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Research Article

Cosmologies with Scalar Fields from Higher Dimensions Applied to Bianchi Type $\mathrm{VI}_{h=-1}$ Model: Classical and Quantum Solutions

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We construct an effective four-dimensional model by compactifying a ten-dimensional theory of gravity coupled with a real scalar dilaton field on a time-dependent torus. The corresponding action in four dimensions is similar to the action of K-essence theories. This approach is applied to anisotropic cosmological Bianchi type $VI_{(h=-1)}$ model for which we study the classical coupling of the anisotropic scale factors with the two real scalar moduli produced by the compactification process. The classical Einstein field equations give us a hidden symmetry, corresponding to the equality between two radii B=C, which permits us to solve exactly the equations of motion. One relation between the scale factors (A,C) via the solutions is found. With this hidden symmetry, then we solve the FRW model, finding that the scale factor goes to B radii. Also the corresponding Wheeler-DeWitt (WDW) equation in the context of Standard Quantum Cosmology is solved, building a wavepacket when the scalar fields have a hyperbolic behavior, obtaining some qualitative results when we analyze the projection plane to the wall formed by the probability density. Bohm's formalism for this cosmological model is revisited too.

1. Introduction

One of the most important things that we have learned from Planck's results [1] is related to the little anisotropies of the universe. The evidence given by these data leads us to the possibility of considering that there is no exact isotropy, since there exist small anisotropy deviations of the CMB radiation and apparent large angle anomalies. In that context, there have been recent attempts to fix constraints on such deviations by using the Bianchi anisotropic models [2]. The basic idea behind these models is to consider the present observational anisotropies and anomalies as imprints of an early anisotropic phase on the CMB which in turn can be explained by the use of different Bianchi models. In particular, Bianchi I model seems to be related to large angle anomalies [2] (and references therein).

The above problems have suggested considering the presence of higher-dimensional degrees of freedom in the cosmology derived from four-dimensional effective theories. Some features of the presence of higher-dimensional effective theories is to consider an effective action with a graviton and a massless scalar field, the dilaton, describing the evolution of the universe [3, 4]. On the other hand, it is well known that relativistic theories of gravity, such as general relativity or string theories, are invariant under reparametrization of time. The quantization of such theories presents a number of problems of principle; one of them is known as the "the problem of time" [5, 6]. This problem is present in all systems, whose classical version is invariant under time reparametrization, leading to its absence at the quantum level. Therefore, the formal question involves how to handle the classical Hamiltonian constraint, $\mathscr{H} \approx 0$, in the quantum

theory. Also, connected with the problem of time is the "Hilbert space problem" [5, 6] referring to the not-at-all-obvious selection of the inner product of states in quantum gravity and whether there is a need for such a structure at all.

In the present work we shall consider an alternative procedure about the role played by the moduli. In particular we shall not consider the presence of fluxes, as in string theory, in order to obtain a moduli-dependent scalar potential in the effective theory. Rather, we are going to promote some of the moduli to time-dependent by considering the particular case of a ten-dimensional gravity coupled to a time-dependent dilaton, compactified on a six-dimensional torus with a time-dependent Kähler modulus. With the purpose of tracking down the role played by such fields, we are going to ignore the dynamics of the complex structure field (for instance, by assuming that it is already stabilized by the presence of a string field in higher scales).

It is very well known that the problem of time is present in all quantum cosmological models [5, 6]. There are some attempts to recover the notion of time for FRW models with matter given by a perfect fluid for an arbitrary barotropic equation of state under the scheme of quantum cosmology (see [7] for more details).

The work is organized in the following form. In Section 2 we present the construction of our effective action by compactification on a time-dependent isotropic torus, where the moduli field σ appears in the internal metric, where we introduce the perfect fluid as first approximation since the fluid in 10D, as a toy model. In Section 2.1 we present the Einstein-Klein-Gordon-like equations in general way and also as an application to the Bianchi type $\mathrm{VI}_{\mathrm{h}=-1}$ case in terms of the radii of the cosmological model. In Section 2.2 we write the same equations using the Misner parameterization, and some results could modify those obtained in the quantum version presented 20 years ago, using this same cosmological model using the Bohm approach. In Section 3 we built its Lagrangian and Hamiltonian descriptions using as a toy model with this Bianchi model and, using the tools of analytical mechanics, we solve for the set of parameters that appear in this cosmological model, $(\mathrm{A}, \mathrm{C}, \mathrm{P_A}, \mathrm{P_C}, \phi, \sigma)$, and when we introduce these results in the set of Einstein-Klein-Gordon-like equations, these are fulfilled. Also we find that the classical initial singularity is avoided in this Bianchi, via the analysis to the solutions. In Section 3.1 we include the isotropic flat FRW to analyze if this cosmological model presents one different behavior in this theory, or whether this cosmological model is related to the Bianchi type $\mathrm{VI}_{\mathrm{h}=-1}$ in some sense, we find that the scale factor of the FRW corresponds to the scale factor C to this Bianchi, in some particular sense we argue that this FRW is embedded into this Bianchi because the other scale factor has an exponential relation with the scale factor C, and also the classical singularity is avoided in this model. In Section 4 we present the quantum scheme, by replacing the classical momenta in the classical Hamiltonian density by differentials operators in the configuration space, building the Wheeler-DeWitt equations, similar to stationary Schroedinger equation in standard quantum mechanics. We introduce the basics ideas to solve this equation in general way using the separation variable method, and using a particular ansatz depending on the coordinate fields $(\mathrm{A}, \mathrm{C}, \phi, \sigma)$ as is common in other works. We find the wave function in the μ state, where μ is a separation constant in this process, and then a wave function for this model is built up as superposition of this set of μ states in the continuum. We write the probability density for this model and present a plot in the space (A,C), including the information of the scalar field in one parameter η; we also obtain that the solutions of the moduli fields are the same for all Bianchi Class A cosmological models. In Section 5 we include the quantum Bohm approach applied to quantum cosmology, and this method appears as the WKB-like approximation. Finally our conclusions are presented in Section 6.

2. Effective Model

We start from a ten-dimensional action coupled with a dilaton (which is the bosonic component common to all superstring theories), which after dimensional reduction can be interpreted as a Brans-Dicke like theory [8]. In the string frame, the effective action depends on two space-time-dependent scalar fields: the dilaton $\Phi(x^\mu)$ and the Kähler modulus $\sigma(x^\mu)$. For simplicity, in this work we shall assume that these fields only depend on time. The high-dimensional (effective) theory is therefore given by

$$S = \frac{1}{2\kappa_{10}^2}\int d^{10}X\sqrt{-\widehat{G}}e^{-2\Phi}\left[\widehat{\mathscr{R}}^{(10)} + 4\widehat{G}^{MN}\nabla_M\Phi\nabla_N\Phi\right] + \int d^{10}X\sqrt{-\widehat{G}}\widehat{\mathscr{L}}_m \tag{1}$$

where all quantities $\widehat{q}$ refer to the string frame while the ten-dimensional metric is described by

$$ds^2 = \widehat{G}_{MN}dX^M dX^N = \widehat{g}_{\mu\nu}dx^\mu dx^\nu + h_{mn}dy^m dy^n, \tag{2}$$

where $M, N, P, \ldots$ are the indices of the ten-dimensional space and Greek indices $\mu, \nu, \ldots = 0, \ldots, 3$ and latin indices $m, n, p, \ldots = 4, \ldots, 9$ correspond to the external and internal space, respectively. We will assume that the six-dimensional internal space has the form of a torus with a metric given by

$$h_{mn} = e^{-2\sigma(t)}\delta_{mn}, \tag{3}$$

with σ a real parameter.

Dimensional reduction of the first term in (1) to four dimensions in the Einstein frame (see Appendix A for details) gives

$$S_4 = \frac{1}{2\kappa_4^2}\int d^4x\sqrt{-g}\left(\mathscr{R} - 2g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - 96g^{\mu\nu}\nabla_\mu\sigma\nabla_\nu\sigma - 36g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\sigma\right), \tag{4}$$

where $\phi = \Phi + (1/2)\ln(\widehat{V})$, with $\widehat{V}$ given by

$$\widehat{V} = e^{6\sigma(t)}Vol\left(X_6\right) = \int d^6y \tag{5}$$

being the volume associated with the six-dimensional space. By considering only a time-dependence on the moduli, one can notice that, for the internal volume $Vol(X_6)$ to be small, the modulus $\sigma(t)$ should be a monotonic increasing function on time (recall that σ is a real parameter), while $\widehat{V}$ is time and moduli independent.

