The DKP Oscillator in Spinning Cosmic String Background

Mansoureh Hosseinpour and Hassan Hassanabadi

Faculty of Physics, Shahrood University of Technology, Shahrood, P.O. Box 3619995161-316, Iran

Correspondence should be addressed to Mansoureh Hosseinpour; hosseinpour.mansoureh@gmail.com

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In this article, we investigate the behaviour of relativistic spin-zero bosons in the space-time generated by a spinning cosmic string. We obtain the generalized beta-matrices in terms of the flat space-time ones and rewrite the covariant form of Duffin-Kemmer-Petiau (DKP) equation in spinning cosmic string space-time. We find the solution of DKP oscillator and determine the energy levels. We also discuss the influence of the topology of the cosmic string on the energy levels and the DKP spinors.

1. Introduction

The Duffin-Kemmer-Petiau (DKP) equation has been used to describe relativistic spin-0 and spin-1 bosons [1–4]. The DKP equation has five- and ten-dimensional representation, respectively, for spin-0 and spin-1 bosons [5]. This equation is compared to the Dirac equation for fermions [6]. The DKP equation has been widely investigated in many areas of physics. The DKP equation has been investigated in the momentum space with the presence of minimal length [7, 8] and for spins 0 and 1 in a noncommutative space [9–12]. Also, the DKP oscillator has been studied in the presence of topological defects [13]. Recently, there has been growing interest in the so-called DKP oscillator [14–23] in particular in the background of a magnetic cosmic string [13]. The cosmic strings and other topological defects can form at a cosmological phase transition [24, 25]. The conical nature of the space-time around the string causes a number of interesting physical effects. Until now, some problems have been investigated in the gravitational fields of topological defects including the one-electron atom problem [26–28]. Spinning cosmic strings are similar usual cosmic string, characterized by an angular parameter \( \alpha \) that depends on their linear mass density \( \mu \). The DKP oscillator is described by performing the nonminimal coupling with a linear potential. The name distinguishes it from the system called a DKP oscillator with Lorentz tensor couplings of [7–12, 14–16]. The DKP oscillator for spin-0 bosons has been investigated by Guo et al. in [10] in noncommutative phase space. The DKP oscillator with spin-0 has been studied by Yang et al. [11]. Exact solution of DKP oscillator in the momentum space with the presence of minimal length has been analysed in [8]. De Melo et al. construct the Galilean DKP equation for the harmonic oscillator in a noncommutative phase space [29]. Falek and Merad investigated the DKP oscillator of spins 0 and 1 bosons in noncommutative space [9]. Recently, there has been an increasing interest on the DKP oscillator [13–16, 22, 29–31]. The nonrelativistic limit of particle dynamics in curved space-time is considered in [32–36]. Also, the dynamics of relativistic bosons and fermions in curved space-time is considered in [17, 20, 31].

The influence of topological defect in the dynamics of bosons via DKP formalism has not been established for spinning cosmic strings. In this way, we consider the quantum dynamics of scalar bosons via DKP formalism embedded in the background of a spinning cosmic string. We solve DKP equation in presence of the spinning cosmic string space-time whose metric has off diagonal terms which involves time and space. The influence of this topological defect in the energy spectrum and DKP spinor presented graphically.

The structure of this paper is as follows: Section 2 describes the covariant form of DKP equation in a spinning cosmic string background. In Section 3, we introduce the DKP oscillator by performing the nonminimal coupling in this space-time, and we obtain the radial equations that are solved. We plotted the DKP spinor, density of probability, and...
the energy spectrum for different conditions involving the deficit angle and the oscillator frequency. In the Section 4 we present our conclusions.

