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Research Article

Christ–Lee Model: Augmented Supervariable Approach

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We derive the complete set of off-shell nilpotent and absolutely anticommuting (anti-)BRST as well as (anti-)co-BRST symmetry transformations for the gauge-invariant Christ–Lee model by exploiting the celebrated (dual-)horizontality conditions together with the gauge-invariant and (anti-)co-BRST invariant restrictions within the framework of geometrical *"augmented"* supervariable approach to BRST formalism. We show the (anti-)BRST and (anti-)co-BRST invariances of the Lagrangian in the context of supervariable approach. We also provide the geometrical origin and capture the key properties associated with the (anti-)BRST and (anti-)co-BRST symmetry transformations (and corresponding conserved charges) in terms of the supervariables and Grassmannian translational generators.

1. Introduction

The dynamics of a given physical system can be described in terms of the differential equations of various degrees. The classical Hamilton's equations, Schrödinger equation in quantum theory, and Maxwell's equations in electrodynamics are a few physically fundamental examples of such systems. In order to get the complete information about the dynamics of a system, one has to solve the equations which describe them. Interestingly, the existence of symmetry further simplifies the solutions of physical system. This is because of the fact that one can describe the properties of a given system without solving all the equations of motion. Thus, the symmetry transformations are the key ingredients of modern physics [1]. It is well-known that the three out of *four* fundamental interactions of nature can be well described by the gauge theories and associated local gauge symmetries. Gauge symmetry is always generated by the first-class constraints present in a given physical theory [2, 3].

Becchi–Rouet–Stora–Tyutin (BRST) formalism is one of the elegant, mathematically rich, and unique ways to covariantly quantize any gauge theory where unitarity and quantum gauge invariance are respected together [4–7]. It is important to mention that, for a given local gauge symmetry at the classical level, we have *two* global supersymmetric types (i.e., BRST and anti-BRST) symmetries at the quantum level [8, 9]. These symmetry transformations have two innate properties: nilpotency of order two and absolute anticommutativity. First property elaborates the fermionic nature of the (anti-)BRST symmetries whereas latter one insures that BRST and anti-BRST transformations are linearly independent of each other. The anti-BRST symmetry is just not an artifact rather it plays an instrumental role in providing the geometrical interpretation of the superfield approach to BRST formalism [10–12]. It is also useful in the investigation of perturbative renormalizability of Yang-Mills theory [13–15]. Thus, it is of utmost interest to study the (anti-)BRST invariant theory.

In our earlier work, we have shown that, in addition to the above fermionic (anti-)BRST symmetries, the nilpotent and absolutely anticommuting (anti-)co-BRST symmetries *also* exist for the Abelian p-form ($p = 1, 2, 3$) gauge theories in a specific $D = 2p$-dimensions of spacetime within the framework of BRST formalism [16] where D is the dimensionality of spacetime and p denotes the degree of the differential form. One of the key differences between the (anti-)BRST and (anti-)co-BRST symmetries is that the former symmetries leave the kinetic term invariant whereas under the latter symmetries the gauge-fixing term remains invariant. The

appropriate anticommutators among the fermionic symmetries lead to a unique bosonic symmetry in the theory. These fermionic and bosonic symmetries (and corresponding charges) provide the physical realizations of the de Rham cohomological operators of differential geometry whereas discrete symmetry plays the role of Hodge duality ($*$) operation (see, e.g., [16–18] for details). In fact, we have conjectured that, in $D = 2p$-dimensions of spacetime, any arbitrary Abelian p-form gauge theory ($p = 1, 2, 3, \ldots$) provides a field-theoretic model for the Hodge theory within the framework of BRST formalism [16]. Furthermore, we point out that the $(0 + 1)D$ rigid rotor and Christ–Lee model in the context of BRST formalism are shown to be the examples of Hodge theory [18, 19].

Christ–Lee (CL) model is one of the simplest examples of gauge-invariant system that is described by a singular Lagrangian. Physically, CL model represents a particle moving in plane with some specific constraints [20]. To be more specific, CL model is endowed with two first-class constraints in the language of Dirac's classification scheme of constraints [2, 3]. CL model has been well studied at the classical and quantum level in many different ways [21–23]. The gauge group of CL model is analogous to the quantum electrodynamics (QED) with a local gauge parameter varying as an arbitrary function of time. This simple physical system has been quantized by exploiting the usual canonical formalism with some specific gauge choices (e.g., temporal and/or Coulomb gauge conditions) [20]. This model has also been quantized by exploiting the BRST formalism [24].

In our earlier work, we have shown that besides the usual off-shell nilpotent (anti-)BRST transformations there also exist (anti-)co-BRST symmetries for CL model. Further, it has been explicitly shown that, in addition to above fermionic transformations, a unique bosonic symmetry transformation is also present for this model within the framework of BRST formalism [19]. We have shown that these transformations (and corresponding conserved charges) obey an algebra which is exactly similar to the algebra satisfied by the de Rham cohomological operators (d, δ, Δ) [25, 26]. Thus, we have been able to show that CL model is a simple toy model for the Hodge theory [19].

As far as the fermionic (anti-)BRST and (anti-)co-BRST symmetries are concerned, their geometrical origin becomes transparent and clear in the superfield formulation [10–12]. Bonora-Tonin superfield approach to BRST formalism is a geometrically intuitive method where the key properties associated with the (anti-)BRST symmetry transformations find their geometrical origin in the language of Grassmannian translational generators in an elegant manner [10, 11]. In this formalism, a D-dimensional Minkowskian manifold is generalized to the $(D, 2)$-dimensional supermanifold. The latter is parametrized by the superspace coordinates $Z^M = (x^\mu, \eta, \overline{\eta})$ where x^μ $(\mu = 0, 1, \ldots, D-1)$ are the bosonic coordinates and $(\eta, \overline{\eta})$ are a pair of Grassmannian variables obeying nilpotency and anticommutativity properties (i.e., $\eta^2 = \overline{\eta} = 0, \eta\overline{\eta} + \overline{\eta}\eta = 0$). The superspace formalism, in general, allows superfields in a given field-theoretic model. One of the simplest examples is the Abelian 1-form gauge theory in 4D of spacetime. In the superfield approach to BRST formalism [10–12], we define a super 1-form connection $\mathscr{A}^{(1)} = dZ^M \mathscr{A}_M(x, \eta, \overline{\eta}) \equiv dx^\mu \mathscr{A}_\mu + d\eta \overline{\mathscr{C}} + d\overline{\eta}\mathscr{C}$ and super exterior derivative $\tilde{d} = dZ^M \partial_M = dx^\mu \partial_\mu + d\eta \partial_\eta + d\overline{\eta}\partial_{\overline{\eta}}$ on the supermanifold corresponding to the $4D$ ordinary 1-form $A^{(1)} = dx^\mu A_\mu(x)$ and exterior derivative $d = dx^\mu \partial_\mu$. Here the superfields $\mathscr{A}_\mu(x, \eta, \overline{\eta})$, $\mathscr{C}(x, \eta, \overline{\eta})$ and $\overline{\mathscr{C}}(x, \eta, \overline{\eta})$, as the supermultiplets of super 1-form, are the generalization of the gauge field $A_\mu(x)$, ghost field $C(x)$, and anti-ghost field $\overline{C}(x)$, respectively. Now we expand these superfields along the Grassmannian directions η and $\overline{\eta}$ with the help of other secondary fields. By exploiting the power and strength of *horizontality condition* (i.e., $\tilde{d}\mathscr{A}^{(1)} \equiv dA^{(1)}$), we precisely determine all the secondary fields in terms of the dynamical/auxiliary fields of the (anti-)BRST invariant theory. We point out that the CL model is a $1D$ model where the dynamical variables (as the generalized coordinates) are only the function of time-evolution parameter. We utilize here the sanctity of the above superfield approach for the present CL model.

For the interacting theories, a more powerful method known as "augmented" version of the superfield approach has been developed where, in addition to the horizontality condition, the gauge-invariant restrictions are also implemented for the derivation of the complete set of proper (anti-)BRST transformations [27–30]. In our present study, we shall utilize the power and strength of the superfield formalism to derive the off-shell nilpotent and absolutely anticommuting (anti-)BRST as well as (anti-)co-BRST symmetry transformations for the $(0 + 1)$-dimensional CL model. In this approach, we have to go beyond the celebrated (dual-)horizontality condition to derive the proper (anti-)BRST and (anti-)co-BRST for all the dynamical variables present in the model. In fact, in addition to the (dual-)horizontality conditions, we use the gauge and (anti-)co-BRST invariant restrictions.

The contents of our present endeavour are as follows. In Section 2, we briefly discuss the CL model and associated local gauge symmetry. We also discuss the supersymmetric type global (anti-)BRST and (anti-)co-BRST symmetry transformations (and corresponding conserved charges). Section 3 is devoted to the derivation of the off-shell nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations with the help of "augmented" supervariable approach. Section 4 deals with the derivation of the proper (anti-)co-BRST transformations where the (anti-)co-BRST restrictions are used, in addition to the dual-horizontality condition. We capture the (anti-)BRST and (anti-)co-BRST invariances of the Lagrangian within the framework of supervariable approach in Section 5. In our Section 6, we show the nilpotency and anticommutativity properties of the (anti-)BRST and (anti-)co-BRST transformations (and corresponding generators) in terms of the translational generators along the directions of Grassmannian variables. Finally, in Section 7, we provide the concluding remarks.