Now, concerning the second term in (1), we shall require to properly define the ten-dimensional stress-energy tensor $\widehat{T}_{MN}$. In the string frame it takes the form

$$\widehat{T}_{MN} = \begin{pmatrix} \widehat{T}_{\mu\nu} & 0 \\ 0 & \widehat{T}_{mn} \end{pmatrix}, \tag{6}$$

where $\widehat{T}_{\mu\nu}$ and $\widehat{T}_{mn}$ denote the four- and six-dimensional components of $\widehat{T}_{MN}$. Observe that we are not considering mixing components of $\widehat{T}_{MN}$ among internal and external components although nonconstrained expressions for $\widehat{T}_{MN}$ have been considered in [9]. In the Einstein frame the four-dimensional components are given by

$$T_{\mu\nu} = e^{2\phi}\widehat{T}_{\mu\nu}. \tag{7}$$

It is important to remark that there are some dilemmas about what the best frame to describe the gravitational theory is. Here in this work we have taken the Einstein frame. Useful references to find a discussion about string and Einstein frames and their relationship in the cosmological context are [10–14].

We can mention that the moduli fields will satisfy a Klein-Gordon-like equation in the Einstein frame as an effective theory. This will be possible to appreciate by taking a variation of action (4) with respect to each of the moduli fields. Now with expression (4) we proceed to build up the Lagrangian and the Hamiltonian of the theory at the classical regime employing the anisotropic cosmological Bianchi type $\mathrm{VI}_{h=-1}$ model. This is the subject of Section 2.1. This Bianchi type cosmological models have been employed in other contexts [15–20]. For example, in [16, 18] are applied in scale invariant theory of gravitation, in [17, 20] to extended gravity theory.

2.1. Effective Einstein Equations in Four Dimensions. The equation of motions associated with the reduced action in four dimensions (4) can be obtained by taking variations with respect to each of the fields. So, in this sense we have that the Einstein equations and the Klein-Gordon-like equations (EKG) associated with the fields (ϕ, σ) are given by

$$\begin{aligned} R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = {} & 2\left(\nabla_\alpha\phi\nabla_\beta\phi - \frac{1}{2}g_{\alpha\beta}\nabla^\gamma\phi\nabla_\gamma\phi\right) \\ & + 96\left(\nabla_\alpha\sigma\nabla_\beta\sigma - \frac{1}{2}g_{\alpha\beta}\nabla^\gamma\sigma\nabla_\gamma\sigma\right) \\ & + 36\left(\nabla_\alpha\phi\nabla_\beta\sigma - \frac{1}{2}g_{\alpha\beta}\nabla^\gamma\phi\nabla_\gamma\sigma\right) \\ & + 8\pi G T_{\alpha\beta}, \end{aligned} \tag{8a}$$

$$\Box\phi = g^{\mu\nu}\phi_{,\mu\nu} - g^{\alpha\beta}\Gamma^\nu_{\alpha\beta}\nabla_\nu\phi = 0, \tag{8b}$$

$$\Box\sigma = g^{\mu\nu}\sigma_{,\mu\nu} - g^{\alpha\beta}\Gamma^\nu_{\alpha\beta}\nabla_\nu\sigma = 0, \tag{8c}$$

where $\Box$ is the d'Alembertian operator in four dimensions. Since we are interested in anisotropic background, we are going to assume that the four-dimensional metric $g_{\alpha\beta}$ is described by the Bianchi type $\mathrm{VI}_{h=-1}$ model whose line element can be read as (we write in usual way and in Misner's parameterization)

$$\begin{aligned} \mathrm{ds}^2 = {} & -\mathrm{N}^2\mathrm{dt}^2 + \mathrm{A}^2(\mathrm{t})\,\mathrm{dx}^2 + \mathrm{B}^2(\mathrm{t})\,\mathrm{e}^{-2\mathrm{x}}\mathrm{dy}^2 \\ & + \mathrm{C}^2(\mathrm{t})\,\mathrm{e}^{2\mathrm{x}}\mathrm{dz}^2, \end{aligned} \tag{9}$$

$$\begin{aligned} = {} & -N^2dt^2 + e^{2\Omega-4\beta_+}dx^2 + e^{2\Omega+2\beta_+ + 2\sqrt{3}\beta_- - 2x}dy^2 \\ & + e^{2\Omega+2\beta_+ - 2\sqrt{3}\beta_- + 2x}dz^2, \end{aligned} \tag{10}$$

where N(t) is the lapse function, the functions A(t), B(t), and C(t) are the corresponding scale factors in the directions (x, y, z), respectively, also using Misner's parameterization for the radii in this model,

$$\begin{aligned} \mathrm{A} &= \mathrm{e}^{\Omega-2\beta_+}, \\ \mathrm{B} &= \mathrm{e}^{\Omega+\beta_+ + \sqrt{3}\beta_-}, \\ \mathrm{C} &= \mathrm{e}^{\Omega+\beta_+ - \sqrt{3}\beta_-}. \end{aligned} \tag{11}$$

Writing the field equations (8a), (8b), and (8c) in the background metric (9), we see that the equation of motions (8a) are given by

$$\begin{aligned} \mathrm{G}^0_0\colon\quad & 2\frac{\mathrm{A}'}{\mathrm{A}}\frac{\mathrm{C}'}{\mathrm{C}} + \left(\frac{\mathrm{C}'}{\mathrm{C}}\right)^2 - \phi'^2 - 18\phi'\sigma' \\ & - 48\sigma'^2 - 8\pi\mathrm{G}\rho - \frac{1}{\mathrm{A}^2} = 0, \end{aligned} \tag{12a}$$

$$\mathrm{G}^0_1 = -\mathrm{G}^1_0\colon\quad \frac{\mathrm{B}'}{\mathrm{B}} - \frac{\mathrm{C}'}{\mathrm{C}} = 0, \tag{12b}$$

$$\begin{aligned} \mathrm{G}^1_1\colon\quad & 2\frac{\mathrm{C}''}{\mathrm{C}} + \left(\frac{\mathrm{C}'}{\mathrm{C}}\right)^2 + \phi'^2 + 18\phi'\sigma' + 48\sigma'^2 \\ & + 8\pi\mathrm{GP} + \frac{1}{\mathrm{A}^2} = 0, \end{aligned} \tag{12c}$$

$$\begin{aligned} \mathrm{G}^2_2\colon\quad & \frac{\mathrm{A}''}{\mathrm{A}} + \frac{\mathrm{A}'}{\mathrm{A}}\frac{\mathrm{C}'}{\mathrm{C}} + \frac{\mathrm{C}''}{\mathrm{C}} + \phi'^2 + 18\phi'\sigma' \\ & + 48\sigma'^2 + 8\pi\mathrm{GP} - \frac{1}{\mathrm{A}^2} = 0, \end{aligned} \tag{12d}$$

$$\Box\phi = 0\colon\quad 2\left(\frac{\mathrm{A}'}{\mathrm{A}} + \frac{\mathrm{C}'}{\mathrm{C}}\right)\phi' + \phi'' = 0, \tag{12e}$$

$$\Box\sigma = 0\colon\quad 2\left(\frac{\mathrm{A}'}{\mathrm{A}} + \frac{\mathrm{C}'}{\mathrm{C}}\right)\sigma' + \sigma'' = 0, \tag{12f}$$

the other components can be seen in Appendix B, and $(\ldots)'$ means derivative with respect to the proper time $\mathrm{d}\tau = \mathrm{N(t)dt}$.

From the last expressions, we can see that it is easy to solve the equation (12b), whose solutions is B(t) = C(t)

(see expression (B.1b) of Appendix B for more detailed derivation). From this result, we see also that the components $G_2^2 = G_3^3$ (as we can observe from the expressions (B.1d) and (B.1e)). On the other hand, the solutions for the fields (ϕ, σ) are obtained from (12e) and (12f) whose solutions can be read as

$$\phi(\tau) = \phi_0 \int \frac{d\tau}{AC^2} + \phi_1, \qquad \sigma(\tau) = \sigma_0 \int \frac{d\tau}{AC^2} + \sigma_1. \tag{13}$$

By taking the relation between the proper and cosmic time ($d\tau = N(t)dt$) we see that the lapse function N(t) can be chosen as $N \sim AC^2$. This tells us that the lapse function plays the role of a gauge transformation. So, under this scheme it is possible to obtain a simpler solution in the cosmic time t, as

$$\phi(\tau) = \phi_0 t + \phi_1, \qquad \sigma(\tau) = \sigma_0 t + \sigma_1, \tag{14}$$

where $(\phi_0, \phi_1, \sigma_0, \sigma_1)$ are integration constants.

So far, we have found from the fields equations (8a), (8b), and (8c) that radii B and C are equal for the Bianchi $VI_{h=-1}$, this classical hidden symmetry is relevant in the quantum level, because, 21 years ago, a generic quantum solution was found for all Bianchi Class A cosmological models in Bohm's formalism [21], and in particular for the Bianchi type $VI_{h=-1}$ it was necessary to modify the general structure of the generic solution with a function over the coordinate $\beta_1 - \beta_2$; with this result, the modification is not necessary, due to the fact that this function is a constant now, as we see using Misner's parameterization of this cosmological model. On the other hand, the moduli fields (ϕ, σ) have the linear behavior in time, as we can see in expressions (14). These solutions were obtained by taking the gauge $N \sim AC^2$. This gauge choice will play a great role in the next section, when we deal with the classical Lagrangian and Hamiltonian which we shall obtain in the next section.