2. Covariant Form of the DKP Equation in the Spinning Cosmic String Background

We choose the cosmic string space-time background, where the line element is given by

$$ds^2 = -dT^2 + dX^2 + dY^2 + dZ^2$$

(1)

The space-time generated by a spinning cosmic string without internal structure, which is termed ideal spinning cosmic string, can be obtained by coordinate transformation as

$$T = t + a \alpha^{-1} \varphi$$

$$X = r \cos(\varphi)$$

$$Y = r \sin(\varphi)$$

$$\varphi = a \alpha'$$

With this transformation, the line element ((1)) becomes

$$ds^2 = -(dt + ad\varphi)^2 + dr^2 + a^2 r^2 d\varphi^2 + dz^2$$

(3)

with $-\infty < z < \infty$, $\rho \geq 0$, and $0 \leq \varphi \leq 2\pi$. From this point on, we will take $c = 1$. The angular parameter $\alpha$ which runs in the interval $[0, 1]$ is related to the linear mass density $\mu$ of the string as $\alpha = 1 - 4\mu$ and corresponds to a deficit angle $\gamma = 2\pi(1 - \alpha)$. We take $a = 4Gj$ where $G$ is the universal gravitation constant and $j$ is the angular momentum of the spinning string; thus $a$ is a length that represents the rotation of the cosmic string. Note that, in this case, the source of the gravitational field relative to a spinning cosmic string possesses angular momentum and the metric (1) has an off-diagonal term involving time and space.

The DKP equation in the cosmic string space-time (1) reads [13, 17, 31]

$$\left(\Gamma^\mu (x) \nabla_\mu - M\right) \Psi(x) = 0.$$  

(4)

The covariant derivative in (4) is

$$\nabla_\mu = \partial_\mu + \Gamma_\mu (x)$$

(5)

where $\Gamma_\mu$ are the spinorial affine connections given by

$$\Gamma_\mu = \frac{1}{2} \omega_{abc} \left[ \beta^a, \beta^b \right].$$

(6)

The matrices $\beta^a$ are the standard Kemmer matrices in Minkowski space-time.

$$\beta^\mu = e_\mu^a \beta^a$$

(7)

The Kemmer matrices are an analogous to Dirac matrices in Dirac equation. There has been an increasing interest Dirac equation for spin half particles [44–47]. The matrices $\beta^a$ satisfies the DKP algebra,

$$\beta^\mu \beta^\nu + \beta^\nu \beta^\mu = g^{\mu\nu} \beta^0 + g^{\mu\lambda} \beta^\lambda.$$

(8)

The conserved four-current is given by

$$J^\mu = \frac{i}{2} \bar{\Psi} \beta^\mu \Psi,$$

(9)

and the conservation law for $J^\mu$ takes the form

$$\nabla_\mu J^\mu + \frac{i}{2} \bar{\Psi} (U - \eta^0 U^\dagger \eta^0) \Psi = \frac{i}{2} \bar{\Psi} (\nabla_\mu \eta^\mu) \Psi.$$

(10)

The algebra expressed by these matrices generates a set of 126 independent matrices whose irreducible representations comprise a trivial representation, a five-dimensional representation describing the spin-zero particles, and a ten-dimensional representation associated with spin-one particles. We choose the $5 \times 5$ beta-matrices as follows [31]:

$$\beta^0 = \begin{pmatrix} \theta & 0_{2 \times 3} \\ 0_{3 \times 2} & 0_{3 \times 3} \end{pmatrix},$$

$$\beta^1 = \begin{pmatrix} 0_{2 \times 2} & \tau \\ \tau^T & 0_{3 \times 3} \end{pmatrix},$$

$$\theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\tau^1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\tau^2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\tau^3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$  

(12)

In (7), $e^\mu_a$ denote the tetrad basis that we can choose as

$$e^\mu_a = \begin{pmatrix} \frac{a \sin(\varphi)}{r \alpha} & -\frac{a \cos(\varphi)}{r \alpha} & 0 \\ \frac{a \sin(\varphi)}{r \alpha} & \frac{a \cos(\varphi)}{r \alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$  

(13)
For the specific tetrad basis given by (13), we find from (7) that the curved-space beta-matrices read

\[
\begin{align*}
\beta^{(0)} &= e^\alpha \beta^a = \beta^0 - \frac{a}{r \alpha} \beta^a \\
\beta^{(1)} &= e^\alpha \beta^a = \beta^a \\
\beta^r &= \cos \phi \beta^1 + \sin \phi \beta^2 \\
\beta^{(2)} &= e^\alpha \beta^a = \frac{\beta^a}{r \alpha} \\
\beta^\phi &= - \sin \phi \beta^1 + \cos \phi \beta^2 \\
\beta^{(3)} &= e^\alpha \beta^a = \beta^3 = \beta^z
\end{align*}
\]  

