N_k = \text{IP} \left[ \frac{y_{\min}^2 T_1}{\omega_k} - \frac{1}{2} \right], \]
where $\text{IP}[x] = \text{integer part of } x$. We thus find $N_k$ listed in Table 1. The functions $f_{n_1, \omega_k}(y)$ are exhibited in Figure 2. The relevant integrals are $I_{n_1, \omega_k}(\omega_k) = (1.170, 0.972, 0.785, 0.621)$ for $n = 0, 1, 2, 3$, $I_{n_1, \omega_k}(\omega_k) = (1.032, 0.609)$ for $n = 0, 1$, $I_{n_1, \omega_k}(\omega_k) = (1.025, 0.592)$, for $n = 0, 1, 2, 3$ (0.979, 0.970, 0.934, 0.932, 0.929) for $k = 4, \ldots, 8$. Thus we obtain a total of 17.8. The event rate takes with a target of mass $m_1$ which takes the form
\[ R = \Phi \nu C_N^2 N^2 \left( \frac{G_F}{2 \sqrt{2}} \right) \frac{1}{8\pi} \frac{m_\nu}{Am_p} m_1^2 17.8. \]

If we restrict the maximum allowed energy to half of that shown in Table 1 by a factor of two, we obtain 15.7 instead of 17.8.

For a $^{130}$TeO$_2$ target ($N = 78$) of 1 kg of mass get
\[ R = 1.7 \times 10^{-5} C_n^2 \text{ per kg-s} = 51C_n^2 \text{ per kg-y} \]

This is much larger than that obtained in the previous section, mainly due to the neutron coherence arising from the $Z$-interaction with the target (the number of scattering centers is approximately the same, $4.5 \times 10^{24}$). In the present case, however, targets can be larger than 1 kg. Next we are going to examine other mechanisms, which promise a better signature.

### 4. Sterile Neutrino Detection via Atomic Excitations

We are going to examine the interesting possibility of excitation of an atom froma level $|j_1, m_1\rangle$ to a nearby level
\(|j_2, m_2\) at energy \(\Delta = E_2 - E_1\), which has the same orbital structure. The excitation energy has to be quite low; that is,

\[
\Delta \leq \frac{1}{2} m_v \nu_{esc}^2 = \frac{1}{2} \times 50 \times 10^3 \times 2.84^2 \left(\frac{2.2}{3}\right)^2 \times 10^{-6} = 0.11 \text{ eV.}
\]

The target is selected so that the two levels \(|j_1, m_1\) and \(|j_2, m_2\) are closer than 0.11 eV. This can result from the splitting of an atomic level by the magnetic field so that they can be connected by the spin operator. The lower one \(|j_1, m_1\) is occupied by electrons but the higher one \(|j_2, m_2\) is completely empty at sufficiently low temperature. It can be populated only by exciting an electron to it from the lower one by the oncoming sterile neutrino. The presence of such an excitation is monitored by a tuned laser which excites such an electron from \(|j_2, m_2\) to a higher state \(|j_3, m_3\), which cannot be reached in any other way, by observing its subsequent decay by emitting photons.

\[
d \sigma = \frac{1}{v} C_v^2 \left(\frac{G_F}{2\sqrt{2}}\right)^2 |\text{ME} (j_1, m_1; j_2, m_2)|^2 \frac{1}{\pi} (E_\nu - \Delta)
\cdot \sqrt{2 (E_\nu - \Delta - m_\nu)} m_\nu = \frac{1}{v} C_v^2 \left(\frac{G_F}{2\sqrt{2}}\right)^2
\cdot |\text{ME} (j_1, m_1; j_2, m_2)|^2 \frac{1}{\pi} m_\nu \sqrt{\frac{2 T_1}{m_\nu}} f \left(y, \frac{\Delta}{T_1}\right)
\]

\[
f \left(y, \frac{\Delta}{T_1}\right) = \left(y^2 - \frac{\Delta}{T_1}\right)^{1/2}, \quad T_1 = \frac{1}{2} m_v \nu_0^2,
\]

where we have set \(E - \Delta = m_\nu + T_1 - \Delta = m_\nu\).