2.2. Misner Parameterization. Using the Misner parameterization for the radii in this model,

$$A = e^{\Omega - 2\beta_+}, \qquad B = e^{\Omega + \beta_+ + \sqrt{3}\beta_-}, \qquad C = e^{\Omega + \beta_+ - \sqrt{3}\beta_-}, \tag{15}$$

with this, The EKG classical equations ((8a),(8b),(8c)), using this parameterization, become

$$G_0^0: \quad 3\frac{\dot{\Omega}^2}{N^2} - 3\frac{\dot{\beta}_+^2}{N^2} - 3\frac{\dot{\beta}_-^2}{N^2} - \frac{\dot{\phi}^2}{N^2} - 18\frac{\dot{\phi}\dot{\sigma}}{N^2} - 48\frac{\dot{\sigma}^2}{N^2} - 8\pi G\rho - e^{-2\Omega+4\beta_+} = 0, \tag{16}$$

$$G_1^1: \quad 2\frac{\ddot{\Omega}}{N^2} + 3\frac{\dot{\Omega}^2}{N^2} + 6\frac{\dot{\Omega}\dot{\beta}_+}{N^2} - 2\frac{\dot{\Omega}\dot{N}}{N^3} + 2\frac{\ddot{\beta}_+}{N^2} + 3\frac{\dot{\beta}_+^2}{N^2} - 2\frac{\dot{\beta}_+\dot{N}}{N^3} + 3\frac{\dot{\beta}_-^2}{N^2} + \frac{\dot{\phi}}{N^2} + 18\frac{\dot{\phi}\dot{\sigma}}{N^2} + 48\frac{\dot{\sigma}^2}{N^2} + 8\pi GP + e^{-2\Omega+4\beta_+} = 0, \tag{17}$$

$$G_2^2: \quad 2\frac{\ddot{\Omega}}{N^2} + 3\frac{\dot{\Omega}^2}{N^2} - 3\frac{\dot{\Omega}\dot{\beta}_+}{N^2} - 2\frac{\dot{\Omega}\dot{N}}{N^3} - 3\sqrt{3}\frac{\dot{\Omega}\dot{\beta}_-}{N^2} - \frac{\ddot{\beta}_+}{N^2} + 3\frac{\dot{\beta}_+^2}{N^2} + \frac{\dot{\beta}_+\dot{N}}{N^3} - \sqrt{3}\frac{\ddot{\beta}_-}{N^2} + 3\frac{\dot{\beta}_-^2}{N^2} + \sqrt{3}\frac{\dot{\beta}_-\dot{N}}{N^3} + \frac{\dot{\phi}}{N^2} + 18\frac{\dot{\phi}\dot{\sigma}}{N^2} + 48\frac{\dot{\sigma}^2}{N^2} + 8\pi GP - e^{-2\Omega+4\beta_+} = 0, \tag{18}$$

$$G_3^3: \quad 2\frac{\ddot{\Omega}}{N^2} + 3\frac{\dot{\Omega}^2}{N^2} - 3\frac{\dot{\Omega}\dot{\beta}_+}{N^2} - 2\frac{\dot{\Omega}\dot{N}}{N^3} + 3\sqrt{3}\frac{\dot{\Omega}\dot{\beta}_-}{N^2} - \frac{\ddot{\beta}_+}{N^2} + 3\frac{\dot{\beta}_+^2}{N^2} + \frac{\dot{\beta}_+\dot{N}}{N^3} + \sqrt{3}\frac{\ddot{\beta}_-}{N^2} + 3\frac{\dot{\beta}_-^2}{N^2} - \sqrt{3}\frac{\dot{\beta}_-\dot{N}}{N^3} + \frac{\dot{\phi}}{N^2} + 18\frac{\dot{\phi}\dot{\sigma}}{N^2} + 48\frac{\dot{\sigma}^2}{N^2} + 8\pi GP - e^{-2\Omega+4\beta_+} = 0, \tag{19}$$

$$G_{01} = G_{10}: \quad 2\sqrt{3}\dot{\beta}_- = 0, \tag{20}$$

$$\Box\phi = 0: \quad -3\frac{\dot{\Omega}}{N}\frac{\dot{\phi}}{N} - \frac{\ddot{\phi}}{N^2} + \frac{\dot{\phi}}{N}\frac{\dot{N}}{N^2} = 0, \tag{21}$$

$$\Box\sigma = 0: \quad -3\frac{\dot{\Omega}}{N}\frac{\dot{\sigma}}{N} - \frac{\ddot{\sigma}}{N^2} + \frac{\dot{\sigma}}{N}\frac{\dot{N}}{N^2} = 0. \tag{22}$$

Equation (20) implies that $\beta_- = \beta_0 =$ constant, then the last set of equations is read as (using the time parametrization $d\tau = Ndt$, $\prime = d/d\tau$)

$$G_0^0: \quad 3\Omega'^2 - 3\beta_+'^2 - \phi'^2 - 18\phi'\sigma' - 48\sigma'^2 - 8\pi G\rho - e^{-2\Omega+4\beta_+} = 0, \tag{23}$$

$$G_1^1: \quad 2\Omega'' + 3\Omega'^2 + 6\Omega'\beta_+' + 2\beta_+'' + 3\beta_+'^2 + \phi' + 18\phi'\sigma' + 48\sigma'^2 + 8\pi GP + e^{-2\Omega+4\beta_+} = 0, \tag{24}$$

$$G_2^2: \quad 2\Omega'' + 3\Omega'^2 - 3\Omega'\beta_+' - \beta_+'' + 3\beta_+'^2 + \phi' + 18\phi'\sigma' + 48\sigma'^2 + 8\pi GP - e^{-2\Omega+4\beta_+} = 0, \tag{25}$$

$$3\Omega'\phi' + \phi'' = 0, \tag{26}$$

$$3\Omega'\sigma' + \sigma'' = 0, \tag{27}$$

The solutions for the fields (ϕ, σ) are obtained from equations ((26), (27)) as

$$\begin{aligned} \phi(\tau) &= \phi_0 \int e^{-3\Omega} d\tau + \phi_1, \\ \sigma(\tau) &= \sigma_0 \int e^{-3\Omega} d\tau + \sigma_1, \end{aligned} \tag{28}$$

that in the gauge $N \sim e^{3\Omega}$, we have the simplest solution in the cosmic time t, as

$$\begin{aligned} \phi(\tau) &= \phi_0 t + \phi_1, \\ \sigma(\tau) &= \sigma_0 t + \sigma_1, \end{aligned} \tag{29}$$

where $(\phi_0, \phi_1, \sigma_0, \sigma_1)$ are integration constants, which are the same solution as in the previous parametrization of the metric.

The algebraic structure of the EKG equations does not allow us to solve this last equation in exact manner, so, in order to do so it is necessary to use another method. In the following section we will implement Hamilton's approach to obtain the exact solutions of these equations, Also, this formalism is useful when we make the quantization of the model.

3. Classical Lagrangian and Hamiltonian

We have found in the last section that the Bianchi $VI_{h=-1}$ has two equal radii. With the idea of reaching the quantum regime of our model, we shall develop the classical Lagrangian and Hamiltonian analysis. Through this picture we will find that we have one conserved quantity and this give us a first constraint in the Hamiltonian formulation. We start by looking for the Lagrangian density for the matter content, and this is given by a perfect fluid as a first approximation in this formalism, whose stress-energy tensor is [22, 23]

$$T_{\mu\nu} = (p + \rho) u_\mu u_\nu + p g_{\mu\nu}, \tag{30}$$

which satisfies the conservation law $\nabla_\nu T^{\mu\nu} = 0$. Taking the barotropic equation of state $p = \gamma\rho$ between the energy density ρ and the pressure p of the comoving fluid, we see that a solution is given by $\rho = C_\gamma (AC^2)^{-(1+\gamma)}$ with C_γ the corresponding constant for different values of γ related to the universe evolution stage. Then, the Lagrangian density for the matter content reads

$$\mathscr{L}_{\text{matter}} = 16\pi G_N \sqrt{-g}\rho = 16\pi N G_N \rho_\gamma \left(AC^2\right)^{-(1+\gamma)}, \tag{31}$$

and then the Lagrangian that describes the fields dynamics is given by

$$\begin{aligned} \mathscr{L}_{VI} = {} & 2\frac{A\dot{C}^2}{N} + 4\frac{C\dot{A}\dot{C}}{N} + 2\frac{AC^2\dot{\phi}^2}{N} + 36\frac{AC^2\dot{\phi}\dot{\sigma}}{N} \\ & + 96\frac{AC^2\dot{\sigma}^2}{N} + 2\frac{C^2}{A}N + 16\pi G A C^2 \rho N. \end{aligned} \tag{32}$$