(14a)

and the spin connections are given by

\[
\Gamma_{\phi} = (1 - \alpha) \left[ \beta^{(1)}, \beta^{(3)} \right]
\]

(15)

We consider only the radial component in the nonminimal substitution. Since the interaction is time-independent, one can write \( \Psi(r, t) \propto e^{i m \phi} e^{ikz} e^{-iEt} \Phi(r) \), where \( E \) is the energy of the scalar boson, \( m \) is the magnetic quantum number, and \( k_z \) is the wave number. The five-component DKP spinor can be written as \( \Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5) \), and the DKP equation (4) leads to the five

\[
\begin{align*}
(r \alpha (-M \Phi_1 (r) + E \Phi_2 (r) + k z \Phi_3 (r)) + \cos \phi \left( (aE + m) \Phi_4 (r) - i \left( (-1 + \alpha) \Phi_3 (r) + r \alpha \Phi'_3 (r) \right) \right) \\
- \sin \phi \left( (aE + m) \Phi_4 (r) \right) \\
+ i \left( (-1 + \alpha) \Phi_4 (r) + r \alpha \Phi'_4 (r) \right) \right) + 0 \\
\left( \Phi_1 (r) - M \Phi_2 (r) \right) = 0 \\
\left( (aE + m) \sin \phi \Phi_1 (r) + r \alpha \left( -M \Phi_3 (r) + i \right) \right) = 0 \\
\left( (aE + m) \cos \phi \Phi_1 (r) + r \alpha \left( -M \Phi_4 (r) + i \right) \right) = 0 \\
\left( k z + M \psi \Phi_3 (r) \right) = 0
\end{align*}
\]

(16)

Then we obtain the following equation of motion for the first component \( \Phi_1 \) of the DKP spinor:

\[
\frac{d^2 \Phi_1 (r)}{dr^2} + \frac{\left( (aE + m)^2 \right)}{\alpha^2 r^2} + \frac{E^2 - k z^2 - M^2}{r} \Phi_1 (r) + \frac{\left( \alpha - 1 \right) \Phi'_1 (r)}{\alpha r} + \Phi''_1 (r) = 0
\]

(17)

Then (17) changes to

\[
\begin{align*}
R''_{nm} (r) + \frac{R'_{nm} (r)}{r} \\
+ \left( E^2 - k z^2 - M^2 - \frac{1 + 4 (aE + m)^2}{4 r^2 \alpha^2} \right) R (r)
\end{align*}
\]

(19)

By the change of variable \( r = x \eta \), we can write (19) in the form

\[
\begin{align*}
R''_{nm} (x) + \frac{R'_{nm} (x)}{x} \\
+ \left( 1 - \frac{\lambda^2}{x^2} \right) R (x) = 0
\end{align*}
\]

(20)

where \( \lambda = \sqrt{(1 + 4 (aE + m)^2)/4 \alpha^2} \) and \( \eta = \sqrt{(E^2 - k z^2 - M^2)^{-1/2}} \). The physical solution of (20) is Bessel \( J \) function. Therefore the general solution to (20) is given by

\[
R (r) = A_{\lambda \eta} Y_{\lambda} \left( \frac{r}{\eta} \right) + B_{\lambda \eta} J_{\lambda} \left( \frac{r}{\eta} \right)
\]

(21)

where \( Y_{\lambda}(r/\eta) \) is the Bessel function of the second kind. Sometimes this family of functions is also called Neumann functions or Weber functions. \( J_{\lambda}(r/\eta) \) is the Bessel function of the first kind, given by

\[
J_{\nu} (x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma (k + \nu + 1)} \left( \frac{x}{2} \right)^{2k+\nu}
\]

(22)

and \( Y_{\nu}(r/\eta) \) is the Bessel function of the second kind, given by

\[
Y_{\nu} (x) = \frac{J_{\nu} (x) \cos (\nu r) - J_{-\nu} (x)}{\sin (\nu r)}
\]

(23)

By considering the boundary condition for (21) such that \( B_{\lambda \eta} = 0 \), we find

\[
R (r) = A_{\lambda \eta} J_{\lambda} \left( \frac{r}{\eta} \right)
\]