2. Preliminaries: Christ–Lee Model and Associated Symmetries

We start off with the first-order as well as gauge-invariant Lagrangian of the $(0+1)$-dimensional Christ–Lee (CL) model as given by [20, 22, 24]

$$L_f = \dot{r}p_r + \dot{\theta}p_\theta - \frac{1}{2}p_r^2 - \frac{1}{2r^2}p_\theta^2 - zp_\theta - V(r), \quad (1)$$

where r, θ are the generalized plane polar coordinates and p_r, p_θ are the corresponding canonical momenta, respectively. The variable z is another generalized coordinate and $V(r)$ is the potential bounded from below. Under the following continuous gauge symmetry transformations

$$\begin{aligned} \delta z &= \dot{\chi}(t), \\ \delta\theta &= \chi(t), \\ \delta[r, p_r, p_\theta] &= 0, \end{aligned} \quad (2)$$

where $\chi(t)$ is an infinitesimal local gauge parameter, the Lagrangian L_f remains invariant (i.e., $\delta L_f = 0$).

The (anti-)BRST invariant Lagrangian for the Christ–Lee model that incorporates the gauge-fixing term and Faddeev–Popov (anti-)ghost variables can be written as [19, 24]

$$\begin{aligned} L = {} & \dot{r}p_r + \dot{\theta}p_\theta - \frac{1}{2}p_r^2 - \frac{1}{2r^2}p_\theta^2 - zp_\theta - V(r) + \frac{1}{2}b^2 \\ & + b(\dot{z} + \theta) - i\dot{\overline{C}}\dot{C} + i\overline{C}C, \end{aligned} \quad (3)$$

where b is the Nakanishi–Lautrup type auxiliary variable and $(\overline{C})C$ are the Faddeev–Popov (anti-)ghost variables (with $C^2 = \overline{C}^2 = 0, C\overline{C} + \overline{C}C = 0$) having ghost numbers $(-1)+1$, respectively. The Lagrangian (3) respects the off-shell nilpotent ($s_{(a)b}^2 = 0, s_{(a)d}^2 = 0$) and absolutely anticommuting ($s_b s_{ab} + s_{ab}s_b = 0, s_d s_{ad} + s_{ad}s_d = 0$) (anti-)BRST ($s_{(a)b}$) and (anti-)co-BRST ($s_{(a)d}$) symmetry transformations. These continuous symmetries are listed as follows [19]:

$$\begin{aligned} s_b z &= \dot{C}, \\ s_b\theta &= C, \\ s_b\overline{C} &= ib, \\ s_b[r, p_r, p_\theta, b, C] &= 0, \\ s_{ab}z &= \dot{\overline{C}}, \\ s_{ab}\theta &= \overline{C}, \\ s_{ab}C &= -ib, \\ s_{ab}[r, p_r, p_\theta, b, \overline{C}] &= 0, \\ s_d z &= \overline{C}, \\ s_d\theta &= -\dot{\overline{C}}, \\ s_d C &= ip_\theta, \\ s_d[r, p_r, p_\theta, b, \overline{C}] &= 0, \\ s_{ad}z &= C, \\ s_{ad}\theta &= -\dot{C}, \\ s_{ad}\overline{C} &= -ip_\theta, \\ s_{ad}[r, p_r, p_\theta, b, C] &= 0. \end{aligned} \quad (4)$$

One can readily check that under the above symmetry transformations L remains quasi-invariant [31]. To be more precise, the Lagrangian transforms to a total time derivative under the above continuous and nilpotent symmetry transformations; namely,

$$\begin{aligned} s_b L &= \frac{d}{dt}\left(b\dot{C}\right), \\ s_{ab}L &= \frac{d}{dt}\left(b\dot{\overline{C}}\right), \\ s_d L &= -\frac{d}{dt}\left(p_\theta\dot{\overline{C}}\right), \\ s_{ad}L &= -\frac{d}{dt}\left(p_\theta\dot{C}\right). \end{aligned} \quad (5)$$

As a consequence, the action integral $S = \int dtL$ remains invariant under the (anti-)BRST and (anti-)co-BRST transformations. According to the Noether theorem, the invariance of the action under the above continuous symmetry transformations leads to the following conserved charges [19]:

$$\begin{aligned} Q_b &= b\dot{C} + p_\theta C \equiv b\dot{C} - \dot{b}C, \\ Q_{ab} &= b\dot{\overline{C}} + p_\theta\overline{C} \equiv b\dot{\overline{C}} - \dot{b}\overline{C}, \\ Q_d &= b\overline{C} - p_\theta\dot{\overline{C}} \equiv b\overline{C} + \dot{b}\dot{\overline{C}}, \\ Q_{ad} &= bC - p_\theta\dot{C} \equiv bC + \dot{b}\dot{C}, \end{aligned} \quad (6)$$

where on the r.h.s., we have used the equation of motion $p_\theta = -\dot{b}$ that has been derived from L. These conserved charges are the generators of the corresponding symmetry transformations. It is also to be noted that these charges are nilpotent of order two (i.e., $Q_{(a)b}^2 = 0, Q_{(a)d}^2 = 0$) and anticommuting ($Q_b Q_{ab} + Q_{ab}Q_b = 0, Q_d Q_{ad} + Q_{ad}Q_d = 0$) in nature.

3. Off-Shell Nilpotent (Anti-)BRST Symmetries: Supervariable Approach

We lay emphasis on the fact that the variable $z(t)$ behaves like a gauge variable [32] because, under the gauge transformation, it transforms as $\delta z(t) = \dot{\chi}(t)$. For example, in QED, the temporal component $A_0(x,t)$ of vector gauge field transforms as $\delta A_0 = \dot{\Lambda}(x,t)$ under the $U(1)$ gauge transformation where $\Lambda(x,t)$ is a local gauge parameter. Thus, we define the exterior derivative d (with $d^2 = 0$) and 1-form connection $Z^{(1)}$ on

(0 + 1)-dimensional space parameterized only by (bosonic) time-evolution parameter t as (see, e.g., [25, 26])

$$d = dt\partial_t, \qquad Z^{(1)} = dtz(t). \tag{7}$$

We note that $dZ^{(1)} = 0$ because of the property of wedge product $(dt \wedge dt) = 0$. In order to derive the proper (anti-)BRST transformations, we generalize the exterior derivative and 1-form to their corresponding super exterior derivative $(\tilde{d})$ and super 1-form $(\tilde{Z}^{(1)})$, respectively, on the $(1,2)$-dimensional superspace parametrized by (bosonic) t and a pair of Grassmannian variables $(\eta, \overline{\eta})$ (with $\eta^2 = \overline{\eta}^2 = 0, \eta\overline{\eta} + \overline{\eta}\eta = 0$) as given by (see, e.g., [10, 11] for details)

$$\begin{aligned} d &\longrightarrow \tilde{d} = dt\partial_t + d\eta\partial_\eta + d\overline{\eta}\partial_{\overline{\eta}}, \quad \left(\tilde{d}^2 = 0\right), \\ Z^{(1)} &\longrightarrow \tilde{Z}^{(1)} \\ &= dt\mathcal{Z}\left(t,\eta,\overline{\eta}\right) + d\eta\overline{\mathcal{F}}\left(t,\eta,\overline{\eta}\right) + d\overline{\eta}\mathcal{F}\left(t,\eta,\overline{\eta}\right), \end{aligned} \tag{8}$$

where $\partial_\eta = \partial/\partial\eta$ and $\partial_{\overline{\eta}} = \partial/\partial\overline{\eta}$ are the Grassmannian derivatives (with $\partial_\eta^2 = \partial_{\overline{\eta}}^2 = 0, \partial_\eta\partial_{\overline{\eta}} + \partial_{\overline{\eta}}\partial_\eta = 0$) corresponding to the variables η and $\overline{\eta}$, respectively. The supermultiplets as the components of super 1-form can be expanded along the directions of Grassmannian variables $(\eta, \overline{\eta})$ as follows:

$$\begin{aligned} \mathcal{Z}\left(t,\eta,\overline{\eta}\right) &= z(t) + \eta\overline{f}_1(t) + \overline{\eta}f_1(t) + i\eta\overline{\eta}B(t), \\ \mathcal{F}\left(t,\eta,\overline{\eta}\right) &= C(t) + i\eta\overline{b}_1(t) + i\overline{\eta}b_1(t) + i\eta\overline{\eta}s(t), \\ \overline{\mathcal{F}}\left(t,\eta,\overline{\eta}\right) &= \overline{C}(t) + i\eta\overline{b}_2(t) + i\overline{\eta}b_2(t) + i\eta\overline{\eta}\overline{s}(t), \end{aligned} \tag{9}$$

where $B, b_1, \overline{b}_1, b_2, \overline{b}_2$ are the bosonic secondary variables and $f_1, \overline{f}_1, s, \overline{s}$ are the fermionic secondary variables. We shall determine these secondary variables in terms of the basics and auxiliary variables by exploiting the following horizontality condition:

$$\tilde{d}\tilde{Z}^{(1)} = dZ^{(1)}. \tag{10}$$

The horizontality condition is also known as *"soul-flatness"* condition where t is a body coordinate and $(\eta, \overline{\eta})$ are the soul coordinates [33]. The horizontality or soul-flatness condition implies that the r.h.s. would remain independent of the soul coordinates when it is generalized onto $(1,2)$D supermanifold. The l.h.s. of (10), in full blaze of glory, can be written as

$$\begin{aligned} \tilde{d}\tilde{Z}^{(1)} &= (dt \wedge d\eta)\left(\partial_t\overline{\mathcal{F}} - \partial_\eta\mathcal{Z}\right) \\ &\quad + (dt \wedge d\overline{\eta})\left(\partial_t\mathcal{F} - \partial_{\overline{\eta}}\mathcal{Z}\right) \\ &\quad - (d\eta \wedge d\overline{\eta})\left(\partial_\eta\mathcal{F} + \partial_{\overline{\eta}}\overline{\mathcal{F}}\right) \\ &\quad - (d\eta \wedge d\eta)\left(\partial_\eta\overline{\mathcal{F}}\right) - (d\overline{\eta} \wedge d\overline{\eta})\left(\partial_{\overline{\eta}}\mathcal{F}\right). \end{aligned} \tag{11}$$