Folding the cross section with the velocity distribution from a minimum \(\sqrt{\Delta/T_1}\) to \(\nu_{esc}\) we obtain

\[
\frac{\langle v \sigma \rangle}{\nu_0} = C_v^2 \left(\frac{G_F}{2\sqrt{2}}\right)^2 m_\nu |\text{ME} (j_1, m_1; j_2, m_2)|^2 \frac{1}{\pi} g \left(\frac{\Delta}{T_1}\right)
\cdot \frac{2}{\sqrt{\pi}} \int_{\sqrt{\Delta/T_1}}^{\nu_{esc}} d y y^2 \left(y^2 - \frac{\Delta}{T_1}\right)^{1/2} e^{-(1+y^2)} \sinh 2y y.
\]

Since this is a one-body transition the relevant matrix element takes the form

\[
|\text{ME} (j_1, m_1; j_2, m_2)|^2 = g_x^2 \delta_{j_1, j_2} \delta_{m_1, m_2} + g_\alpha^2 (C_{\ell j_1, \nu_1 j_2, m_1, m_2})^2
\]

(in the case of the axial current we have \(g_\alpha = 1\) and we need evaluate the matrix element of \(\sigma \cdot \sigma\) and then square it and sum and average over the neutrino polarization).

\[
C_{\ell j_1, \nu_1 j_2, m_1, m_2} = \langle n \ell j_2 m_2 | \sigma | n \ell j_1 m_1 \rangle
= \langle j_1 m_1, 1 m_2 - m_1 | j_2 m_2 \rangle
\cdot \sqrt{(2 j_1 + 1) 3 \sqrt{2 \ell + 1} \sqrt{5}} \begin{pmatrix} \ell & \frac{1}{2} & j_1 \\ \frac{1}{2} & j_2 \\ 0 & 1 & 1 \end{pmatrix}
\]

expressed in terms of the Clebsch-Gordan coefficient and the nine-j symbol. It is clear that in the energy transfer of interest only the axial current can contribute to excitation.

The cross section takes the form

\[
\sigma = \frac{1}{v} C_v^2 \left(\frac{G_F}{2\sqrt{2}}\right)^2 |\text{ME} (j_1, m_1; j_2, m_2)|^2 \frac{1}{\pi} (E_\nu - \Delta)
\cdot \sqrt{2 (E_\nu - \Delta - m_\nu)} m_\nu = \frac{1}{v} C_v^2 \left(\frac{G_F}{2\sqrt{2}}\right)^2
\cdot |\text{ME} (j_1, m_1; j_2, m_2)|^2 \frac{1}{\pi} m_\nu \sqrt{\frac{2 T_1}{m_\nu}} f \left(y, \frac{\Delta}{T_1}\right)
\]

Clearly the maximum excitation energy that can be reached is \(\Delta_{\text{max}} = 2.84^2 T_1 = 0.108 \text{ eV.}\) The function \(g(\Delta/T_1)\) is exhibited in Figure 3.

Proceeding as in Section 2 and noting that for small excitation energy \(g(\Delta/T_1) \approx 1.4\) we find

\[
R = 1.8 \times 10^{-2} C_v^2 \frac{1}{A} (C_{\ell j_1, \nu_1 j_2, m_1, m_2})^2 \text{ kg·y.}
\]

The expected rate will be smaller after the angular momentum factor \(C_{\ell j_1, \nu_1 j_2, m_1, m_2}\) is included (see Appendix A). Anyway, leaving aside this factor, which can only be determined after a specific set of levels is selected, we see that the obtained rate is comparable to that expected from electron recoil (see (18)). In fact for a target with \(A = 100\) we obtain

\[
R = 1.8 \times 10^{-4} C_v^2 \left(C_{\ell j_1, \nu_1 j_2, m_1, m_2}\right)^2 \text{ kg·y.}
\]

This rate, however, decreases as the excitation energy increases (see Figure 3). In the present case, however, we have two advantages.