In the last expression we have employed the result that radii B and C are equal. Using the standard definition of the momenta $\Pi_{q^\mu} = \partial\mathscr{L}/\partial\dot{q}^\mu$, where q^μ are the coordinate fields $q^\mu = (A, C, \phi, \sigma)$ we obtain the momenta associated with each field

$$\begin{aligned} \Pi_A &= \frac{4C\dot{C}}{N}, \\ \dot{C} &= \frac{N\Pi_A}{4C}, \\ \Pi_C &= 4\frac{A\dot{C}}{N} + 4\frac{C\dot{A}}{N}, \\ \dot{A} &= \frac{N}{4AC^2}\left[C\Pi_C - A\Pi_A\right], \\ \Pi_\phi &= \frac{AC^2}{N}\left[36\dot{\sigma} + 4\dot{\phi}\right], \\ \dot{\phi} &= \frac{N}{44AC^2}\left[3\Pi_\sigma - 16\Pi_\phi\right], \\ \Pi_\sigma &= \frac{AC^2}{N}\left[192\dot{\sigma} + 36\dot{\phi}\right], \\ \dot{\sigma} &= \frac{N}{132AC^2}\left[9\Pi_\phi - \Pi_\sigma\right], \end{aligned} \tag{33}$$

and, introducing them into the Lagrangian density, we obtain the canonical Lagrangian as $\mathscr{L}_{canonical} = \Pi_{q^\mu}\dot{q}^\mu - N\mathscr{H}$. When we perform the variation of this canonical Lagrangian with respect to N, $\delta\mathscr{L}_{canonical}/\delta N = 0$, we obtain the constraint $\mathscr{H} = 0$. In our model the only constraint corresponds to Hamiltonian density, which is weakly zero. So, we obtain the Hamiltonian density for this model

$$\begin{aligned} \mathscr{H}_{VI} = {} & \frac{1}{8AC^2}\Big[-A^2\Pi_A^2 + 2A\Pi_A C\Pi_C - \frac{16}{11}\Pi_\phi^2 \\ & + \frac{6}{11}\Pi_\phi\Pi_\sigma - \frac{1}{33}\Pi_\sigma^2 - 128\pi G_N \rho_\gamma \left(AC^2\right)^{1-\gamma} \\ & - 16C^4\Big]. \end{aligned} \tag{34}$$

The last Hamiltonian can be rewritten in a simpler way by taking the transformations related to the generalized coordinates and momenta as $Q = e^q$, and $\Pi_Q = \partial S/\partial Q$. So we see that the momenta can be transformed as follows $\Pi_A \longrightarrow \Pi_A = e^{-a}P_a$, where $P_a = \partial S/\partial a$, and we write in similar way the other momenta in the field (ϕ, σ), is say $\Pi_\phi = P_\phi$ and $\Pi_\sigma = P_\sigma$. Now, we develop our analysis for the case when the matter content is taken as a stiff fluid, $\gamma = 1$. In this case the Hamiltonian density takes the following form:

$$\begin{aligned} \mathscr{H} = {} & -P_a^2 + 2P_aP_c - \frac{16}{11}P_\phi^2 + \frac{6}{11}P_\phi P_\sigma - \frac{1}{33}P_\sigma^2 - 16e^{4c} \\ & - 128\pi G_N \rho_1, \end{aligned} \tag{35}$$

where we have used the gauge transformation $N = 8AC^2$. By using the Hamilton equations for the momenta $\dot{P}_\mu = -\partial\mathscr{H}/\partial q^\mu$ and coordinates $\dot{q}^\mu = \partial\mathscr{H}/\partial P_\mu$, we have

$$\dot{a} = -2P_a + 2P_c, \tag{36a}$$

$$\dot{c} = 2P_a, \tag{36b}$$

$$\dot{\phi} = -\frac{32}{11}P_\phi + \frac{6}{11}P_\sigma, \tag{36c}$$

$$\dot{\sigma} = \frac{6}{11}P_\phi - \frac{2}{33}P_\sigma, \tag{36d}$$

$$\begin{aligned} \dot{P}_a &= 0, \Longrightarrow \\ P_a &= p_a = \text{constant}, \end{aligned} \tag{36e}$$

$$\dot{P}_c = 64e^{4c}, \tag{36f}$$

$$\begin{aligned} \dot{P}_\phi &= 0, \Longrightarrow \\ P_\phi &= p_\phi = \text{constant}, \end{aligned} \tag{36g}$$

$$\begin{aligned} \dot{P}_\sigma &= 0, \Longrightarrow \\ P_\sigma &= p_\sigma = \text{constant}. \end{aligned} \tag{36h}$$

From the last system of differential equations it is possible to solve the equations (36b) by taking the solution associated with the momenta P_a which is constant. We have $c(t) = 2p_a t + c_o$, and, by transforming this solution to the original generalized coordinate, we obtain $C(t) = C_0 e^{2p_a t}$. On the other hand, to solve the momenta P_c we use the Hamiltonian density to obtain $16e^{4c} = 2P_a P_c + \alpha$, where $\alpha = -p_a^2 - (16/11)p_\phi^2 + (6/11)p_\phi p_\sigma - (1/33)p_\sigma^2 - 128\pi G_N \rho_1$ is a constant. In this way, we have

$$\begin{aligned} \dot{P}_c &= 4\alpha + 8p_a P_c, \\ P_c &= \frac{P_0}{8p_a}e^{8p_a t} - \frac{\alpha}{2p_a}, \end{aligned} \tag{37}$$

with P_0 an integration constant. So, by reintroducing this one into the last system of differential equations it is possible to solve them

$$\begin{aligned} c(t) &= 2p_a t + c_o, \longrightarrow \\ C(t) &= C_0 e^{2p_a t} \\ a(t) &= a_0 + \frac{b_0}{4p_a}t + \frac{P_0}{32p_a^2}e^{8p_a t}, \\ A(t) &= A_0 e^{(b_0/4p_a)t}\exp\left[2\frac{C_0^4}{p_a^2}e^{8p_a t}\right]. \\ \phi(\tau) &= \phi_0 t + \phi_1, \\ \sigma(\tau) &= \sigma_0 t + \sigma_1, \end{aligned} \tag{38}$$

where (ϕ_1, σ_1) are integration constants, whereas $\phi_0 = -(32/11)p_\phi + (6/11)p_\sigma$ and $\sigma_0 = (6/11)p_\phi - (2/33)p_\sigma$ and the constant $b_0 = \phi_0^2 + 18\phi_0\sigma_0 + 512\pi G\rho_1 - 4P_a^2 + 48\sigma_0^2$, and C_0 is an integration constant. The last solutions were introduced in the Einstein field equation, and it was found that they are satisfied when $P_0 = 64C_0^4$.

Therefore, radii A(t) and C(t) have the following behavior:

$$\begin{aligned} A(t) &= A_0 e^{(b_0/4p_a)t}\exp\left[2\frac{C_0^4}{p_a^2}e^{8p_a t}\right], \\ C(t) &= C_0 e^{2p_a t}. \end{aligned} \tag{39}$$

existing the following relation between these scale factors

$$A(t) = A_0 e^{(b_0/4p_a)t}\exp\left[2\frac{1}{p_a^2}C^4\right], \tag{40}$$

is say, for a fixed time, the scale factor A goes as an exponential function on C^4. Using this result we argue that the classical initial singularity in this theory is avoided, having a initials values in its radii, (A_0, C_0).

In the frame of classical analysis we obtained that two radii of the Bianchi $VI_{h=-1}$ type are equal ($B(t) = C(t)$), presenting a hidden symmetry, which say that the volume function of this cosmological model goes as a jet in the x direction, whereas in the (y,z) plane it goes as a circle. This hidden symmetry allows us to simplify the analysis in the classical context, and, as a consequence, the quantum version will be simplified.

3.1. Flat FRW versus Bianchi Type $VI_{h=-1}$. In this subsection we explore the FRW case in this classical scheme, finding the following.