Exploiting (10) and (11) with the help of (9), we obtain the following interesting relationships among the basic and secondary variables; namely,

$$\begin{aligned} f_1 &= \dot{C}, \\ \overline{f}_1 &= \dot{\overline{C}}, \\ b_2 + \overline{b}_1 &= 0, \\ B &= \dot{b}, \\ b_1 &= 0, \\ \overline{b}_2 &= 0, \\ s &= 0, \\ \overline{s} &= 0, \end{aligned} \tag{12}$$

where we have made the choice $b_2 = -\overline{b}_1 = b$ for the Nakanishi–Lautrup type auxiliary variable. Substituting the values of secondary variables from (12) in (9), we yield the following expressions for the supervariables:

$$\begin{aligned} \mathcal{Z}^{(h)}\left(t,\eta,\overline{\eta}\right) &= z(t) + \eta\dot{\overline{C}}(t) + \overline{\eta}\dot{C}(t) + i\eta\overline{\eta}\dot{b}(t), \\ \mathcal{F}^{(h)}\left(t,\eta,\overline{\eta}\right) &= C(t) - i\eta b(t), \\ \overline{\mathcal{F}}^{(h)}\left(t,\eta,\overline{\eta}\right) &= \overline{C}(t) + i\overline{\eta}b(t), \end{aligned} \tag{13}$$

where the superscript (h) on supervariables implies the super expansions of supervariables obtained after the application of horizontality condition (10).

At this juncture, we lay emphasis on the fact that the quantity $(z - \dot{\theta})$ remains invariant under the gauge transformations (2). Thus, it would also be independent of the Grassmannian variables when we generalize it onto $(1,2)$-dimensional superspace. This gauge-invariant quantity will serve our purpose to derive the off-shell nilpotent (anti-)BRST transformations for θ variable [27–29]. In the language of differential form, we can write this gauge-invariant quantity as follows:

$$Z^{(1)} - d\theta^{(0)} = dt\left(z(t) - \partial_t\theta(t)\right), \tag{14}$$

which is clearly a 1-form object. Here $\theta^{(0)} = \theta$ is a zero-form. Now, we generalize this 1-form object onto $(1,2)$-dimensional supermanifold as

$$\tilde{Z}^{(1)} - \tilde{d}\tilde{\theta}^{(0)} = Z^{(1)} - d\theta^{(0)}, \tag{15}$$

where the super zero-form $\tilde{\theta}^{(0)}$ is defined in the following fashion:

$$\tilde{\theta}^{(0)} = \Theta\left(t,\eta,\overline{\eta}\right) = \theta(t) + \eta\overline{f}_2 + \overline{\eta}f_2 + i\eta\overline{\eta}\overline{B}. \tag{16}$$

In the above, $\overline{B}$ is a bosonic secondary variable whereas $f_2, \overline{f}_2$ are the fermionic secondary variables. The l.h.s. of (15) can be explicitly written as

$$\widetilde{Z}^{(1)} - \widetilde{d}\widetilde{\theta}^{(0)} = dt\left[\mathscr{Z}^{(h)} - \partial_t\Theta\right] + d\eta\left[\overline{\mathscr{F}}^{(h)} - \partial_\eta\Theta\right] + d\overline{\eta}\left[\mathscr{F}^{(h)} - \partial_{\overline{\eta}}\Theta\right]. \tag{17}$$

Using (15) and (17) together with (13), we obtain the precise value of the secondary variables

$$\begin{aligned} f_2 &= C, \\ \overline{f}_2 &= \overline{C}, \\ \overline{B} &= b. \end{aligned} \tag{18}$$

Furthermore, we point out that the dynamical variables r, p_r, and p_θ are also gauge-invariant as one can see from (2). These gauge-invariant variables would also remain unaffected by the presence of Grassmannian variables. As a result, we obtain the following super expansions; namely,

$$\begin{aligned} \Theta^{(h)}(t,\eta,\overline{\eta}) &= \theta(t) + \eta\overline{C} + \overline{\eta}C + i\eta\overline{\eta}b, \\ \mathscr{R}^{(h)}(t,\eta,\overline{\eta}) &= r(t), \\ \mathscr{P}_r^{(h)}(t,\eta,\overline{\eta}) &= p_r(t), \\ \mathscr{P}_\theta^{(h)}(t,\eta,\overline{\eta}) &= p_\theta(t). \end{aligned} \tag{19}$$

It is to be noted that if we look carefully at the super expansions given in (13) and (19), we can easily find out the proper (anti-)BRST transformations for all the dynamic variables. In fact, the BRST and anti-BRST transformations can be obtained for any generic dynamical variable $\phi(t)$ from its corresponding supervariable $\Phi^{(h)}(t,\eta,\overline{\eta})$ in the following manner:

$$\begin{aligned} s_b\phi(t) &= \frac{\partial}{\partial\overline{\eta}}\Phi^{(h)}(t,\eta,\overline{\eta})\Big|_{\eta=0}, \\ s_{ab}\phi(t) &= \frac{\partial}{\partial\eta}\Phi^{(h)}(t,\eta,\overline{\eta})\Big|_{\overline{\eta}=0}, \\ s_b s_{ab}\phi(t) &= \frac{\partial}{\partial\overline{\eta}}\frac{\partial}{\partial\eta}\Phi^{(h)}(t,\eta,\overline{\eta}). \end{aligned} \tag{20}$$

Using the above equations, we obtain the off-shell nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations as listed in (4) [19]. However, the (anti-)BRST transformations for the Nakanishi–Lautrup variable b have been derived from the requirements of nilpotency and anticommutativity of the (anti-)BRST transformations.

Exploiting the basic tenets of BRST formalism, we can write the Lagrangian (3) in three different ways by using the (anti-)BRST ($s_{(a)b}$) transformations as

$$\begin{aligned} L &= \dot{r}p_r + \dot{\theta}p_\theta - \frac{1}{2}p_r^2 - \frac{1}{2r^2}p_\theta^2 - zp_\theta - V(r) \\ &\quad - s_b\left[i\overline{C}\left(\dot{z} + \theta + \frac{b}{2}\right)\right] \\ &\equiv \dot{r}p_r + \dot{\theta}p_\theta - \frac{1}{2}p_r^2 - \frac{1}{2r^2}p_\theta^2 - zp_\theta - V(r) \\ &\quad + s_{ab}\left[iC\left(\dot{z} + \theta + \frac{b}{2}\right)\right] \\ &\equiv \dot{r}p_r + \dot{\theta}p_\theta - \frac{1}{2}p_r^2 - \frac{1}{2r^2}p_\theta^2 - zp_\theta - V(r) \\ &\quad + s_b s_{ab}\left[\frac{i}{2}\left(z^2 - \theta^2\right) - \frac{1}{2}\overline{C}C\right], \end{aligned} \tag{21}$$

modulo a total time derivative term. It is clear from the above that, due to the nilpotency property ($s_{(a)b}^2 = 0$) of $s_{(a)b}$, the (anti-)BRST invariance of L can now be proven in a simple and straightforward manner.