(i) The characteristic signature of photons spectrum is following the deexcitation of the level \(|j_3, m_3\) mentioned above. The photon energy can be changed if the target is put in a magnetic field by a judicious choice of \(|j_3, m_3\).

(ii) The target now can be much larger, since one can employ a solid at very low temperatures. The ions
5. Sterile Neutrino Capture by a Nucleus Undergoing Electron Capture

This is essentially the process:

\[ \nu_e + e^+_\nu + A (N, Z) \rightarrow A (N + 1, Z - 1)^* \]  

(36)

involving the absorption of a neutrino with the simultaneous capture of a bound electron. It has already been studied [62] in connection with the detection of the standard relic neutrinos. It involves modern technological innovations like the Penning Trap Mass Spectrometry (PT-MS) and the Microcalorimetry (MC). The former should provide an answer to the question of accurately measuring the nuclear binding energies and how strong the resonance enhancement is expected, whereas the latter should analyze the bolometric spectrum in the tail of the peak corresponding to L-capture to the excited state in order to observe the relic antineutrino events. They also examined the suitability of \(^{157}\)Tb for relic antineutrino detection via the resonant enhancement to be considered by the PT-MS and MC teams. In the present case the experimental constraints are expected to be less stringent since the sterile neutrino is much heavier.

Let us measure all energies from the ground state of the final nucleus and assume that \( \Delta \) is the mass difference of the two neutral atoms. Let us consider a transition to the final state with energy \( E_{x'} \). The cross section for a neutrino (here as well as in the following we may write neutrino, but it is understood that we mean antineutrino) of given velocity \( v \) and kinetic energy \( E_v \) is given by

\[ \sigma (E_v) = C_v^2 \frac{1}{v} |ME(E_v)|^2 \langle \phi_\nu \rangle^2 \]

(37)

\[ \langle \phi_\nu \rangle = \frac{G_F}{2 \sqrt{2}} (2\pi)^3 d^3 p_A (2\pi)^4 \delta (p_A - p_v) \delta (E_v + m_\nu + \Delta - E_x - b), \]

where \( p_A \) is the recoiling nucleus momentum. Integrating over the recoil momentum using the \( \delta \) function we obtain

\[ \sigma (E_v) = C_v^2 \frac{1}{v} |ME(E_v)|^2 \langle \phi_\nu \rangle^2 \left( \frac{G_F}{2 \sqrt{2}} \right)^2 \cdot \delta (E_v + m_\nu + \Delta - E_x - b). \]  

(38)

We note that since the oncoming neutrino has a mass, the excited state must be higher than the highest excited state at \( E_x' = \Delta - b \). With indicating by \( \epsilon = E_x - E_x' \) the above equation can be written as

\[ \sigma (E_v) = C_v^2 \frac{1}{v} |ME(E_x)|^2 \langle \phi_\nu \rangle^2 \left( \frac{G_F}{2 \sqrt{2}} \right)^2 \cdot \delta (E_v + m_\nu - \epsilon). \]  

(39)

Folding it with the velocity distribution as above we obtain

\[ \langle \nu \sigma (E_v) \rangle = C_v^2 \frac{1}{v} |ME(E_x)|^2 \langle \phi_\nu \rangle^2 \left( \frac{G_F}{2 \sqrt{2}} \right)^2 \cdot \int_0^\infty dy y^2 \frac{2}{\sqrt{\pi}} e^{-1+y^2} \sinh 2y \frac{y}{y} \]

(40)

\[ \cdot \delta \left( m_\nu + \frac{1}{2} m_\nu y_0 y^2 - \epsilon \right) \]

or using the delta function

\[ \langle \nu \sigma (E_v) \rangle = 2 \pi C_v^2 \frac{1}{v} |ME(E_x)|^2 \langle \phi_\nu \rangle^2 \left( \frac{G_F}{2 \sqrt{2}} \right)^2 F (X), \]

(41)

\[ F (X) = \frac{2}{\sqrt{\pi}} e^{-1+X^2} \sinh 2X, \]

\[ X = \frac{1}{y_0} \left( 2 \left( \frac{\epsilon}{m_\nu} - 1 \right) \right). \]