Solving the standard flat FRW cosmological model in this proposal, we employed the usual metric

$$\begin{aligned} ds^2 &= -N(t)^2 dt^2 \\ &\quad + a(t)^2\left[dr^2 + r^2 d\theta^2 + r^2\sin^2(\theta)\, d\phi^2\right], \end{aligned} \tag{41}$$

where a(t) is the scale factor in this model. Equation (30) gives the energy density $\rho_{FRW} = M_\gamma a^{-3(\gamma+1)}$, and the lagrangian density is written as

$$\begin{aligned} \mathscr{L}_{FRW} &= 6\frac{a\dot{a}^2}{N} + 2\frac{a^3\dot{\phi}^2}{N} + 36\frac{a^3\dot{\phi}\dot{\sigma}}{N} + 96\frac{a^3\dot{\sigma}^2}{N} \\ &\quad + 16\pi G a^3 \rho_{FRW} N \end{aligned} \tag{42}$$

and, with the corresponding Hamiltonian density in the gauges $N = 24a^3$ and $\gamma = 1$, we use the transformation $a = e^A$,

$$\mathscr{H}_{FRW} = -P_A^2 - \frac{48}{11}P_\phi^2 + \frac{18}{11}P_\phi P_\sigma - \frac{1}{11}P_\sigma^2 - c_1, \tag{43}$$

with $c_1 = 384\pi G M_1$. The Hamilton equations are

$$\dot{A} = -2P_A, \tag{44a}$$

$$\dot{\phi} = -\frac{96}{11}P_\phi + \frac{18}{11}P_\sigma, \tag{44b}$$

$$\dot{\sigma} = \frac{18}{11}P_\phi - \frac{2}{11}P_\sigma, \tag{44c}$$

$$\begin{aligned} \dot{P}_A &= 0, \Longrightarrow \\ P_A &= p_A = \text{constant}, \end{aligned} \tag{44d}$$

$$\begin{aligned} \dot{P}_\phi &= 0, \Longrightarrow \\ P_\phi &= p_\phi = \text{constant}, \end{aligned} \tag{44e}$$

$$\begin{aligned} \dot{P}_\sigma &= 0, \Longrightarrow \\ P_\sigma &= p_\sigma = \text{constant}. \end{aligned} \tag{44f}$$

The solutions for the moduli fields are

$$\begin{aligned} \phi &= \varphi_0 t + \varphi_1, \\ \varphi_{0\text{FRW}} &= -\frac{96}{11}p_\phi + \frac{18}{11}p_\sigma = 3\left(-\frac{32}{11}p_\phi + \frac{6}{11}p_\sigma\right) \\ &= 3\phi_{0(\text{VI}_{h=-1})}, \\ \sigma &= \xi_0 t + \xi_1, \\ \xi_{0\text{FRW}} &= \frac{18}{11}p_\phi - \frac{2}{11}p_\sigma = 3\left(-\frac{2}{33}P_\sigma + \frac{6}{11}P_\phi\right) \\ &= 3\sigma_{0(\text{VI}_{h=-1})}, \end{aligned} \tag{45}$$

and the scale factor A, solving (44a), is

$$\begin{aligned} A(t) &= 2p_A t + A_1, \Longrightarrow \\ a(t) &= a_0 e^{2p_A t} \end{aligned} \tag{46}$$

where $p_A = \sqrt{(4608\pi G M_1 + \varphi_{0(\text{FRW})}^2 + 18\varphi_{0(\text{FRW})}\xi_{0(\text{FRW})} + 48\xi_{0(\text{FRW})}^2)/12}$. This solution is similar at the radii C=B in the Bianchi type VI, studied in the previous case. We can see that the projection at the spherical universe to flat FRW yields in the circle formed for the radii C=B in the Bianchi type $\text{VI}_{h=-1}$. With this result, we can infer that the flat FRW cosmological model is embedded in this anisotropic cosmological Bianchi type model. Also, this cosmological model avoids the classical initial singularity in this theory in the beginning of the universe, having initial radio a_0.

4. Quantum Scheme

In the absence of a complete theory of quantum gravity, it is relevant to try their quantum cosmological version within the approach of canonical quantization, in sense that the wave function for a quantum cosmological model under consideration is found. By doing so in canonical quantization the wave function is acquired through solving the Wheeler-DeWitt equation (WDW), with the appropriate boundary condition, the best one known are the Hartle-Hawking no-boundary proposal [24] and the Vilenkin tunneling proposal [25].

In order to achieve the WDW equation for this model we shall replace the generalized momenta $\Pi_{q^\mu} \longrightarrow i\hbar\partial_{q^\mu}$ in the Hamiltonian (34), these momenta are associated with the scale factors A and C and the moduli fields (ϕ, σ). The WDW equation for this model can be built by doing $A^2(\partial^2/\partial A^2) \longrightarrow A^{p+2}(\partial/\partial A)(A^{-p}(\partial/\partial A)) = A^2(\partial^2/\partial A^2) - pA(\partial/\partial A)$, where p is a parameter which measures the ambiguity in the factor ordering. The wave function is a functional $\Psi(\phi, \sigma, A, B)$, where (ϕ, σ, A, B) are the coordinates of the superspace. The last ideas are the basis of the canonical quantization and the equation $\mathscr{H}\Psi = 0$ is known as the WDW equation. This equation is a second-order differential equation on superspace, which means we have one differential equation in each hypersurface of the extended space-time (for more details see [26, 27]). On the other hand, the WDW equation has factor-ordering ambiguities, and the derivatives are the Laplacian in the supermetric $\mathscr{G}_{ij}$ [27].

The WDW equation has been treated in many different ways and there are a lot of papers that deal with different approaches to solve it, the most remarkable question that has been dealt with the WDW equation is to find a typical wave function of the universe, this subject was nicely addressed in [24, 28] and related to the problem of how the universe emerged from a big bang singularity can be read in [5, 29]. On the other hand, the best candidates for quantum solutions are those that have a damping behavior with respect to the scale factor, since only such wave functions allow for good classical solutions when using the WKB approximation for any scenario in the evolution of our universe [24, 30].

In this way, we obtain

$$\begin{aligned} \widehat{\mathscr{H}}\Psi = {} & \hbar^2 A^2 \frac{\partial^2\Psi}{\partial A^2} - \hbar^2 pA\frac{\partial\Psi}{\partial A} - 2\hbar^2 AC\frac{\partial^2\Psi}{\partial A\partial C} \\ & + \frac{16\hbar^2}{11}\frac{\partial^2\Psi}{\partial\phi^2} - \frac{6\hbar^2}{11}\frac{\partial^\Psi}{\partial\phi\partial\sigma} + \frac{\hbar^2}{33}\frac{\partial^2\Psi}{\partial\sigma^2} \\ & + \left(16C^4 + c_1\right)\Psi = 0, \end{aligned} \tag{47}$$

where the constant c_1 is associated with $\gamma = 1$ in Hamiltonian (34). The last partial differential equation can be rewritten by using the transformations $A = e^a$ and $C = e^c$, and this can be read as

$$\begin{aligned} \widehat{\mathscr{H}}\Psi = {} & \hbar^2\frac{\partial^2\Psi}{\partial a^2} - \hbar^2(p+1)\frac{\partial\Psi}{\partial a} - 2\hbar^2\frac{\partial^2\Psi}{\partial a\partial c} \\ & + \frac{16\hbar^2}{11}\frac{\partial^2\Psi}{\partial\phi^2} - \frac{6\hbar^2}{11}\frac{\partial^2\Psi}{\partial\phi\partial\sigma} + \frac{\hbar^2}{33}\frac{\partial^2\Psi}{\partial\sigma^2} \\ & + \left(16e^{4c} + c_1\right)\Psi = 0. \end{aligned} \tag{48}$$

By looking at the last expression, we can write the wave function $\Psi(a, c, \phi, \sigma) = \Theta(a, c)\Phi(\phi, \sigma)$ and this gives us two equations in a separated way

$$\begin{aligned} & \frac{\partial^2\Theta}{\partial a^2} - (p+1)\frac{\partial\Theta}{\partial a} - 2\frac{\partial^2\Theta}{\partial a\partial c} \\ & - \frac{1}{\hbar^2}\left(16e^{4c} + c_1 + \mu^2\right)\Theta = 0, \end{aligned} \tag{49a}$$

$$-\frac{16}{11}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{6}{11}\frac{\partial^2\Phi}{\partial\phi\partial\sigma} - \frac{1}{33}\frac{\partial^2\Phi}{\partial\sigma^2} + \frac{\mu^2}{\hbar^2}\Phi = 0, \tag{49b}$$

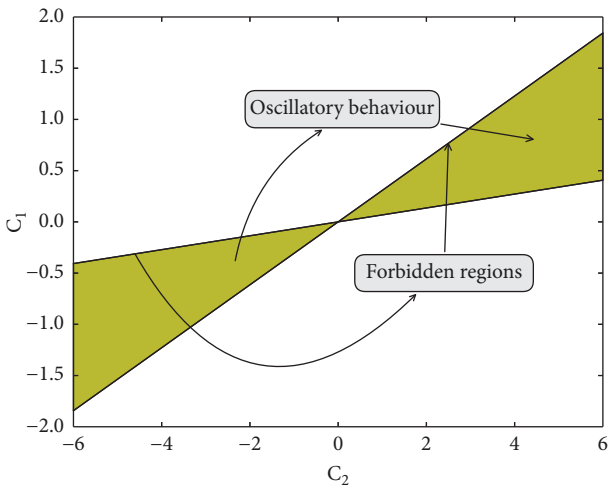

Figure 1: Regions where the behaviors of the moduli fields are oscillatory or hyperbolic. The regions outside of the oscillatory and forbidden regions correspond to the hyperbolic behavior.