4. Off-Shell Nilpotent (Anti-)Co-BRST Symmetries: Supervariable Approach

In this section, we shall derive the off-shell nilpotent (i.e., $s_{(a)d}^2 = 0$) and absolutely anticommuting (i.e., $s_d s_{ad} + s_{ad} s_d = 0$) (anti-)co-BRST symmetry transformations ($s_{(a)d}$). We accomplish this goal by exploiting the power and strength of the dual-horizontality condition together with (anti-)co-BRST invariant restrictions. The action of co-exterior derivative $\delta = *d*$ (with $\delta^2 = 0$) on 1-form $Z^{(1)}$ yields

$$\delta Z^{(1)} = *d*Z^{(1)} = \dot{z}(t), \tag{22}$$

where $(*)$ is the Hodge duality operation defined on $(0+1)$-dimensional manifold. The gauge-fixing term $(\dot{z} + \theta)$, which remains invariant under (anti-)co-BRST symmetries, can be written in the following fashion:

$$\delta Z^{(1)} + \theta^{(0)} = \dot{z}(t) + \theta(t). \tag{23}$$

The invariance of gauge-fixing term under the (anti-)co-BRST transformations can be captured in the following (anti-)co-BRST invariant restriction [17, 28]

$$\star\widetilde{d}\star\widetilde{Z}^{(1)} + \Theta = *d*Z^{(1)} + \theta, \tag{24}$$

which tells us that the r.h.s. is independent of the Grassmannian variables η and $\overline{\eta}$ when we generalize it onto $(1,2)$D supermanifold. In the above, the super co-exterior derivative $\widetilde{\delta} = \star\widetilde{d}\star$ (with $\widetilde{\delta}^2 = 0$) and the Hodge duality $(\star)$ operation are defined onto $(1,2)$-dimensional supermanifold. In terms of the supervariables, one can simplify (24) as

$$\begin{aligned} &\left(\partial_t\mathscr{Z} + \partial_\eta\overline{\mathscr{F}} + \partial_{\overline{\eta}}\mathscr{F}\right) + S^{\eta\eta}\partial_\eta\mathscr{F} + S^{\overline{\eta}\overline{\eta}}\partial_{\overline{\eta}}\overline{\mathscr{F}} + \Theta \\ &\quad = \dot{z} + \theta, \end{aligned} \tag{25}$$

where $S^{\eta\eta}$ and $S^{\overline{\eta}\overline{\eta}}$ are symmetric in η and $\overline{\eta}$. In the above equation, we have used the following mathematical definitions defined on $(1,2)$-dimensional supermanifold [17, 28]:

$$\begin{aligned}
\star dt &= (d\eta \wedge d\overline{\eta}), \\
\star (dt \wedge d\eta \wedge d\overline{\eta}) &= 1, \\
\star d\eta &= (dt \wedge d\overline{\eta}), \\
\star (dt \wedge d\eta \wedge d\eta) &= S^{\eta\eta}, \\
\star d\overline{\eta} &= (dt \wedge d\eta), \\
\star (dt \wedge d\overline{\eta} \wedge d\overline{\eta}) &= S^{\overline{\eta}\overline{\eta}}, \\
(dt \wedge dt \wedge d\eta) &= 0, \\
(d\eta \wedge d\eta \wedge d\overline{\eta}) &= 0, \\
(d\eta \wedge d\overline{\eta} \wedge d\overline{\eta}) &= 0,
\end{aligned} \tag{26}$$

From (25), we finally yield the following relationships:

$$\begin{aligned}
\overline{f}_2 &= -\dot{\overline{f}}_1, \\
f_2 &= -\dot{f}_1, \\
\overline{B} &= -\dot{B}, \\
s &= 0, \\
b_1 &= -\overline{b}_2, \\
\overline{b}_1 &= 0, \\
b_2 &= 0. \\
\overline{s} &= 0.
\end{aligned} \tag{27}$$

Substituting these relationships in (9), we obtain the following super expansions for the supervariables

$$\begin{aligned}
\Theta^{(r)}(t,\eta,\overline{\eta}) &= \theta(t) - \eta\dot{\overline{f}}_1(t) - \overline{\eta}\dot{f}_1(t) - i\eta\overline{\eta}\dot{B}(t), \\
\mathscr{F}^{(r)}(t,\eta,\overline{\eta}) &= C(t) + i\overline{\eta}\mathscr{B}(t), \\
\overline{\mathscr{F}}^{(r)}(t,\eta,\overline{\eta}) &= \overline{C}(t) - i\eta\mathscr{B}(t),
\end{aligned} \tag{28}$$

where we have chosen $b_1 = -\overline{b}_2 = \mathscr{B}$ for our algebraic convenience and the superscript (r) denotes the reduced form of the supervariables [cf. (9) and (16)].

It is clear that we have not yet obtained the super expansions in terms of the basic variables of the present theory. In fact, the coefficients of $\eta, \overline{\eta}$, and $\eta\overline{\eta}$ in the expression of supervariables are still unknown. Thus, to accomplish this goal, we invoke the (anti-)co-BRST invariant restrictions on the dynamical variables. These restrictions are (see, e.g., [28, 30] for details)

$$\begin{aligned}
s_{(a)d}\left[zp_\theta - i\overline{C}C\right] &= 0, \\
s_{(a)d}\left[\theta p_\theta - i\dot{\overline{C}}C\right] &= 0.
\end{aligned} \tag{29}$$

We demand that these (anti-)co-BRST invariant restrictions would remain intact when we generalize them onto $(1,2)$-dimensional supermanifold. As a result, we can write

$$\begin{aligned}
\mathscr{Z}\mathscr{P}_\theta - i\overline{\mathscr{F}}^{(r)}\mathscr{F}^{(r)} &= zp_\theta - i\overline{C}C, \\
\Theta^{(r)}\mathscr{P}_\theta - i\partial_t\overline{\mathscr{F}}^{(r)}\mathscr{F}^{(r)} &= \theta p_\theta - i\dot{\overline{C}}C.
\end{aligned} \tag{30}$$

Exploiting (28) and (30), we yield the following relationships:

$$\begin{aligned}
p_\theta\overline{f}_1 - \mathscr{B}C &= 0, \\
p_\theta f_1 - \mathscr{B}\overline{C} &= 0, \\
Bp_\theta - \mathscr{B}\mathscr{B} &= 0, \\
p_\theta\dot{\overline{f}}_1 - \dot{\mathscr{B}}C &= 0, \\
p_\theta\dot{f}_1 - \mathscr{B}\dot{\overline{C}} &= 0, \\
\dot{B}p_\theta - \dot{\mathscr{B}}\mathscr{B} &= 0.
\end{aligned} \tag{31}$$

Here we again emphasize the fact that the restrictions in (29) are not enough to determine the precise values of secondary variables. We further note that $s_d(z\overline{C}) = 0$, $s_{ad}(zC) = 0$. These co-BRST and anti-co-BRST invariant restrictions would remain independent of η and $\overline{\eta}$. The generalization of these restrictions onto $(1,2)$-dimensional manifold yields the following interesting relationships:

$$\begin{aligned}
&\mathscr{Z}\overline{\mathscr{F}}^{(r)} = z\overline{C} \Longrightarrow \\
&\begin{cases} f_1\overline{C} = 0, \\ B\overline{C} - f_1\mathscr{B} = 0, \\ \overline{f}_1\overline{C} - iz\mathscr{B} = 0, \end{cases} \\
&\text{and } \mathscr{Z}\mathscr{F}^{(r)} = zC \Longrightarrow \\
&\begin{cases} \overline{f}_1 C = 0, \\ BC - \overline{f}_1\mathscr{B} = 0, \\ \overline{f}_1 C + iz\mathscr{B} = 0. \end{cases}
\end{aligned} \tag{32}$$

It is clear that the relations $f_1\overline{C} = 0$ and $\overline{f}_1 C = 0$ fix the value of secondary variables f_1 and $\overline{f}_1$ as $f_1 \propto \overline{C}$ and $\overline{f}_1 \propto C$ [28, 30]. The simplest solutions that satisfy the relationships appear in (31) and (32) are

$$\begin{aligned}
f_1 &= \overline{C}, \\
\overline{f}_1 &= C, \\
\mathscr{B} &= p_\theta = B.
\end{aligned} \tag{33}$$

As a consequence, we obtain the precise values of the secondary variables in terms of the basic and auxiliary variables. Further, it is to be noted that the dynamical variables r, p_r, and p_θ are (anti-)co-BRST invariant and, thus, the supervariables

corresponding to them would remain independent of the Grassmannian variables. The supervariables now have the following expansions along the Grassmannian directions as follows:

$$\begin{aligned}
\mathscr{Z}^{(d)}\left(t,\eta,\overline{\eta}\right) &= z\left(t\right)+\eta C\left(t\right)+\overline{\eta}\overline{C}\left(t\right)+i\eta\overline{\eta}p_{\theta}\left(t\right),\\
\Theta^{(d)}\left(t,\eta,\overline{\eta}\right) &= \theta\left(t\right)-\eta\dot{C}\left(t\right)-\overline{\eta}\dot{\overline{C}}\left(t\right)-i\eta\overline{\eta}\dot{p}_{\theta}\left(t\right),\\
\mathscr{F}^{(d)}\left(t,\eta,\overline{\eta}\right) &= C\left(t\right)+i\overline{\eta}p_{\theta}\left(t\right),\\
\overline{\mathscr{F}}^{(d)}\left(t,\eta,\overline{\eta}\right) &= \overline{C}\left(t\right)-i\eta p_{\theta}\left(t\right),\\
\mathscr{R}^{(d)}\left(t,\eta,\overline{\eta}\right) &= r\left(t\right),\\
\mathscr{P}_{r}^{(d)}\left(t,\eta,\overline{\eta}\right) &= p_{r}\left(t\right),\\
\mathscr{P}_{\theta}^{(d)}\left(t,\eta,\overline{\eta}\right) &= p_{\theta}\left(t\right),
\end{aligned} \tag{34}$$

where the superscript (d) denotes that the above expressions for the supervariables obtained after the application of dual-horizontality conditions together with the (anti-)co-BRST invariant restrictions. Now, from the above expansions of the supervariables, we obtain the complete set of off-shell nilpotent and absolutely anticommuting (anti-)co-BRST symmetry transformations [cf. (4)] (see, [30] for details). To be more specific, the co-BRST (s_d) and anti-co-BRST (s_{ad}) transformations for any generic variable can be obtained from their corresponding supervariable as

$$\begin{aligned}
s_d\phi\left(t\right) &= \frac{\partial}{\partial\overline{\eta}}\Phi^{(d)}\left(t,\eta,\overline{\eta}\right)\Big|_{\eta=0},\\
s_{ad}\phi\left(t\right) &= \frac{\partial}{\partial\eta}\Phi^{(d)}\left(t,\eta,\overline{\eta}\right)\Big|_{\overline{\eta}=0},\\
s_d s_{ad}\phi\left(t\right) &= \frac{\partial}{\partial\overline{\eta}}\frac{\partial}{\partial\eta}\Phi^{(d)}\left(t,\eta,\overline{\eta}\right).
\end{aligned} \tag{35}$$

In other words, the co-BRST symmetry transformation (s_d) is equivalent to the translation of the generic supervariable $\Phi^{(d)}(t,\eta,\overline{\eta})$ along $\overline{\eta}$-direction while keeping η-direction fixed. Similarly, the anti-co-BRST transformation (s_{ad}) can be obtained by taking the translation of the generic supervariable $\Phi^{(d)}(t,\eta,\overline{\eta})$ along η-direction while $\overline{\eta}$-direction remains intact.