As expected the cross section exhibits resonance behavior though the normalized function \( F(X) \) as shown in Figure 4. It is, of course, more practical to exhibit the function \( F(X) \) as a function of the energy \( \epsilon \). This is exhibited in Figure 5. From this figure we see that the cross section resonance is quite narrow. We find that the maximum occurs at \( \epsilon = m_\nu (1 + 2.8 \times 10^{-7}) = 50 \text{keV} + 0.014 \text{eV} \) and has a width \( \Gamma = m_\nu (1 + 9.1 \times 10^{-7}) - m_\nu (1 + 0.32 \times 10^{-7}) = 0.04 \text{eV} \). So for all practical purposes it is a line centered at the neutrino

![Figure 3: The function \( g(\Delta/T_\nu) \) for sterile neutrino scattering by an atom as a function of the excitation energy in eV.](image-url)
mass. The width may be of some relevance in the special case whereby the excited state can be determined by atomic deexcitations at the sub-eV level, but it will not show up in the nuclear deexcitations.

If there is a resonance in the final nucleus at the energy \( E_x = \epsilon + (\Delta - b) \) with a width \( \Gamma \) then perhaps it can be reached even if \( \epsilon \) is a bit larger than \( m_\gamma \); for example, \( \epsilon = m_\gamma + \Gamma/2 \). The population of this resonance can be determined by measuring the energy of the deexcitation \( \gamma \)-ray, which should exceed by \( \epsilon \) the maximum observed in ordinary electron capture.

For antineutrinos having zero kinetic energy the atom in the final state has to have an excess energy \( \Delta - (b - m_\gamma) \) and this can only happen if this energy can be radiated out via photon or phonon emission. The photon emission takes place either as atomic electron or nuclear level transitions. In the first case photon energies are falling in the eV-keV energy region and this implies that only nuclei with a very small \( \Delta \)-value could be suitable for this detection. In the second case, there should exist a nuclear level that matches the energy difference \( E_x = \Delta - (b - m_\gamma) \) and therefore the incoming antineutrino has no energy threshold. Moreover, spontaneous electron capture decay is energetically forbidden, since this is allowed for \( E_x < \Delta - (b + m_\gamma) \).

As an example we consider the capture of a very low energy \( \bar{\nu} \) by the \( ^{157}\text{ Tb} \) nucleus:

\[
\bar{\nu} + e^- + ^{157}\text{ Tb} \rightarrow ^{157}\text{ Gd}^* \tag{42}
\]

taking the allowed transitions from the ground state \((3/2^+)\) of parent nucleus, \( ^{157}\text{ Tb} \), to the first excited \( 5/2^+ \) state of the daughter nucleus \( ^{157}\text{ Gd} \). The spin and parity of the nuclei involved obey the relations \( \Delta I = 1 \), \( \Pi_f \Pi_i = +1 \), and the transition is dubbed as allowed. The nuclear matrix element \( ME \) can be written as

\[
|ME|^2 = \left( \frac{g_A}{g_V} \right)^2 (GT)^2, \tag{43}
\]

where \( g_A = 1.2695 \) and \( g_V = 1 \) are the axial and vector coupling constants, respectively. The nuclear matrix element is calculated using the microscopic quasi-particle-phonon (MQPM) model \([63, 64]\) and it is found to be \( |ME|^2 = 0.96 \). The experimental value of first excited \( 5/2^+ \) is at 64 keV \([65]\) while that predicted by the model is at 65 keV. The \( \Delta \)-value is ranging from 60 to 63 keV \([65]\).