where μ^2 is the separation constant. The partial differential equation (49a) associated with the variables A and C has a solution by taking the following ansatz

$$\Theta(a,c) = e^{\alpha a} G(c), \tag{50}$$

where α is a real constant. With this in mind, we obtain

$$\frac{dG}{dc} + \frac{1}{2\hbar^2\alpha}\left(16e^{4c} + \alpha_0\right) G = 0, \tag{51}$$

where we have defined the constant $\alpha_0 = c_1 + \mu^2 - \hbar^2\alpha^2 + \hbar^2(p+1)\alpha$, and the solution is

$$\Theta(A,C) = G_0\left(\frac{A}{C^{\beta/\alpha}}\right)^{\alpha} e^{-(2/\hbar^2\alpha)C^4}, \tag{52}$$

where $\beta = \alpha_0/2\hbar^2\alpha$, and we used the scale factors $A = e^a$ and $C = e^c$. On the other hand, the solutions associated with the moduli fields corresponds to the hyperbolic partial differential equation (49b), whose solution is given by

$$\begin{aligned}\Phi(\phi,\sigma) = {} & C_3 \sin\left[\Lambda\left(C_1\phi + C_2\sigma\right)\right] \\ & + C_4 \cos\left[\Lambda\left(C_1\phi + C_2\sigma\right)\right],\end{aligned} \tag{53}$$

where $\{C_i\}_{i=1}^4$ are integration constants and $\Lambda = \mu\sqrt{-33/(48C_1^2 - 18C_1C_2 + C_2^2)}$. The last expression (53) has two different behaviors, these behaviors are given by the cases $48C_1^2 - 18C_1C_2 + C_2^2 < 0$ and $48C_1^2 - 18C_1C_2 + C_2^2 > 0$. For the first case, we have that the behavior of the wave function associated with the moduli fields (ϕ,σ) is oscillatory, and this happens when $C_1 \in [((9-\sqrt{33})/48)C_2, ((9+\sqrt{33})/48)C_2]$, with $C_2 > 0$. On the other hand, if $C_2 < 0$ we have that the quantum solution becomes hyperbolic functions, and this represents the second kind of solution, and we summarize the solutions in the Figure 1.

On the other hand, we see that the solution associated with (49b) is the same for all Bianchi Class A cosmological models. It was the main result obtained in [3, 4]. This can be appreciated by observing the Hamiltonian operator (35), which can be split as $\widehat{H}(a,c,\phi,\sigma)\Psi = \widehat{H}_g(a,c)\Psi + \widehat{H}_m(\phi,\sigma)\Psi = 0$, where $\widehat{H}_g$ y $\widehat{H}_m$ represents the Hamiltonian for gravitational sector and the moduli fields, respectively, and the scale factors have the transformations $A = e^a, C = e^c$.

With the last results, we obtain the solution in the state μ to the WDW equation (48) whose wave function Ψ_μ can be built by taking the superposition of functions (52) and (53) by the hyperbolic behavior, taking that $C_3 = C_4$; that is

$$\begin{aligned}&\Psi_\mu(A,C,\eta,\nu) \\ &\quad = \mathscr{K}\frac{A^\alpha}{C^{\nu/2\alpha\hbar^2}} e^{-(2/\alpha\hbar^2)C^4} \exp\left[-\left(\frac{\ln C}{2\alpha\hbar^2}\mu^2 - \eta\mu\right)\right],\end{aligned} \tag{54}$$

where the parameters are $\nu = c_1 - \alpha\hbar^2 + (p+1)\hbar^2$ and $\eta = \sqrt{-33/(48C_1^2 - 18C_1C_2 + C_2^2)}(C_1\phi + C_2\sigma)$, where we have taken the constant $G_0 = 1$ for simplicity and $\mathscr{K}$ is a constant the allows us to normalize the wave function.

This wave function is not normalized, but we can build the wave packet as

$$\begin{aligned}\Psi(A,C,\eta,\nu) &= \mathscr{K}\frac{A^\alpha}{C^{\nu/2\alpha\hbar^2}} \\ &\cdot e^{-(2/\alpha\hbar^2)C^4}\int_{-\infty}^{\infty} G(\mu)\exp\left[-\left(\frac{\mu^2}{2\alpha\hbar^2} - \mu\eta\right)\right] d\mu,\end{aligned} \tag{55}$$

employing the integral from [31]:

$$\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)}dx = \sqrt{\frac{\pi}{a}}e^{(b^2-4ac)/4a} \tag{56}$$

and considering the relations $a = \ln C/2\alpha\hbar^2$, $b = -\eta$, $c = 0$, and $G(\mu) = 1$, we obtain

$$\begin{aligned}&\Psi(A,C,\eta,\nu) \\ &\quad = \mathscr{K}\sqrt{\pi}\frac{A^\alpha}{C^{\nu\lambda}} e^{-4\lambda C^4}\sqrt{\frac{1}{\ln(C^\lambda)}}\exp\left[\frac{\eta^2}{4}\frac{1}{\ln C^\lambda}\right], \\ &\qquad \lambda = 2\alpha\hbar^2,\end{aligned} \tag{57}$$

and, then, computing the probability density,

$$\rho = \Psi\Psi^\star = \mathscr{K}^2\pi\frac{A^{2\alpha}}{C^{2\nu\lambda}} e^{-8\lambda C^4}\frac{1}{\ln C^\lambda}\exp\left[\frac{\eta^2}{2}\frac{1}{\ln C^\lambda}\right], \tag{58}$$

whose behavior can be seen in Figure 2,

We can see that the projection plane gives an exponential-like behavior between the scale factor A versus the scale factor C; in similar way that in the classical solutions found in the equation (40), when we consider $\eta = 1$, we are frozen the two dilatonic scalar fields.

5. Bohm's Formalism

So far, we have solved the WDW equation (48), and we have found that the wave function Ψ can be split as the product

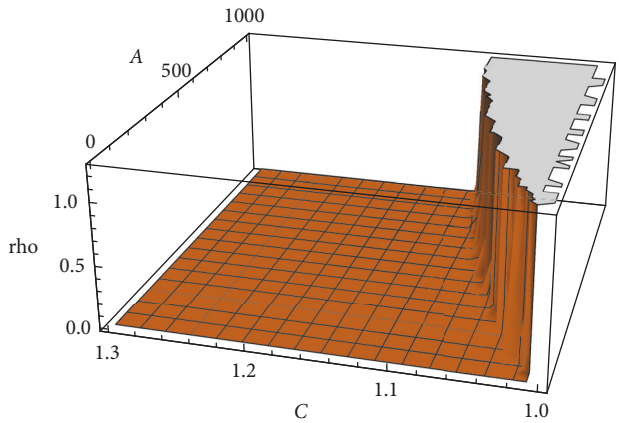

Figure 2: Probability density for equation (58), where we take the following values for the parameters $\lambda = 10$, $\alpha = 12.5$, $\nu = 10$, and $\eta = 1$.

of two wave functions Θ and Φ, which are given by (52) and (53). On the other hand, we know that WKB approximation is very important in quantum mechanics and, therefore, we shall introduce an ansatz for the wave function Ψ to take the form

$$\Theta\left(\ell^{\mu}\right) = W\left(\ell^{\mu}\right) e^{-S(\ell^{\mu})/\hbar}, \tag{59}$$

where $W(\ell^{\mu})$ is an amplitude which varies slowly and $S(\ell^{\mu})$ is the phase whose variation is faster that the amplitude, and this allows us to obtain *eikonal*-like equations. The term ℓ^{μ} is the dynamical variables of the minisuperspace which are $\ell^{\mu} = a, c$. This formalism is known as Bohm's formalism too [27, 32–34].

So, (49a) is transformed under expression (59) into

$$\begin{aligned}
&\hbar^2\left[\frac{\partial^2 W}{\partial a^2} - (p+1)\frac{\partial W}{\partial a} - 2\frac{\partial^2 W}{\partial a \partial c}\right] + \hbar\left[-2\frac{\partial W}{\partial a}\frac{\partial S}{\partial a}\right.\\
&\quad - W\frac{\partial^2 S}{\partial a^2} + (p+1)W\frac{\partial S}{\partial a} + 2\frac{\partial W}{\partial a}\frac{\partial S}{\partial c} + 2\frac{\partial W}{\partial c}\frac{\partial S}{\partial a}\\
&\quad \left. + 2W\frac{\partial^2 S}{\partial a \partial c}\right] + W\left[\left(\frac{\partial S}{\partial a}\right)^2 - 2\left(\frac{\partial S}{\partial a}\right)\left(\frac{\partial S}{\partial c}\right)\right.\\
&\quad \left. - \left(16e^{4c} + C_1 + \mu^2\right)\right] = 0.
\end{aligned} \tag{60}$$

The last expression (60) can be written as the following set of partial differential equations, as a WKB-like procedure:

$$\left(\frac{\partial S}{\partial a}\right)^2 - 2\left(\frac{\partial S}{\partial a}\right)\left(\frac{\partial S}{\partial c}\right) - \left(16e^{4c} + C_1 + \mu^2\right) = 0, \tag{61a}$$

$$\begin{aligned}
&-2\frac{\partial W}{\partial a}\frac{\partial S}{\partial a} - W\frac{\partial^2 S}{\partial a^2} + (p+1)W\frac{\partial S}{\partial a} + 2\frac{\partial W}{\partial a}\frac{\partial S}{\partial c}\\
&\quad + 2\frac{\partial W}{\partial c}\frac{\partial S}{\partial a} + 2W\frac{\partial^2 S}{\partial a \partial c} = 0,
\end{aligned} \tag{61b}$$

$$\frac{\partial^2 W}{\partial a^2} - (p+1)\frac{\partial W}{\partial a} - 2\frac{\partial^2 W}{\partial a \partial c} = 0. \tag{61c}$$