Before we wrap this section, we point out that the total gauge-fixing term $(1/2)b^2 + b(\dot{z}-\theta)$ remains invariant under (anti-)co-BRST transformations. Furthermore, the three terms $\dot{r}p_r - (1/2r^2)p_\theta^2 + V(r)$ do not transform under (anti-)co-BRST transformations because the dynamical variables r, p_r, p_θ remain invariant under the off-shell nilpotent (anti-)co-BRST symmetry transformations (4). The rest of terms in L, we can write as the co-BRST exact term and anti-co-BRST exact term. As a consequence, the Lagrangian can be written in two different ways in terms of s_d and s_{ad} as follows:

$$\begin{aligned}
L &= \dot{r}p_r-\frac{1}{2r^2}p_\theta^2+V\left(r\right)+\frac{1}{2}b^2+b\left(\dot{z}-\theta\right)\\
&\quad+s_d\left[+iC\left(\dot{z}-\theta\right)\right]\\
&\equiv \dot{r}p_r-\frac{1}{2r^2}p_\theta^2+V\left(r\right)+\frac{1}{2}b^2+b\left(\dot{z}-\theta\right)\\
&\quad+s_{ad}\left[-i\overline{C}\left(\dot{z}-\theta\right)\right],
\end{aligned} \tag{36}$$

modulo a total time derivative. It is now clear from the above that the (anti-)co-BRST invariance of L can be proven in a simpler way because of the nilpotency property of the (anti-)co-BRST transformations.

5. Invariance of Lagrangian

In this section, we capture the (anti-)BRST and (anti-)co-BRST invariances of the Lagrangian in terms of the Grassmannian translational generators $(\partial_\eta, \partial_{\overline{\eta}})$. To accomplish this goal, we generalize the total Lagrangian (L) from $(0+1)$-dimensional manifold to super Lagrangian $(\mathscr{L})$ defined onto $(1,2)-$dimensional supermanifold.

We note that the gauge-invariant (first-order) Lagrangian (1) can be generalized to super Lagrangian in terms of the supervariables (13) and (19) as

$$\begin{aligned}
L_f &\longrightarrow \mathscr{L}_f\\
&= \dot{r}p_r+\dot{\Theta}^{(h)}p_\theta-\frac{1}{2}p_r^2-\frac{1}{2r^2}p_\theta^2-\mathscr{Z}^{(h)}p_\theta-V\left(r\right).
\end{aligned} \tag{37}$$

One can check that the super Lagrangian $\mathscr{L}_f$, defined onto (1, 2)-dimensional supermanifold, is independent of the Grassmannian variables (*i.e.*, $\mathscr{L}_f = L_f$) and this is the reason behind the invariance of L_f under the (anti-)BRST transformations. This statement, mathematically, can be corroborated in terms of the translational generators along the Grassmannian directions as follows:

$$\begin{aligned}
&\frac{\partial}{\partial\overline{\eta}}\mathscr{L}_f=0\Longleftrightarrow\\
&\quad s_bL_f=0,\\
&\frac{\partial}{\partial\eta}\mathscr{L}_f=0\Longleftrightarrow\\
&\quad s_{ab}L_f=0.
\end{aligned} \tag{38}$$

Similarly, the total Lagrangian L onto $(1,2)$-dimensional supermanifold can be written as

$$\begin{aligned}
\mathscr{L} &= \mathscr{L}_f+\frac{1}{2}b^2+b\left(\dot{\mathscr{Z}}^{(h)}+\Theta^{(h)}\right)-i\dot{\overline{\mathscr{F}}}^{(h)}\dot{\mathscr{F}}^{(h)}\\
&\quad+i\overline{\mathscr{F}}^{(h)}\mathscr{F}^{(h)}.
\end{aligned} \tag{39}$$

The quasi-(anti-)BRST invariance of the total Lagrangian L [cf. (5)] can be translated in terms of the above super Lagrangian and the Grassmannian derivatives as

$$\begin{aligned}
\frac{\partial}{\partial\overline{\eta}}\mathscr{L}\bigg|_{\eta=0} &= \frac{d}{dt}\left(b\dot{C}\right) \Longleftrightarrow \\
s_b L &= \frac{d}{dt}\left(b\dot{C}\right), \\
\frac{\partial}{\partial\eta}\mathscr{L}\bigg|_{\overline{\eta}=0} &= -\frac{d}{dt}\left(b\dot{\overline{C}}\right) \Longleftrightarrow \\
s_{ab} L &= -\frac{d}{dt}\left(b\dot{\overline{C}}\right).
\end{aligned} \tag{40}$$

Mention should be made here that the super Lagrangian (39), after a bit algebraic computation, leads to the Lagrangian (3) plus total time derivative terms which contain Grassmannian variables $(\eta, \overline{\eta})$. This is why, the actions of Grassmannian derivatives on (39) lead to total derivatives.

Furthermore, there are two more ways to write the super Lagrangian (39) in the language of supervariables. These are listed as follows:

$$\begin{aligned}
L \longrightarrow \mathscr{L} &= \mathscr{L}_f \\
&+ \frac{\partial}{\partial\overline{\eta}}\left[-i\overline{\mathscr{F}}^{(h)}\left(\dot{\mathscr{Z}}^{(h)} + \Theta^{(h)} + \frac{b}{2}\right)\right]\bigg|_{\eta=0} = \mathscr{L}_f \\
&+ \frac{\partial}{\partial\eta}\left[i\mathscr{F}^{(h)}\left(\dot{\mathscr{Z}}^{(h)} + \Theta^{(h)} + \frac{b}{2}\right)\right]\bigg|_{\overline{\eta}=0} = \mathscr{L}_f + \frac{\partial}{\partial\overline{\eta}} \\
&\cdot \frac{\partial}{\partial\eta}\left[\frac{i}{2}\left(\mathscr{Z}^{(h)}\mathscr{Z}^{(h)} - \Theta^{(h)}\Theta^{(h)}\right) - \frac{1}{2}\overline{\mathscr{F}}^{(h)}\mathscr{F}^{(h)}\right].
\end{aligned} \tag{41}$$

We notice that (anti-)BRST invariance of total Lagrangian L can also be captured in terms of the Grassmannian derivatives as follows:

$$\begin{aligned}
\frac{\partial}{\partial\overline{\eta}}\mathscr{L}\bigg|_{\eta=0} &= 0 \Longleftrightarrow \\
s_b L &= 0, \\
\frac{\partial}{\partial\eta}\mathscr{L}\bigg|_{\overline{\eta}=0} &= 0 \Longleftrightarrow \\
s_{ab} L &= 0,
\end{aligned} \tag{42}$$

where we have used the nilpotency property $(\partial_\eta^2 = \partial_{\overline{\eta}}^2 = 0)$ of the Grassmannian derivatives ∂_η and $\partial_{\overline{\eta}}$. At this juncture, we point out that there is a little difference between (40) and (42). This happens because of the fact that we have discarded the total time derivative term while deriving the Lagrangian (3) from (21) (without any loss of generality). Because we know that the total time derivative term in the Lagrangian (or action) does not affect the dynamics of the system.

In an exactly similar fashion, one can also prove the (anti-)co-BRST invariance of the Lagrangian. For this purpose, we generalize the Lagrangian (3) from an ordinary 1-dimensional space to the $(1,2)$-dimensional superspace in terms of the supervariables (34) as

$$\begin{aligned}
L &\longrightarrow \mathscr{L} \\
&= \dot{r}p_r + \dot{\Theta}^{(d)}p_\theta - \frac{1}{2}p_r^2 - \frac{1}{2r^2}p_\theta^2 - \mathscr{Z}^{(d)}p_\theta - V(r) \\
&+ \frac{1}{2}b^2 + b\left(\dot{\mathscr{Z}}^{(d)} + \Theta^{(d)}\right) - i\dot{\overline{\mathscr{F}}}^{(d)}\dot{\mathscr{F}}^{(d)} \\
&+ i\overline{\mathscr{F}}^{(d)}\mathscr{F}^{(d)}.
\end{aligned} \tag{43}$$

Upon simplifying the above super Lagrangian, we note that it is independent of the Grassmannian variables (η and $\overline{\eta}$). In fact, it leads to the Lagrangian (3) modulo a total time derivative term. As a consequence, we yield

$$\begin{aligned}
\frac{\partial}{\partial\overline{\eta}}\mathscr{L}\bigg|_{\eta=0} &= -\frac{d}{dt}\left(p_\theta\dot{\overline{C}}\right) \Longleftrightarrow \\
s_d L &= -\frac{d}{dt}\left(p_\theta\dot{\overline{C}}\right), \\
\frac{\partial}{\partial\eta}\mathscr{L}\bigg|_{\overline{\eta}=0} &= -\frac{d}{dt}\left(p_\theta\dot{C}\right) \Longleftrightarrow \\
s_{ad} L &= -\frac{d}{dt}\left(p_\theta\dot{C}\right),
\end{aligned} \tag{44}$$

which are consistent with the equations given in (5).