For K-shell electron capture where \( \langle \phi_e \rangle^2 = ((\alpha Z/\pi)m_\epsilon)^3 \) (1s capture) with binding energy \( b = 50.24 \) keV, the velocity averaged cross section takes the value

\[
\langle \sigma v \rangle = 8.98 \times 10^{-46} C_r^2 \text{ cm}^2 \tag{44}
\]

and the event rate we expect for mass \( m_t = 1 \) kg is

\[
R = 8.98 \times 10^{-46} C_r^2 \times 6 \times 10^3 \times 6.023 \times 10^{23} \times \frac{m_t}{A} \times 9.28 \times 10^{17} \text{ y}^{-1} = 19 C_r^2 \text{ y}^{-1}. \tag{45}
\]

The lifetime of the source should be suitable for the experiment to be performed. If it is too short, the time available will not be adequate for the execution of the experiment. If it is too long, the number of counts during the data taking will be too small. Then one will face formidable backgrounds and/or large experimental uncertainties.

The source should be cheaply available in large quantities. Clearly a compromise has to be made in the selection of the source. One can be optimistic that such adequate quantities can be produced in Russian reactors. The nuclide parameters

\[
X
\]

\[
F
\]

\[
\epsilon/m_\gamma - 1
\]

Figure 4: The cross section exhibits resonance behavior. Shown is the resonance properly normalized as a function of \( X = (1/\lambda_0) \sqrt{2(\epsilon/m_\gamma - 1)} \). The width is \( \Gamma = 1.49 \) and the location of the maximum is at 1.03.
relevant to our work can be found in [66] (see also [67]), summarized in Table 2.

6. Modification of the End Point Spectra of $\beta$ Decaying Nuclei

The end point spectra of $\beta$ decaying nuclei can be modified by the reaction involving sterile (anti)neutrinos:

$$\nu + A (N, Z) \rightarrow A (N - 1, Z + 1) + e^- \quad (46)$$

where $\Delta$ is the atomic mass difference. Integrating over the nuclear recoil momentum and the direction of the electron momentum we get

$$\sigma(E_\gamma) = C_{\gamma}^2 \frac{1}{v} |\text{ME}(E_\gamma)|^2 \left( \frac{G_F}{2\sqrt{2}} \right)^2 \frac{d^3p_A}{(2\pi)^3} \frac{d^3p_e}{(2\pi)^3} \left( \frac{2\pi}{\Delta} \right) \delta(p - p_A - p_e) \delta(E_\gamma + \Delta - E_e), \quad (50)$$

where

$$E_e = m_\nu + \frac{1}{2}m_e v^2 + \Delta + m_e,$$

$$p_e = \sqrt{E_e^2 - m_e^2}.$$  

Folding the cross section with the velocity distribution we find

$$\langle \sigma v \rangle = C_{\gamma}^2 \left( \frac{G_F}{2\sqrt{2}} \right)^2 \frac{2}{\pi^3/2} \int_0^\infty dy f(y), \quad (52)$$

where

$$f(y) = |\text{ME}|^2 y \sinh(2y) E_e P_e e^{-i(1+y)} F(Z_f, E_e)$$

with

$$y = \frac{v}{v_0}, \quad (54)$$

Table 2: Nuclides relevant for the search of the keV sterile neutrinos in the electron capture process. We give the life time $T_{1/2}$, the $Q$-value, the electron binding energy $B_i$ for various captures, and the value of $\Delta = Q - B_i$ For details see [66].