This set of equations are solved in a different way from the previous works, because the constraint equation was (61c) in the past; however now it is solved in first instance. In this set of equations, the first partial differential equation is known as the Hamilton-Jacobi equation for the gravitational field, with (61c) we obtain the W function, and expression (61b) is the constraint equation. From the last system of partial differential equation (61a), (61b), and (61c), we observe that (61c) has a solution given by

$$\begin{aligned}
\mathrm{W}(\mathrm{a},\mathrm{c}) = \frac{1}{\mathrm{S}_0^2}&\left[4\mathrm{S}_4 \mathrm{e}^{4\mathrm{c}} + \left(2\mathrm{S}_4\mathrm{a} - \mathrm{S}_5\right)\mathrm{S}_0^2\right.\\
&\left. + \mathrm{S}_4\left(\mathrm{C}_1 + \mu^2 + \mathrm{S}_0^2\right)\mathrm{c}\right]\mathrm{e}^{-((1+\mathrm{p})/2)\mathrm{c}}
\end{aligned} \tag{62}$$

and the solutions for the function S are

$$\begin{aligned}
\mathrm{S} &= \mathrm{S}_0\mathrm{a} + \mathrm{S}_2\mathrm{c} + \mathrm{S}_3\mathrm{e}^{4\mathrm{c}},\\
\mathrm{S}_3 &= -\frac{2}{\mathrm{S}_0},\\
\mathrm{S}_2 &= \frac{\mathrm{S}_0^2 - \mathrm{C}_1 - \mu^2}{2\mathrm{S}_0},
\end{aligned} \tag{63}$$

where S_0, S_4, and S_5 are integration constants. When we introduce these results into (49a), with $\Theta(\ell^{\mu}) = \mathrm{W}(\ell^{\mu})\mathrm{e}^{-\mathrm{S}(\ell^{\mu})/\hbar}$, this partial differential equation is satisfied identically.

6. Final Remarks

In this work we have developed the anisotropic Bianchi $VI_{h=-1}$ model from a higher-dimensional theory of gravity, and our analysis covers the classical and quantum aspects. In the frame of classical analysis we obtained that two radii of the Bianchi $VI_{h=-1}$ type are proportional, and we choose the equality (B(t) = C(t)) presenting a hidden symmetry, which says that the volume function of this cosmological model goes as a jet in x direction, whereas in the (y,z) plane it goes as a circle. This hidden symmetry allows us to simplify the analysis in the classical context, and as a consequence the quantum version was simplified too. From the analysis of the solutions to the scale factors, we find that there exists a relation between these, and the scale factor A goes an exponential function on the scale factor C; see (40). We can compare the last result with the jet emission that occurs in some stars, and in this sense we explore in this context the flat FRW cosmological model, finding that the scale factor goes to the C radii corresponding to Bianchi type VI cosmological model, and we can infer that the flat FRW cosmological model is embedded in this anisotropic cosmological Bianchi type model. We argue that the classical initial singularity is avoided in this theory, having initials scale factors different to zero, in both cases.

It could be interesting to study other types of matter in this context, beyond the barotropic matter. An example is the Chaplygin gas, where a proper time, characteristic to this matter, leads to the presence of singularities types I, II, III, and IV (generalizations to these models are presented in [35–37]) which also appear in the phantom scenario to

dark/energy matter. In our case, since the matter we are considering is barotropic, initial singularities of these types do not emerge from our analysis, implying also that phantom fields are absent. It is important to remark that this model is a very simplified one in the sense that we do not consider the presence of moduli-dependent matter and we do not analyze under which conditions inflation is present or how it starts.

Concerning the quantum scheme, we can observe that this anisotropic model is completely integrable with no need to use numerical methods. We built the wave function as a wavepacket via the parameter μ, which appears as the separation constant in the method used. When we write the probability density and take the projection plane (A,C), we observe that this projection shows that the scale factor A have an exponential behavior with the scale factor C, in a similar way that the classical relation presented in (40). Some results have been obtained by considering just the gravitational variables in [38]. On the other hand, we obtain that the solutions in the moduli fields are the same for all Bianchi Class A cosmological models (53), and the last conclusion is possible because the Hamiltonian operator in (34) can be written in separated way as $\widehat{H}(A,C,\phi,\sigma)\Psi = \widehat{H}_g(A,C)\Psi + \widehat{H}_m(\phi,\sigma)\Psi = 0$, where $\widehat{H}_g$ and $\widehat{H}_m$ are the Hamiltonian for the gravitational sector and the moduli fields, respectively. The full wave function given by $\Psi = \Phi(\phi,\sigma)\Theta(A,C)$ is a superposition of the gravitational variables and the moduli fields. There are some recent researches related on this line [39], they built a wave packet for an arbitrary anisotropic background. However, one of the main problems in quantum cosmology is how to build a wave packet that allows determining the possible states of the classical universe [40–48].

Appendix

A. Dimensional Reduction

With the purpose of being consistent, here we present the main ideas for the dimensional reduction of action (1). By the use of the conformal transformation

$$\widehat{G}_{MN} = e^{\Phi/2} G^E_{MN}, \tag{A.1}$$

action (1) can be written as

$$S = \frac{1}{2\kappa_{10}^2}\int d^{10}X\sqrt{-\widehat{G}}\left(e^{\Phi/2}\widehat{\mathscr{R}} + 4G^{E_{MN}}\nabla_M\Phi\nabla_N\Phi\right) + \int d^{10}X\sqrt{-G^E}e^{5\Phi/2}\mathscr{L}_{\text{matt}}, \tag{A.2}$$

where the ten-dimensional scalar curvature $\widehat{\mathscr{R}}$ transforms by the conformal transformation as

$$\widehat{\mathscr{R}} = e^{-\Phi/2}\left(\mathscr{R}^E - \frac{9}{2}G^{E_{MN}}\nabla_M\nabla_N\Phi - \frac{9}{2}G^{E_{MN}}\nabla_M\Phi\nabla_N\Phi\right). \tag{A.3}$$

By substituting expression (A.3) in (A.2) we obtain

$$S = \frac{1}{2\kappa_{10}^2}\int d^{10}X\sqrt{-G^E}\left(\mathscr{R}^E - \frac{9}{2}G^{E_{MN}}\nabla_M\nabla_N\Phi - \frac{1}{2}G^{E_{MN}}\nabla_M\Phi\nabla_N\Phi\right) + \int d^{10}X\sqrt{-G^E}e^{5\Phi/2}\mathscr{L}_{\text{matt}}. \tag{A.4}$$

The last expression is the ten-dimensional action in the Einstein frame. Expressing the metric determinant in four-dimensions in terms of the moduli field σ we have

$$\det\widehat{G}_{MN} = \widehat{G} = e^{-12\sigma}\widehat{g}. \tag{A.5}$$

By substituting the last expression (A.5) in (1) and considering that

$$\widehat{\mathscr{R}}^{(10)} = \widehat{\mathscr{R}}^{(4)} - 42\widehat{g}^{\mu\nu}\nabla_\mu\sigma\nabla_\nu\sigma + 12\widehat{g}^{\mu\nu}\nabla_\mu\nabla_\nu\sigma, \tag{A.6}$$

we obtain that

$$S = \frac{1}{2\kappa_{10}^2}\int d^4x d^6y\sqrt{-\widehat{g}}e^{-6\sigma}e^{-2\Phi}\left[\widehat{\mathscr{R}}^{(4)} - 42\widehat{g}^{\mu\nu}\nabla_\mu\sigma\nabla_\nu\sigma + 12\widehat{g}^{\mu\nu}\nabla_\mu\nabla_\nu\sigma + 4\widehat{g}^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi\right]. \tag{A.7}$$

Let us redefine in the last expression the dilaton field Φ as

$$\Phi = \phi - \frac{1}{2}\ln\left(\widehat{V}\right), \tag{A.8}$$

where $\widehat{V} = \int d^6y$. So, expression (A.7) can be written as

$$S = \frac{1}{2\kappa_{10}^2}\int d^4x\sqrt{-\widehat{g}}e^{-2(\phi+3\sigma)}\left[\widehat{\mathscr{R}}^{(4)} - 42\widehat{g}^{\mu\nu}\nabla_\mu\sigma\nabla_\nu\sigma + 12\widehat{g}^{\mu\nu}\nabla_\mu\nabla_\nu\sigma + 4\widehat{g}^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right]. \tag{A.9}$$