(24)

3. The DKP Oscillator in Spinning Cosmic String Background

The DKP oscillator is introduced via the nonminimal substitution [17, 30, 31]

\[
\frac{1}{i} \nabla_\alpha \longrightarrow \frac{1}{i} \nabla_\alpha - i M \omega \eta_0 \tau
\]

(25)

where \( \omega \) is the oscillator frequency, \( M \) is the mass of the boson already found in (4), and \( \overline{\nabla} \) is defined in (5). We consider
only the radial component in the nonminimal substitution. The DKP equation (4) leads to the five equations:

\[
\begin{align*}
(r \alpha ( - M \Phi_1 (r) + E \Phi_2 (r) + k z \Phi_3 (r) ) + (\cos \varphi (a E + m) \Phi_4 (r) + \alpha (1 + r^2 M \omega) \Phi_5 (r)) - \sin \varphi (a E + m) \Phi_3 (r) - i(1 - \alpha + r^2 \alpha M \omega) \Phi_4 (r) + \alpha (1 + r^2 M \omega) \Phi_5 (r)) &= 0, \\
(a E + m) \sin \varphi \Phi_1 (r) - M r \alpha \Phi_3 (r) + i(r^2 \alpha M \omega) \Phi_1 (r) + \alpha (1 - \alpha + r^2 \alpha M \omega) \Phi_2 (r) + \alpha (1 + r^2 M \omega) \Phi_5 (r)) &= 0, \\
(\Phi_2 (r) = \frac{E}{M} \Phi_1 (r)) \\
(\Phi_3 (r) = - \frac{k z}{M} \Phi_1 (r)) \\
(\Phi_4 (r) = \frac{-a E \cos \varphi \Phi_1 (r) - m \sin \varphi \Phi_1 (r) + i(r^2 \alpha M \omega) \sin \varphi \Phi_2 (r) + \alpha \sin \varphi \Phi_1 (r)}{M r \alpha}) \\
(\Phi_5 (r) = \frac{a E \sin \varphi \Phi_1 (r) + m \sin \varphi \Phi_2 (r) + i(r^2 \alpha M \omega \cos \varphi \Phi_1 (r) + \alpha \cos \varphi \Phi_1 (r)}{M r \alpha})
\end{align*}
\]

By solving the above system of (26) in favour of \( \Phi_1 \) we get

\[
\Phi_2 (r) = \frac{E}{M} \Phi_1 (r) \\
\Phi_3 (r) = - \frac{k z}{M} \Phi_1 (r) \\
\Phi_4 (r) = \frac{-a E \cos \varphi \Phi_1 (r) - m \sin \varphi \Phi_1 (r) + i(r^2 \alpha M \omega) \sin \varphi \Phi_2 (r) + \alpha \sin \varphi \Phi_1 (r)}{M r \alpha} \\
\Phi_5 (r) = \frac{a E \sin \varphi \Phi_1 (r) + m \sin \varphi \Phi_2 (r) + i(r^2 \alpha M \omega \cos \varphi \Phi_1 (r) + \alpha \cos \varphi \Phi_1 (r)}{M r \alpha}
\]

Combining these results we obtain (23) of motion for the first component of the DKP spinor:

\[
\begin{align*}
\Phi''_1 (r) + \left( \frac{-1 + \alpha}{r \alpha} \Phi'_1 (r) \right) + \left( E^2 - k z^2 - M^2 + 2 M \omega - \frac{4 a E + m}{4 r^2 \alpha^2} - r^2 M^2 \omega^2 \right) \Phi_1 (r) &= 0, \\
(28)
\end{align*}
\]

Let us take \( \Phi_1 \) as

\[
\Phi_1 = r^{1/2 \alpha} R_{n,k} (r). \\
(29)
\]

Then (28) changes to

\[
\begin{align*}
R''_{nm} (r) + \left( \frac{R'_{nm} (r)}{r R_{nm} (r)} \right) + \left( E^2 - k z^2 - M^2 + 2 M \omega - \frac{4 a E + m}{4 r^2 \alpha^2} - r^2 M^2 \omega^2 \right) R_{nm} (r) &= 0, \\
(30)
\end{align*}
\]