<table>
<thead>
<tr>
<th>Nuclide</th>
<th>$T_{1/2}$</th>
<th>EC transition</th>
<th>$Q$ (keV)</th>
<th>$B_i$ (keV)</th>
<th>$B_f$ (keV)</th>
<th>$Q - B_f$ (keV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{137}$Tb</td>
<td>71 y</td>
<td>$3/2^- \rightarrow 3/2^-$</td>
<td>60.04(30)</td>
<td>K: 50.239(5)</td>
<td>L1: 8.3756(5)</td>
<td>9.76</td>
</tr>
<tr>
<td>$^{135}$Ho</td>
<td>4570 y</td>
<td>$7/2^- \rightarrow 5/2^-$</td>
<td>2.55(16)</td>
<td>MI: 2.0468(5)</td>
<td>NI: 0.4163(5)</td>
<td>0.51</td>
</tr>
<tr>
<td>$^{179}$Ta</td>
<td>1.82 y</td>
<td>$7/2^- \rightarrow 9/2^+$</td>
<td>105.6(4)</td>
<td>K: 65.3508(6)</td>
<td>L1: 11.2707(4)</td>
<td>40.2</td>
</tr>
<tr>
<td>$^{193}$Pt</td>
<td>50 y</td>
<td>$1/2^- \rightarrow 3/2^+$</td>
<td>56.63(30)</td>
<td>L1: 13.4185(3)</td>
<td>MI: 3.1737(17)</td>
<td>43.2</td>
</tr>
<tr>
<td>$^{202}$Pb</td>
<td>52 ky</td>
<td>$0^+ \rightarrow 2^+$</td>
<td>46(14)</td>
<td>L1: 15.3467(4)</td>
<td>MI: 3.7041(4)</td>
<td>30.7</td>
</tr>
<tr>
<td>$^{205}$Pb</td>
<td>13 My</td>
<td>$5/2^- \rightarrow 1/2^+$</td>
<td>50.6(5)</td>
<td>L1: 15.3467(4)</td>
<td>MI: 3.7041(4)</td>
<td>35.3</td>
</tr>
<tr>
<td>$^{235}$Np</td>
<td>396 d</td>
<td>$5/2^- \rightarrow 7/2^+$</td>
<td>124.2(9)</td>
<td>K: 115.6061(16)</td>
<td>L1: 21.7574(3)</td>
<td>8.6</td>
</tr>
</tbody>
</table>

The Fermi function, $F(Z_f, E_e)$, encapsulates the effects of the Coulomb interaction for a given lepton energy $E_e$ and final state proton number $Z_f$. The function $f(y)$ is exhibited in Figure 6.

In transitions happening inside the same isospin multiplet ($J^p \rightarrow J', J \neq 0$) both the vector and axial form factors contribute and in this case the nuclear matrix element $\text{ME}(E_e)$ can be written as

$$|\text{ME}|^2 = (\langle F \rangle^2 + \left( \frac{g_A}{g_V} \right)^2 \langle GT \rangle^2), \quad (55)$$

where $g_A = 1.2695$ and $g_V = 1$ are the axial and vector coupling constants, respectively. In case of $^3$H target we adopt $\langle F \rangle = 0.9987$ and $\langle GT \rangle = 2.788$ from [68]. Thus $|\text{ME}|^2 = 5.49$.

Thus the velocity averaged cross section takes the value

$$\langle \sigma v \rangle = 3.44 \times 10^{-46} C_{\gamma}^2 \text{cm}^2 \quad (56)$$

and the expected event rate becomes

$$R = 3.44 \times 10^{-46} C_{\gamma}^2 \times 6.10^3 \times 6.023 \times 10^{23} \times \frac{m_n}{A} \times 9.28 \times 10^{17} \text{y}^{-1}. \quad (57)$$
For a mass of the current KATRIN target, that is, about 1 gr, we get

\[ R = 0.380 C^2 \gamma^{-1}. \]  

(58)

It is interesting to compare the neutrino capture rate

\[ R_{\nu} = \langle \sigma v \rangle \frac{\rho}{m_{\nu}}, \]

\[ = 3.44 \times 10^{-46} C^2 \gamma \times 6 \times 10^3 \times 9.28 \times 10^{17} \]

\[ = 1.91 \times 10^{-24} C^2 \gamma^{-1} \]  

(59)

with that of beta decay process \[^3\text{H} \rightarrow ^3\text{He} + e^- + \bar{\nu}_\beta\], whose rate \( R_{\beta} \) is given by

\[ R_{\beta} = \frac{G_F^2}{2\pi^3} \int_{m_e}^W p_e E_p F(Z, E_p) |\mathbf{M}E|^2 E_p dE_p, \]  

(60)

where \( W_o \) is the maximal electron energy or else beta decay endpoint

\[ W_o = K_{\text{end}} + m_e \]  

(61)

with

\[ K_{\text{end}} = \frac{(m_{^3\text{H}} - m_e)^2 - (m_{^3\text{He}} + m_e)^2}{2m_{^3\text{H}}} = \Delta \]  