The four-dimensional metric $\widehat{g}_{\mu\nu}$ means that the metric is in the string frame, and we should take a conformal transformation linking the string and Einstein frames. We label by $g_{\mu\nu}$ the metric of the external space in the Einstein frame, and this conformal transformation is given by

$$\widehat{g}_{\mu\nu} = e^{2\Theta}g_{\mu\nu}, \tag{A.10}$$

which after some algebra gives the four-dimensional scalar curvature

$$\widehat{\mathscr{R}}^{(4)} = e^{-2\Theta}\left(\mathscr{R}^{(4)} - 6g^{\mu\nu}\nabla_\mu\nabla_\nu\Theta - 6g^{\mu\nu}\nabla_\mu\Theta\nabla_\nu\Theta\right), \tag{A.11}$$

where the function Θ is given by the transformation

$$\Theta = \phi + 3\sigma + \ln\left(\frac{\kappa_{10}^2}{\kappa_4^2}\right). \tag{A.12}$$

Now, we must replace expressions (A.10), (A.11), and (A.12) in expression (A.9) and we find

$$S = \frac{1}{2\kappa_4^2}\int d^4x\sqrt{-g}\left(\mathscr{R} - 6g^{\mu\nu}\nabla_\mu\nabla_\nu\phi - 6g^{\mu\nu}\nabla_\mu\nabla_\nu\sigma - 2g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - 96g^{\mu\nu}\nabla_\mu\sigma\nabla_\nu\sigma - 36g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\sigma\right). \tag{A.13}$$

At first glance, it is important to clarify one point related to the stress-energy tensor which has the matrix form (6); this tensor belongs to the string frame. In order to write the stress-energy tensor in the Einstein frame we need to work with the last integrand of expression (1). So, after taking the variation with respect to the ten-dimensional metric, we see that

$$\begin{aligned}\int d^{10}X\sqrt{-\widehat{G}}e^{-2\Phi}\kappa_{10}^{2}e^{2\Phi}\widehat{T}_{MN}\widehat{G}^{MN} &= \int d^{4}x\sqrt{-g}e^{2(\Theta+\phi)}\widehat{T}_{MN}\widehat{G}^{MN}, \\ &= \int d^{4}x\sqrt{-g}\left(e^{2\phi}\widehat{T}_{\mu\nu}g^{\mu\nu}+e^{2(\Theta+\phi)}\widehat{T}_{mn}\widehat{g}^{mn}\right),\end{aligned} \tag{A.14}$$

where we can observe that the four-dimensional stress-energy tensor in the Einstein frame is defined as we wrote in expression (7).

B. The Explicit Einstein Equations

The Einstein equations and the equation of motions are given by expressions (8a), (8b), and (8c). In detail they are

$$\begin{aligned}G_0^0:\quad & \frac{\dot{A}}{NA}\frac{\dot{B}}{NB}+\frac{\dot{A}}{NA}\frac{\dot{C}}{NC}+\frac{\dot{B}}{NB}\frac{\dot{C}}{NC} \\ & -\left(\frac{\dot{\phi}}{N}\right)^{2}-18\frac{\dot{\phi}}{N}\frac{\dot{\sigma}}{N}-48\left(\frac{\dot{\sigma}}{N}\right)^{2} \\ & -8\pi G\rho-\frac{1}{A^{2}}=0,\end{aligned} \tag{B.1a}$$

$$G_1^0=-G_0^1:\quad \frac{\dot{B}}{NB}-\frac{\dot{C}}{NC}=0, \tag{B.1b}$$

$$\begin{aligned}G_1^1:\quad & \frac{\ddot{B}}{N^{2}B}+\frac{\ddot{C}}{N^{2}C}+\frac{\dot{B}}{NB}\frac{\dot{C}}{NC}-\frac{\dot{B}}{NB}\frac{\dot{N}}{N^{2}} \\ & -\frac{\dot{C}}{NC}\frac{\dot{N}}{N^{2}}+\left(\frac{\dot{\phi}}{N}\right)^{2}+18\frac{\dot{\phi}}{N}\frac{\dot{\sigma}}{N} \\ & +48\left(\frac{\dot{\sigma}}{N}\right)^{2}+8\pi Gp+\frac{1}{A^{2}}=0,\end{aligned} \tag{B.1c}$$

$$\begin{aligned}G_2^2:\quad & \frac{\ddot{A}}{N^{2}A}+\frac{\ddot{C}}{N^{2}C}+\frac{\dot{A}}{NA}\frac{\dot{C}}{NC}-\frac{\dot{A}}{NA}\frac{\dot{N}}{N^{2}} \\ & -\frac{\dot{C}}{NC}\frac{\dot{N}}{N^{2}}+\left(\frac{\dot{\phi}}{N}\right)^{2}+18\frac{\dot{\phi}}{N}\frac{\dot{\sigma}}{N} \\ & +48\left(\frac{\dot{\sigma}}{N}\right)^{2}+8\pi Gp-\frac{1}{A^{2}}=0,\end{aligned} \tag{B.1d}$$

$$\begin{aligned}G_3^3:\quad & \frac{\ddot{A}}{N^{2}A}+\frac{\ddot{B}}{N^{2}B}+\frac{\dot{A}}{NA}\frac{\dot{B}}{NB}-\frac{\dot{A}}{NA}\frac{\dot{N}}{N^{2}} \\ & -\frac{\dot{B}}{NB}\frac{\dot{N}}{N^{2}}+\left(\frac{\dot{\phi}}{N}\right)^{2}+18\frac{\dot{\phi}}{N}\frac{\dot{\sigma}}{N} \\ & +48\left(\frac{\dot{\sigma}}{N}\right)^{2}+8\pi Gp-\frac{1}{A^{2}}=0,\end{aligned} \tag{B.1e}$$

$$\begin{aligned}\Box\phi=0:\quad & \left(\frac{\dot{A}}{NA}+\frac{\dot{B}}{NB}+\frac{\dot{A}}{NA}+\frac{\dot{C}}{NC}\right)\frac{\dot{\phi}}{N}+\frac{\ddot{\phi}}{N^{2}} \\ & -\frac{\dot{\phi}}{N}\frac{\dot{N}}{N^{2}}=0,\end{aligned} \tag{B.1f}$$

$$\begin{aligned}\Box\sigma=0:\quad & \left(\frac{\dot{A}}{NA}+\frac{\dot{B}}{NB}+\frac{\dot{A}}{NA}+\frac{\dot{C}}{NC}\right)\frac{\dot{\sigma}}{N}+\frac{\ddot{\sigma}}{N^{2}} \\ & -\frac{\dot{\sigma}}{N}\frac{\dot{N}}{N^{2}}=0.\end{aligned} \tag{B.1g}$$

The expressions that involve $\dot{q}$ means derivative with respect to the cosmic time, and the relation between the cosmic and proper times is given by

$$d\tau=N(t)\,dt. \tag{B.2}$$

With the last consideration we see that it is possible to rewrite the last system of differential equations (B.1a), (B.1b), (B.1c), (B.1d), (B.1e), (B.1f), and (B.1g) as follows:

$$\begin{aligned}\dot{\phi}&=N\phi', \\ \ddot{\phi}&=N^{2}\phi''+\dot{N}\phi',\end{aligned} \tag{B.3}$$

just to mention the transformation of the scalar field. Under this change, we see that the system (B.1a), (B.1b), (B.1c), (B.1d), (B.1e), (B.1f), and (B.1g) can be transformed into

$$\begin{aligned}G_0^0:\quad & 2\frac{A'C'}{AC}+\left(\frac{C'}{C}\right)^{2}-\phi'^{2}-18\phi'\sigma' \\ & -48\sigma'^{2}-8\pi G\rho-\frac{1}{A^{2}}=0,\end{aligned} \tag{B.4a}$$

$$\begin{aligned}G_1^1:\quad & 2\frac{C''}{C}+C'^{2}+\phi'^{2}+18\phi'\sigma'+48\sigma'^{2} \\ & +8\pi Gp+\frac{1}{A^{2}}=0,\end{aligned} \tag{B.4b}$$

$$\begin{aligned}G_2^2=G_3^3:\quad & \frac{A''}{A}+\frac{C''}{C}+A'C'+\phi'^{2}+18\phi'\sigma' \\ & +48\sigma'^{2}+8\pi Gp-\frac{1}{A^{2}}=0,\end{aligned} \tag{B.4c}$$

$$\Box\phi=0:\quad 2\left(\frac{A'}{A}+\frac{C'}{C}\right)\phi'+\phi''=0, \tag{B.4d}$$

$$\Box\sigma=0:\quad 2\left(\frac{A'}{A}+\frac{C'}{C}\right)\sigma'+\sigma''=0, \tag{B.4e}$$

where we used the fact that the scale factors $B(t)$ and $C(t)$ are equal. This result can be obtained from expression (B.1b), whose differential equation can be transformed into the proper time as

$$\begin{aligned} \frac{B'}{B} - \frac{C'}{C} = 0 &\Longrightarrow \\ \frac{d}{dt}\ln B - \frac{d}{dt}\ln C = 0 &\Longrightarrow \\ \frac{d}{dt}\ln\left(\frac{B}{C}\right) = 0 &\Longrightarrow \\ \ln\left(\frac{B}{C}\right) = 1. \end{aligned} \tag{B.5}$$

In the last expression we have fixed the integration constant equal to one.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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