As we already know that the total gauge-fixing term $b^2/2 + b(\dot{z} + \theta)$ is (anti-)co-BRST invariant [cf. (4)]. Thus, the total gauge-fixing super Lagrangian

$$\mathscr{L}_{GF} = \frac{1}{2}b^2 + b\left(\dot{\mathscr{Z}}^{(d)} + \Theta^{(d)}\right), \tag{45}$$

as one can easily check, is independent of the Grassmannian variables. In fact, we have

$$\begin{aligned}
\frac{\partial}{\partial\eta}\mathscr{L}_{GF} &= 0, \\
\frac{\partial}{\partial\overline{\eta}}\mathscr{L}_{GF} &= 0,
\end{aligned} \tag{46}$$

which reflect the fact that the total gauge-fixing terms remain invariant under the off-shell nilpotent (anti-)co-BRST transformations. Furthermore, the dynamical variables p_r and p_θ remain invariant under the (anti-)co-BRST transformations. Thus, we can write the total super Lagrangian in two more different ways in terms of the supervariables (34) as

$$\begin{aligned}
\mathscr{L} &= \dot{r}p_r - \frac{1}{2r^2}p_\theta^2 + V(r) + \frac{1}{2}b^2 + b\left(\dot{\mathscr{Z}}^{(d)} + \Theta^{(d)}\right) \\
&+ \frac{\partial}{\partial\eta}\left[+i\mathscr{F}^{(d)}\left(\mathscr{Z}^{(d)} - \dot{\Theta}^{(d)}\right)\right]\bigg|_{\overline{\eta}=0} \\
&\equiv \dot{r}p_r - \frac{1}{2r^2}p_\theta^2 + V(r) + \frac{1}{2}b^2 + b\left(\dot{\mathscr{Z}}^{(d)} + \Theta^{(d)}\right) \\
&+ \frac{\partial}{\partial\overline{\eta}}\left[-i\overline{\mathscr{F}}^{(d)}\left(\mathscr{Z}^{(d)} + \dot{\Theta}^{(d)}\right)\right]\bigg|_{\eta=0}.
\end{aligned} \tag{47}$$

It is clear from the above super Lagrangian that the (anti-)co-BRST invariance of the Lagrangian can now be proven in a simpler way due to the nilpotency property ($\partial_\eta^2 = \partial_{\overline{\eta}}^2 = 0$) of the Grassmannian derivatives ($\partial_\eta, \partial_{\overline{\eta}}$).

6. Nilpotency and Absolute Anticommutativity Check

The (anti-)BRST as well as (anti-)co-BRST symmetry transformations obey two key properties: (i) nilpotency of order two and (ii) absolute anticommutativity. The nilpotency property for any generic variable can be translated into superspace in terms of the corresponding supervariable and Grassmannian translational generators as follows:

$$\begin{aligned} s_b^2\phi(t) = 0 &\Longleftrightarrow \\ \frac{\partial}{\partial\overline{\eta}}\frac{\partial}{\partial\overline{\eta}}\Phi^{(h)}(t,\eta,\overline{\eta}) &= 0, \\ s_{ab}^2\phi(t) = 0 &\Longleftrightarrow \\ \frac{\partial}{\partial\eta}\frac{\partial}{\partial\eta}\Phi^{(h)}(t,\eta,\overline{\eta}) &= 0, \\ s_d^2\phi(t) = 0 &\Longleftrightarrow \\ \frac{\partial}{\partial\overline{\eta}}\frac{\partial}{\partial\overline{\eta}}\Phi^{(d)}(t,\eta,\overline{\eta}) &= 0, \\ s_{ad}^2\phi(t) = 0 &\Longleftrightarrow \\ \frac{\partial}{\partial\eta}\frac{\partial}{\partial\eta}\Phi^{(d)}(t,\eta,\overline{\eta}) &= 0. \end{aligned} \tag{48}$$

Similarly, the absolute anticommutativity property of the above nilpotent symmetry transformations can also be captured in terms of the supervariables and Grassmannian derivatives as given below:

$$\begin{aligned} (s_b s_{ab} + s_{ab} s_b)\phi(t) = 0 &\Longleftrightarrow \\ \left(\frac{\partial}{\partial\overline{\eta}}\frac{\partial}{\partial\eta} + \frac{\partial}{\partial\eta}\frac{\partial}{\partial\overline{\eta}}\right)\Phi^{(h)}(t,\eta,\overline{\eta}) &= 0, \\ (s_d s_{ad} + s_{ad} s_d)\phi(t) = 0 &\Longleftrightarrow \\ \left(\frac{\partial}{\partial\overline{\eta}}\frac{\partial}{\partial\eta} + \frac{\partial}{\partial\eta}\frac{\partial}{\partial\overline{\eta}}\right)\Phi^{(d)}(t,\eta,\overline{\eta}) &= 0, \end{aligned} \tag{49}$$

where $\phi(t)$ is any generic variable and $\Phi^{(h)}(t,\eta,\overline{\eta})$ and $\Phi^{(d)}(t,\eta,\overline{\eta})$ are the corresponding supervariables listed in (13), (19), and (34), respectively.

It is worthwhile to mention that the conserved BRST and anti-BRST charges can be written in terms of the (anti-)BRST symmetry transformations as follows:

$$\begin{aligned} Q_b &= -is_b\left(\overline{C}\dot{C} - \dot{\overline{C}}C\right) = -is_{ab}\left(\dot{C}C\right), \\ Q_{ab} &= +is_{ab}\left(\overline{C}\dot{C} - \dot{\overline{C}}C\right) = +is_b\left(\dot{\overline{C}}\overline{C}\right). \end{aligned} \tag{50}$$

Exploiting the expressions of the supervariables given in (13) and (19), one can generalize these conserved charges onto $(1,2)-$dimensional supermanifold as

$$\begin{aligned} Q_b &= -i\frac{\partial}{\partial\overline{\eta}}\left[\overline{\mathcal{F}}^{(h)}\dot{\mathcal{F}}^{(h)} - \dot{\overline{\mathcal{F}}}^{(h)}\mathcal{F}^{(h)}\right]\Big|_{\eta=0} \\ &\equiv -i\int d\overline{\eta}\left[\overline{\mathcal{F}}^{(h)}\dot{\mathcal{F}}^{(h)} - \dot{\overline{\mathcal{F}}}^{(h)}\mathcal{F}^{(h)}\right]\Big|_{\eta=0} \\ &= -i\frac{\partial}{\partial\eta}\left[\dot{\mathcal{F}}^{(h)}\mathcal{F}^{(h)}\right] \equiv -i\int d\eta\left[\dot{\mathcal{F}}^{(h)}\mathcal{F}^{(h)}\right], \\ Q_{ab} &= +i\frac{\partial}{\partial\eta}\left[\overline{\mathcal{F}}^{(h)}\dot{\mathcal{F}}^{(h)} - \dot{\overline{\mathcal{F}}}^{(h)}\mathcal{F}^{(h)}\right]\Big|_{\overline{\eta}=0} \\ &\equiv +i\int d\eta\left[\overline{\mathcal{F}}^{(h)}\dot{\mathcal{F}}^{(h)} - \dot{\overline{\mathcal{F}}}^{(h)}\mathcal{F}^{(h)}\right]\Big|_{\overline{\eta}=0} \\ &= +i\frac{\partial}{\partial\overline{\eta}}\left[\dot{\overline{\mathcal{F}}}^{(h)}\overline{\mathcal{F}}^{(h)}\right] \equiv +i\int d\overline{\eta}\left[\dot{\overline{\mathcal{F}}}^{(h)}\overline{\mathcal{F}}^{(h)}\right]. \end{aligned} \tag{51}$$

Using the basic principles of BRST formalism, we can also write the BRST and anti-BRST charges in the following fashion; namely,

$$\begin{aligned} Q_b &= is_b s_{ab}(zC) = \frac{i}{2}s_b s_{ab}\left(\dot{\theta}C - \theta\dot{C}\right), \\ Q_{ab} &= is_b s_{ab}\left(z\overline{C}\right) = \frac{i}{2}s_b s_{ab}\left(\dot{\theta}\overline{C} - \theta\dot{\overline{C}}\right). \end{aligned} \tag{52}$$