In order to solve the above equation, we employ the change of variable: \( s = r^2 \); thus we rewrite the radial equation (34) in the form

\[
\begin{align*}
R''_{nm} (s) + \frac{1}{s} R'_{nm} (s) + \frac{1}{s^2} \left( - \xi_1 s^2 + \xi_2 s - \xi_3 \right) R_{nm} (r) &= 0, \\
(31)
\end{align*}
\]

If we compare with this second-order differential equation with the Nikiforov-Uvarov (NU) form, given in (A.1) of Appendix, we see that

\[
\begin{align*}
\xi_1 &= \frac{M^2 \omega^2}{4} \\
\xi_2 &= \frac{1}{4} \left( E^2 - k z^2 - M^2 + 2 M \omega - \frac{M \omega}{\alpha} \right), \\
\xi_3 &= \frac{1 - \alpha^2 + 4 (a E + m)^2}{16 \alpha^2}
\end{align*}
\]

which gives the energy levels of the relativistic DKP equation from

\[
(2n + 1) \sqrt{\xi_1 - \xi_2 + 2 \sqrt{\xi_3 \xi_1}} = 0, \\
(33)
\]

where

\[
\begin{align*}
\alpha_1 &= 1, \\
\alpha_2 &= \alpha_3 = \alpha_4 = \alpha_5 = 0, \\
\alpha_6 &= \xi_1, \\
\alpha_7 &= -\xi_2, \\
\alpha_8 &= \xi_3, \\
\alpha_9 &= \xi_1, \\
\alpha_{10} &= 1 + 2 \sqrt{\xi_3},
\end{align*}
\]
$\alpha_{11} = 2\sqrt{\xi_1}$

$\alpha_{12} = \sqrt{\xi_3}$.

$\alpha_{13} = -\sqrt{\xi_1}$. \hfill (34)

As the final step, it should be mentioned that the corresponding wave function is

$$R_{nm}(r) = N r^{\alpha_{12}-\alpha_{11}} e^{\alpha_{12} r^2} L_n^{\alpha_{12}-\alpha_{11}}(\alpha_{11} r^2).$$ \hfill (35)

where $N$ is the normalization constant. In limit $a \to 0$ we have the usual metric in cylindrical coordinates where described by the line element

$$ds^2 = -dt^2 + dr^2 + \alpha^2 r^2 d\phi^2 + dz^2$$ \hfill (36)

as pointed out by authors in [17] dynamic of DKP oscillator in the presence of this metric describe by

$$\varphi''(r) + \frac{\alpha - 1}{\alpha r} \varphi'(r) + \left( E^2 - M^2 - k z^2 \right. \left. + \frac{(2\alpha - 1) M \omega}{\alpha} \frac{m^2}{\alpha^2 r^2} - M^2 \omega^2 r^2 \right) \varphi_1(r) = 0$$ \hfill (37)

and the corresponding wave function is

$$\varphi(r) = N r^{2A} e^{Br^2} L_n^{C-1}(Dr^2)$$ \hfill (38)

where $A, B, C,$ and $D$ are constant and $L_n^{C-1}$ denotes the generalized Laguerre polynomial. In Figure 1, $\Phi_1(r)$ is plotted versus $r$ for different quantum number with the parameters listed under it. The density of probability $|\Phi_1|^2$ is shown in Figure 2. The negative and positive solution of energy versus $\alpha$ is shown in Figures 3 and 4 for $n = 1, 5$ and 10. As in Figures 3 and 4, we observe that the absolute value of energy decreases with $\alpha$. Also in Figure 5 energy is plotted versus $\omega$ for quantum numbers. We see that absolute value of energy increases with $\omega$. The negative and positive solution of energy versus $n$ is shown in Figure 6 for different parameter $\alpha$. We obtained the energy levels of the DKP oscillator in that background and observed that the energy increases with the level number. In Figure 7, energy is plotted versus $a$ for different quantum numbers. We see energy increases with parameter $a$. Also we observed that the energy levels of the...
DKP oscillator in that background increases with the level number.