(62)

the electron kinetic energy at the endpoint, and

\[ m_e = 510.998910 \text{ keV} \]  

(63)

\[ m_{^3\text{H}} = 2808920.8205 \text{ keV} \]  

(63)

\[ m_{^3\text{He}} = 2808391.2193 \text{ keV}. \]  

(63)

Masses \( m_{^3\text{H}} \) and \( m_{^3\text{He}} \) are nuclear masses \([58, 69, 70]\). The calculation of (60) gives \( R_{\beta} = 0.055 \gamma^{-1} \). The ratio of \( R_{\nu} \) to corresponding beta decay \( R_{\beta} \) is very small.

\[ R_{\nu} = 0.034 \cdot 10^{-21} C^2 R_{\beta}. \]  

(64)

The situation is more optimistic in a narrow interval \( W_o - \delta < E_e < W_o \) near the endpoint. As an example, we consider an energy resolution \( \delta = 0.2 \text{ eV} \) close to the expected sensitivity of the KATRIN experiment \([57]\). Then the ratio of the event rate \( R_{\beta}(\delta = 0.2 \text{ eV}) \) to that of neutrino capture \( R_{\nu} \) gives

\[ R_{\nu} = 5.75 \cdot 10^{-9} C^2 R_{\beta}(\delta = 0.2 \text{ eV}). \]  

(65)

In Figure 7 we present the ratio of the event rate decay rate of \( R_{\beta}(\delta) \) for the beta decay compared with the neutrino capture rate \( R_{\nu} \) as a function of the energy resolution \( \delta \) in the energy region \( W_o - \delta < E_e < W_o \).

Moreover, the electron kinetic energy \( K_e \) due to neutrino capture process (48) is

\[ K_e = E_{\gamma} + K_{\text{end}} > m_e + 18.591 \text{ keV}; \]  

(66)

7. Discussion

In the present paper we examined the possibility of direct detection of sterile neutrinos of a mass 50 keV, in dark matter searches. This depends on finding solutions to two problems. The first is the amount of energy expected to be deposited in the detector and the second one is the expected event rate. In connection with the energy we have seen that, even though these neutrinos are quite heavy, their detection is not easy, since like all dark matters candidates move in our galaxies with not relativistic velocities, \( 10^{-3} \text{ c} \) on the average, and with energies about 0.05 eV, not all of which can be deposited in the detectors. Thus the detection techniques employed in
the standard dark matter experiments, like those looking for heavy WIMP candidates, are not applicable in this case.

We started our investigation by considering neutrino-electron scattering. Since the energy of the sterile neutrino is very small one may have to consider systems with very small electron binding, for example, electron pairs in superconductors, which are limited to rather small number of electron pairs. Alternatively one may use low temperature bolometers, which can be larger in size resulting in a higher expected event rate. These experiments must be able to detect very small amount of energy.

Then we examined more exotic options by exploiting atomic and nuclear physics. In atomic physics we examined the possibility of spin induced excitations. Again to avoid background problems the detector has a crystal operating at low temperatures. Then what matters is the atomic structure of the ions of the crystal or of suitably implanted impurities. The rate in this case is less than that obtained in the case of bolometers, but one may be able to exploit the characteristic feature of the spectrum of the emitted photons.

From the nuclear physics point of view, we consider the antineutrino absorption on an electron capturing nuclear system leading to a fine resonance in the \((N+1, Z-1)\) system, centered 50 keV above the highest excited state reached by the ordinary electron capture. The deexcitation of this resonance will lead to a very characteristic \(\gamma\) ray. Finally the sterile neutrino will lead to \(\nu + A(N, Z) \rightarrow e^- + A(N-1, Z+1)\) reaction. The produced electrons will have a maximum energy which goes beyond the end point energy of the corresponding \(\beta\) decay essentially by the neutrino mass. The signature is less profound than in the case of antineutrino absorption.