From the expressions given in (50) and (52), it is quite easy to show that $s_b Q_b = 0, s_{ab}Q_{ab} = 0$ which imply the nilpotency property: $Q_b^2 = 0, Q_{ab}^2 = 0$ whereas $s_b Q_{ab} = 0, s_{ab}Q_b = 0$ show the anticommutativity $Q_{ab}Q_b + Q_b Q_{ab} = 0$ of the (anti-)BRST charges $Q_{(a)b}$. The above expressions for the (anti-)BRST charges in terms of the supervariables are listed as follows:

$$\begin{aligned} Q_b &= i\frac{\partial}{\partial\overline{\eta}}\frac{\partial}{\partial\eta}\left[\mathcal{Z}^{(h)}\mathcal{F}^{(h)}\right] \equiv i\int d\overline{\eta}\int d\eta\left[\mathcal{Z}^{(h)}\mathcal{F}^{(h)}\right] \\ &= \frac{i}{2}\frac{\partial}{\partial\overline{\eta}}\frac{\partial}{\partial\eta}\left[\dot{\Theta}^{(h)}\mathcal{F}^{(h)} - \Theta^{(h)}\dot{\mathcal{F}}^{(h)}\right] \\ &\equiv \frac{i}{2}\int d\overline{\eta}\int d\eta\left[\dot{\Theta}^{(h)}\mathcal{F}^{(h)} - \Theta^{(h)}\dot{\mathcal{F}}^{(h)}\right], \\ Q_{ab} &= i\frac{\partial}{\partial\overline{\eta}}\frac{\partial}{\partial\eta}\left[\mathcal{Z}^{(h)}\overline{\mathcal{F}}^{(h)}\right] \equiv i\int d\overline{\eta}\int d\eta\left[\mathcal{Z}^{(h)}\overline{\mathcal{F}}^{(h)}\right] \\ &= \frac{i}{2}\frac{\partial}{\partial\overline{\eta}}\frac{\partial}{\partial\eta}\left[\dot{\Theta}^{(h)}\overline{\mathcal{F}}^{(h)} - \Theta^{(h)}\dot{\overline{\mathcal{F}}}^{(h)}\right] \\ &\equiv \frac{i}{2}\int d\overline{\eta}\int d\eta\left[\dot{\Theta}^{(h)}\overline{\mathcal{F}}^{(h)} - \Theta^{(h)}\dot{\overline{\mathcal{F}}}^{(h)}\right]. \end{aligned} \tag{53}$$

As a consequence of the expressions (51) and (53), we can capture the nilpotency and anticommutativity of the (anti-)BRST charges in terms of the Grassmannian generators as given below:

$$\begin{aligned}
\frac{\partial}{\partial\overline{\eta}}Q_b &= 0 \Longleftrightarrow \\
Q_b^2 &= 0, \\
\frac{\partial}{\partial\eta}Q_{ab} &= 0 \Longleftrightarrow \\
Q_{ab}^2 &= 0, \\
\frac{\partial}{\partial\overline{\eta}}Q_{ab} &= \frac{\partial}{\partial\eta}Q_b = 0 \Longleftrightarrow \\
Q_bQ_{ab} + Q_{ab}Q_b &= 0.
\end{aligned} \tag{54}$$

This algebra is true because of the fact that $\partial_\eta^2 = 0, \partial_{\overline{\eta}}^2 = 0$ and $\partial_\eta\partial_{\overline{\eta}} + \partial_{\overline{\eta}}\partial_\eta = 0$.

In a similar fashion, we can write the co-BRST and anti-co-BRST charges, in four different ways; namely,

$$\begin{aligned}
Q_d &= is_d\left(\overline{C}\dot{C} - \dot{\overline{C}}C\right) = is_{ad}\left(\dot{\overline{C}}\,\overline{C}\right) = is_ds_{ad}\left(\theta\overline{C}\right) \\
&= \frac{i}{2}s_ds_{ad}\left(z\dot{\overline{C}} - \dot{z}\overline{C}\right), \\
Q_{ad} &= -is_{ad}\left(\overline{C}\dot{C} - \dot{\overline{C}}C\right) = -is_d\left(\dot{C}C\right) = is_ds_{ad}\left(\theta C\right) \\
&= \frac{i}{2}s_ds_{ad}\left(z\dot{C} - \dot{z}C\right).
\end{aligned} \tag{55}$$

It is clear that, from the above expressions for the conserved (i.e., $\dot{Q}_{(a)d} = 0$) (anti-)co-BRST charges $Q_{(a)d}$, one can now again easily show $Q_d^2 = 0, Q_{ad}^2 = 0$ and $Q_dQ_{ad} + Q_{ad}Q_d = 0$ by exploiting the definition of a generator. For instance, the following relation $s_dQ_d = -i\{Q_d, Q_d\} = 0$ leads to $Q_d^2 = 0$ which shows the nilpotency of co-BRST charge.

In terms of the supervariables (34), the (anti-)co-BRST charges given in (55) take the following forms:

$$\begin{aligned}
Q_d &= +i\frac{\partial}{\partial\overline{\eta}}\left[\overline{\mathcal{F}}^{(d)}\dot{\mathcal{F}}^{(d)} - \dot{\overline{\mathcal{F}}}^{(d)}\mathcal{F}^{(d)}\right]\Bigg|_{\eta=0} \\
&\equiv +i\int d\overline{\eta}\left[\overline{\mathcal{F}}^{(d)}\dot{\mathcal{F}}^{(d)} - \dot{\overline{\mathcal{F}}}^{(d)}\mathcal{F}^{(d)}\right]\Bigg|_{\eta=0} \\
&= +i\frac{\partial}{\partial\eta}\left[\dot{\overline{\mathcal{F}}}^{(d)}\overline{\mathcal{F}}^{(d)}\right] \equiv +i\int d\eta\left[\dot{\overline{\mathcal{F}}}^{(d)}\overline{\mathcal{F}}^{(d)}\right] \\
&= i\frac{\partial}{\partial\overline{\eta}}\frac{\partial}{\partial\eta}\left[\Theta^{(d)}\overline{\mathcal{F}}^{(d)}\right] \equiv i\int d\overline{\eta}\int d\eta\left[\Theta^{(d)}\overline{\mathcal{F}}^{(d)}\right] \\
&= \frac{i}{2}\frac{\partial}{\partial\overline{\eta}}\frac{\partial}{\partial\eta}\left[\mathcal{Z}^{(d)}\dot{\overline{\mathcal{F}}}^{(d)} - \dot{\mathcal{Z}}^{(d)}\overline{\mathcal{F}}^{(d)}\right] \\
&\equiv \frac{i}{2}\int d\overline{\eta}\int d\eta\left[\mathcal{Z}^{(d)}\dot{\overline{\mathcal{F}}}^{(d)} - \dot{\mathcal{Z}}^{(d)}\overline{\mathcal{F}}^{(d)}\right],
\end{aligned}$$

$$\begin{aligned}
Q_{ad} &= -i\frac{\partial}{\partial\eta}\left[\overline{\mathcal{F}}^{(d)}\dot{\mathcal{F}}^{(d)} - \dot{\overline{\mathcal{F}}}^{(d)}\mathcal{F}^{(d)}\right]\Bigg|_{\overline{\eta}=0} \\
&\equiv -i\int d\eta\left[\overline{\mathcal{F}}^{(d)}\dot{\mathcal{F}}^{(d)} - \dot{\overline{\mathcal{F}}}^{(d)}\mathcal{F}^{(d)}\right]\Bigg|_{\overline{\eta}=0} \\
&= -i\frac{\partial}{\partial\overline{\eta}}\left[\dot{\mathcal{F}}^{(d)}\mathcal{F}^{(d)}\right] \equiv -i\int d\overline{\eta}\left[\dot{\mathcal{F}}^{(d)}\mathcal{F}^{(d)}\right] \\
&= i\frac{\partial}{\partial\overline{\eta}}\frac{\partial}{\partial\eta}\left[\Theta^{(d)}\mathcal{F}^{(d)}\right] \equiv i\int d\overline{\eta}\int d\eta\left[\Theta^{(d)}\mathcal{F}^{(d)}\right] \\
&= \frac{i}{2}\frac{\partial}{\partial\overline{\eta}}\frac{\partial}{\partial\eta}\left[\mathcal{Z}^{(d)}\dot{\mathcal{F}}^{(d)} - \dot{\mathcal{Z}}^{(d)}\mathcal{F}^{(d)}\right] \\
&\equiv \frac{i}{2}\int d\overline{\eta}\int d\eta\left[\mathcal{Z}^{(d)}\dot{\mathcal{F}}^{(d)} - \dot{\mathcal{Z}}^{(d)}\mathcal{F}^{(d)}\right].
\end{aligned} \tag{56}$$

It clear from the above equation that the following relations are true; namely,

$$\begin{aligned}
\frac{\partial}{\partial\overline{\eta}}Q_d &= 0 \Longleftrightarrow \\
Q_d^2 &= 0, \\
\frac{\partial}{\partial\eta}Q_{ad} &= 0 \Longleftrightarrow \\
Q_{ad}^2 &= 0, \\
\frac{\partial}{\partial\overline{\eta}}Q_{ad} &= \frac{\partial}{\partial\eta}Q_d = 0 \Longleftrightarrow \\
Q_dQ_{ad} + Q_{ad}Q_d &= 0,
\end{aligned} \tag{57}$$

where we have used the properties of the translational generators ∂_η and $\partial_{\overline{\eta}}$. It is now clear that we have captured the nilpotency as well as anticommutativity properties of the fermionic symmetry transformations (and corresponding charges) in the language of translational generators along the Grassmannian directions $(\eta, \overline{\eta})$.