4. Conclusion

The overall objective of this paper is the study of the relativistic quantum dynamics of a DKP oscillator field for spin-0 particle in the spinning cosmic string space-time. The line element in this background is obtained by coordinate transformation of Cartesian coordinate. The metric has off diagonal terms which involves time and space. We considered the covariant form of DKP equation in the spinning cosmic string background and obtained the solutions of DKP equation for spin-0 bosons. Second we introduced DKP oscillator via the nonminimal substitution and considered DKP oscillator in that background. From the corresponding DKP equation, we obtained a system of five equations. By
combining the results of this system we obtained a second-order differential equation for first component of DKP spinor that the solutions are Laguerre polynomials. We see that the results are dependent on the linear mass density of the cosmic string. In the limit case of \( \alpha = 0 \) and \( \alpha = 1 \), i.e., in the absence of a topological defect, recover the general solution for flat space-time. We plotted \( \Phi(r) \), for \( n = 1,2 \). We examined the behaviour of the density of probability \( |\Phi|^2 \). We observed that \( |\Phi|^2 \) for any parameter by increasing \( r \) it tends to zero. We obtained the behaviour of energy spectrum as a function of \( \alpha \). We see that the absolute value of energy decreases as \( \alpha \) increasing.

### Appendix

**Nikiforov-Uvarov (NU) Method**

The Nikiforov-Uvarov method is helpful in order to find eigenvalues and eigenfunctions of the Schrödinger equation, as well as other second-order differential equations of physical interest. More details can be found in [48, 49]. According to this method, the eigenfunctions and eigenvalues of a second-order differential equation with potential are

\[
\Phi''(s) + \frac{\alpha_1 - \alpha_2 s}{s(1 - \alpha_3 s)} \Phi'(s) + \frac{1}{(s - 1)(s - \alpha_2 s)} \left(-\xi_1 s^2 + \xi_2 s - \xi_3\right) \Phi(s) = 0
\]  

(A.1)

According to the NU method, the eigenfunctions and eigenenergies, respectively, are

\[
\Phi(s) = s^{\alpha_1} (1 - \alpha_3 s)^{-\alpha_1 - (\alpha_1 + \alpha_2)/\alpha_3} p_n^{(\alpha_1 - 1, \alpha_2/\alpha_3, -\alpha_1 - 1)} (1 - 2\alpha_3 s)
\]  

(A.2)

and

\[
\alpha_2 n - (2n + 1) \alpha_3 + (2n + 1) \left(\sqrt{\alpha_3} + \alpha_3 \sqrt{\alpha_2} \right)
\]

\[
+ n(n - 1) \alpha_3 \alpha_5 + 2\alpha_3 \alpha_6 + 2\sqrt{\alpha_3 \alpha_5} = 0
\]  

(A.3)

where

\[
\alpha_1 = \frac{1}{2} (1 - \alpha_1),
\]

\[
\alpha_2 = \frac{1}{2} \left(\alpha_2 - 2\alpha_3\right),
\]

\[
\alpha_3 = \alpha_3 + \xi_1,
\]

\[
\alpha_4 = 2\alpha_3 \alpha_5 - \xi_2,
\]

\[
\alpha_5 = \alpha_4^2 + \xi_3,
\]

\[
\alpha_6 = \alpha_3 \alpha_7 + \alpha_2^2 \alpha_6 + \alpha_6,
\]

\[
\alpha_{10} = \alpha_1 + 2\alpha_2 + 2\sqrt{\alpha_3 \alpha_5},
\]

\[
\alpha_{11} = \alpha_2 - 2\alpha_5 + 2\sqrt{\alpha_3 \alpha_5}.
\]

(A.4)

In the rather more special case of \( \alpha = 0 \),

\[
\lim_{\alpha \to 0} p_n^{(\alpha_1 - 1, \alpha_2/\alpha_3, -\alpha_1 - 1)} (1 - 2\alpha_3 s) = L_n^{\alpha_1 - 1} (\alpha_{11} s)
\]  

(A.5)

and, from (14a), we find for the wave function

\[
\Phi(s) = s^{\alpha_1} e^{\alpha_3 s} L_n^{\alpha_1 - 1} (\alpha_{11} s)
\]  

(A.6)

where \( L_n^{\alpha_1 - 1} \) denotes the generalized Laguerre polynomial.

### Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

### References