Regarding the event rate, as we have mentioned before, it is proportional to the coupling of the sterile neutrino to the usual electron neutrino indicated above as \(C_{\nu e}^2\). This parameter is not known. In neutrino oscillation experiments a value of \(C_{\nu e}^2 \approx 10^{-2}\) has been employed. With such a value our results show that the 50 keV neutrino is detectable in the experiments discussed above. This large value of \(C_{\nu e}^2\) is not consistent, however, with a sterile 50 keV neutrino. In fact such a neutrino would have a life time \([71]\) of \(2 \times 10^7\) y, much shorter than the age of the universe. A cosmologically viable sterile 50 keV neutrino is allowed to couple to the electron neutrino with coupling \(C_{\nu e}^2 < 1.3 \times 10^{-7}\). Our calculations indicate that such a neutrino is not directly detectable with experiments considered in this work. The results, however, obtained for the various physical processes considered in this work, can be very useful in the analysis of the possible experimental searches of lighter sterile neutrinos in the mass range of 1–10 keV.

Appendix

A. Angular Momentum Coefficients Entering Atomic Excitations

The angular momentum coefficients entering single particle transitions are shown in (A.1) and (B.4).

Equation (A.1). The coefficients \((C_{\ell j_1 m_1, j_2 m_2} e^{-1})^2\) connect via the spin operator a given initial state \(|i\rangle = |n\ell, j_1, m_1\rangle\) with all possible states \(|f\rangle = |n\ell, j_2, m_2\rangle\) for \(\ell = 0, 1\). Note s-states are favored.

\[
\begin{pmatrix}
\ell & j_1 & m_1 & j_2 & m_2 & C_{\ell j_1 m_1 j_2 m_2}^2 \\
0 & 1 & -1 & 1 & 1 & 2 \\
0 & 2 & 2 & 2 & 2 & 2
\end{pmatrix}
\]

(B.1)

B. Exotic Atomic Experiments

As we have mentioned the atomic experiment has to be done at low temperatures. It may be difficult to find materials exhibiting atomic structure at low temperatures. It amusing to note that one may be able to employ at low temperatures some exotic materials used in quantum technologies (for a recent review see [72]) like nitrogen-vacancy (NV), that is, materials characterized by spin \(S = 1\), which in a magnetic field allow transitions between \(m = 0, m = 1\) and \(m = -1\). These states are spin symmetric. Antisymmetry requires the space part to be antisymmetric, that is, a wave function of the form

\[
\psi = \phi_{n\ell}^2(r) |L = \text{odd}, S = 1\rangle \quad J = L - 1, L, L + 1.
\]

Of special interest are

\[
\psi = \phi_{n\ell}^2(r)^3 P_J, \quad \phi_{n\ell}^2(r)^3 F_J.
\]

Then the spin matrix element takes the form

\[
\langle \ell j_1, m_1 | \sigma | \ell j_2, m_2 \rangle \\
= \frac{1}{\sqrt{2j_1 + 1}} \langle j_1 m_1, 1m_2 - m_1 | j_2 m_2 \rangle
\]

(B.3)
The reduced matrix elements are given in (B.5), as well as the full matrix element $\langle P_{\ell m_\ell} | \sigma | P_{\ell m_\ell} \rangle$ of the most important component.

**Equation (B.4).** The same as in equation (A.1), the coefficients $(C_{\ell_1 j_1 m_1 j_2 m_2})^2$ for $\ell = 2$ are

$$
\begin{pmatrix}
0 & 1 & 2 & 2 \\
2 & 2 & 2 & 2 \\
3 & 2 & 2 & 2 \\
4 & 2 & 2 & 2 \\
5 & 2 & 2 & 2 \\
6 & 2 & 2 & 2 \\
\end{pmatrix}
$$

**Equation (B.5).** The coefficients are $\langle P_{\ell_1 m_1} | \sigma | P_{\ell_2 m_2} \rangle$, $\langle F_{\ell_1} | \sigma | F_{\ell_2} \rangle$, and $\langle P_{\ell_1 m_1} | \sigma | P_{\ell_2 m_2} \rangle$. For the notation see text.

**Disclosure**

Permanent address of John D. Vergados is as follows: University of Ioannina, 45110 Ioannina, Greece.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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**References**