7. Conclusions

In our present endeavour, we have derived the *proper* off-shell nilpotent and absolutely anticommuting (anti-)BRST as well as (anti-)co-BRST symmetry transformations within the framework of "augmented" supervariable approach. For the derivation of (anti-)BRST symmetry transformations, we have used, on one hand, horizontality condition and gauge-invariant restriction [cf. (10) and (17)]. On the other hand, we have exploited the dual-horizontality condition together with (anti-)co-BRST invariant restrictions for the precise derivation of (anti-)co-BRST transformations [cf. (24), (30), and (32)]. The (anti-)BRST and (anti-)co-BRST transformations for the Nakanishi–Lautrup type variable b have been derived from the requirements of the nilpotency and absolute anticommutativity properties of these transformations.

We point out that the first-order Lagrangian L_f is gauge-invariant and (anti-)BRST invariant. As a consequence, L_f is independent of the Grassmannian variables [cf. (38)] when it is generalized onto $(1,2)$D superspace. Further, the total gauge-fixing terms remain invariant under the (anti-)co-BRST symmetry transformations. In the superspace, one can see from (46) that it is also independent of Grassmannian variables $(\eta, \overline{\eta})$. We have expressed the total Lagrangian (3) in terms of the continuous and nilpotent symmetry transformations $s_{(a)b}$ and $s_{(a)d}$. In fact, for the (anti-)BRST invariance, the total gauge-fixing and Faddeev–Popov ghost terms can be written as the BRST exact, anti-BRST exact and anti-BRST exact followed by BRST exact [cf. (21)]. Similarly, for the (anti-)co-BRST invariance, the total Lagrangian (3) is also written in terms of the (anti-)co-BRST symmetry transformations [cf. (36)]. Thus, the (anti-)BRST and (anti-)co-BRST invariances of the Lagrangian (3) become straightforward because of the nilpotency property of the symmetry transformations $s_{(a)b}$ and $s_{(a)d}$.

We have provided the geometrical origin of the continuous (anti-)BRST and (anti-)co-BRST transformations within the framework of superspace formalism [cf. (20) and (35)]. By using the basic tenets of supervariable approach, we have written the Lagrangian in many different ways in terms of the supervariables (13) and (19) for the (anti-)BRST invariance and in terms of supervariables (34) for the (anti-)co-BRST invariance. Thus, we have been able to capture the invariance of the Lagrangian in the language of translational generators ∂_η and $\partial_{\overline{\eta}}$ (cf., Section 5).

Further, we have expressed the (anti-)BRST and (anti-)co-BRST charges in terms of the nilpotent symmetry transformations [cf. (50), (52), and (55), respectively]. In view of these, it is easy for us to write the conserved charges in terms of the supervariables, as one can see, in (51), (53) and (56). The key properties (i.e., nilpotency and anticommutativity) associated with the (anti-)BRST and (anti-)co-BRST transformations (and corresponding conserved charges) are translated in the properties of Grassmannian translational generators $\partial_\eta, \partial_{\overline{\eta}}$ along $\eta, \overline{\eta}$ directions.

Data Availability

No data were used to support this study.

Disclosure

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Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] C. N. Yang, "Physics today," *Einstein's Impact on Theoretical Physics*, vol. 33, no. 6, p. 42, 1980.

[2] P. A. M. Dirac, *Lectures on Quantum Mechanics*, Belfer Graduate School of Science, Yeshiva University Press, New York, NY, USA, 1964.

[3] K. Sundermeyer, *Constrained Dynamics: Lecture Notes in Physics*, vol. 169, Springer, Berlin, Germany, 1982.

[4] C. Becchi, A. Rouet, and R. Stora, "The abelian Higgs Kibble model, unitarity of the S-operator," *Physics Letters B*, vol. 52, no. 3, pp. 344–346, 1974.

[5] C. Becchi, A. Rouet, and R. Stora, "Renormalization of the abelian Higgs-Kibble model," *Communications in Mathematical Physics*, vol. 42, pp. 127–162, 1975.

[6] C. Becchi, A. Rouet, and R. Stora, "Renormalization of gauge theories," *Annals of Physics*, vol. 98, no. 2, pp. 287–321, 1976.

[7] I. V. Tyutin, "Gauge invariance in field theory and statistical physics in operator formalism," 1975, https://arxiv.org/abs/0812.0580.

[8] G. Curci and R. Ferrari, "Slavnov transformations and supersymmetry," *Physics Letters B*, vol. 63, no. 1, pp. 91–94, 1976.

[9] I. Ojima, "Another BRS Transformation," *Progress of Theoretical and Experimental Physics*, vol. 64, no. 2, pp. 625–638, 1980.

[10] L. Bonora and M. Tonin, "Superfield formulation of extended BRS symmetry," *Physics Letters B*, vol. 98, no. 1-2, pp. 48–50, 1981.

[11] L. Bonora, P. Pasti, and M. Tonin, "Geometric description of extended BRS symmetry in superfield formulation," *Nuovo Cimento A*, vol. 63, no. 3, pp. 353–364, 1981.

[12] R. Delbourgo and P. D. Jarvis, "Extended BRS invariance and OSp (4/2) supersymmetry," *Journal of Physics A: Mathematical and Theoretical*, vol. 15, no. 2, article no. 611, 1981.

[13] F. R. Ore Jr and P. van Nieuwenhuizen, "Local BRST symmetry and the geometry of gauge-fixing," *Nuclear Physics B*, vol. 204, no. 2, pp. 317–332, 1982.

[14] L. Alvarez-Gaumé and L. Baulieu, "The two quantum symmetries associated with a classical symmetry," *Nuclear Physics B*, vol. 212, no. 2, pp. 255–267, 1983.

[15] S. Hwang, "On the consistency of a BRST quantization — The rôle of the antiBRST symmetry," *Nuclear Physics B*, vol. 322, no. 1, pp. 107–126, 1989.

[16] R. Kumar, S. Krishna, A. Shukla, and R. P. Malik, "Abelian p-form (p = 1, 2, 3) gauge theories as the field theoretic models for the Hodge theory," *International Journal of Modern Physics A*, vol. 29, no. 24, Article ID 1450135, 2014.

[17] R. P. Malik, "One-form abelian gauge theory as the hodge theory," *International Journal of Modern Physics A*, vol. 22, no. 21, pp. 3521–3562, 2007.

[18] S. Gupta and R. P. Malik, "Rigid rotor as a toy model for Hodge theory," *The European Physical Journal C*, vol. 68, no. 1-2, pp. 325–335, 2010.

[19] R. Kumar and A. Shukla, "Novel symmetries in Christ-Lee model," *Europhysics Letters*, vol. 115, no. 2, article no. 21003, 2016.

[20] N. H. Christ and T. D. Lee, "Operator ordering and Feynman rules in gauge theories," *Physical Review D: Particles, Fields, Gravitation and Cosmology*, vol. 22, no. 4, pp. 939–958, 1980.

[21] M. E. Costa and H. O. Girotti, "Quantization of gauge-invariant theories through the Dirac-bracket formalism," *Physical Review D: Particles, Fields, Gravitation and Cosmology*, vol. 24, no. 12, pp. 3323–3325, 1981.

[22] M. E. V. Costa, H. O. Girotti, and T. J. M. Simes, "Dynamics of gauge systems and Dirac's conjecture," *Physical Review D*, vol. 32, no. 2, article no. 405, 1985.

[23] S. K. Blau, "The Feynman path integral for constrained systems," *Annals of Physics*, vol. 205, no. 2, pp. 392–417, 1991.

[24] U. Kulshreshtha, D. S. Kulshreshtha, and H. J. W. Muler-Kirsten, "BRST formulation of the christ–lee model," *Canadian Journal of Physic*, vol. 73, pp. 386–392, 1995.

[25] T. Eguchi, P. B. Gilkey, and A. J. Hanson, "Gravitation, gauge theories and differential geometry," *Physics Reports*, vol. 66, no. 6, pp. 213–393, 1980.

[26] S. Mukhi and N. Mukunda, *Introduction to Topology, Differential Geometry and Group Theory for Physicists*, Wiley Eastern Private Limited, New Delhi, India, 2010.

[27] R. P. Malik, "Unique nilpotent symmetry transformations for matter fields in QED: augmented superfield formalism," *The European Physical Journal C*, vol. 47, no. 1, pp. 227–234, 2006.

[28] D. Shukla, T. Bhanja, and R. P. Malik, "Self-dual chiral boson: augmented superfield approach," *The European Physical Journal C*, vol. 74, no. 9, article no. 3025, 2014.

[29] S. Gupta and R. Kumar, "Nilpotent symmetries in Jackiw-Pi model: augmented superfield approach," *International Journal of Theoretical Physics*, vol. 55, no. 2, pp. 927–948, 2016.

[30] S. Gupta and R. Kumar, "On augmented superfield approach to vector Schwinger model," *International Journal of Modern Physics A*, vol. 31, no. 34, Article ID 1650173, 2016.

[31] E. C. G. Sudarshan and N. Mukunda, *Classical Dynamics: A Modern Perspective*, John Wiley & Sons, New York, NY, USA, 1974.

[32] D. Nemeschansky, C. Preitschopf, and M. Weinstein, "A BRST primer," *Annals of Physics*, vol. 183, no. 2, pp. 226–268, 1988.

[33] N. Nakanishi and I. Ojima, *Covariant Operator Formalism of Gauge Theories and Quantum Gravity*, World Scientific, Singapore, 1990.