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Research Article

Fractional Derivative Regularization in QFT

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We propose in this paper a new regularization, where integer-order differential operators are replaced by fractional-order operators. Regularization for quantum field theories based on application of the Riesz fractional derivatives of noninteger orders is suggested. The regularized loop integrals depend on parameter that is the order $\alpha > 0$ of the fractional derivative. The regularization procedure is demonstrated for scalar massless fields in φ^4-theory on n-dimensional pseudo-Euclidean space-time.

1. Introduction

A characteristic feature of quantum field theories, which are used in high-energy physics, is ultraviolet divergence [1, 2]. These divergences arise in momentum space from modes of very high moment, i.e., the structure of the field theories at very short distances. In the narrow class of quantum theories, which are called "renormalizable", the divergences can be removed by a singular redefinition of the parameters of the theory. This process is called the renormalization [3], and it defines a quantum field theory as a nontrivial limit of theory with an ultraviolet cut-off.

The renormalization requires the regularization of the loop integrals in momentum space. These regularized integrals depend on parameters such as the momentum cut-off, the Pauli-Villars masses, and the dimensional regularization parameter, which are used in the corresponding regularization procedure. This regularized integration is ultraviolet finite.

We suggest a regularization procedure based on fractional-order derivatives of the Riesz type. In momentum space, these derivatives are represented by power functions of momentum.

Fractional calculus and fractional differential equations [4, 5] have a wide application in mechanics and physics. The theory of integrodifferential equations of noninteger orders is a powerful tool to describe the dynamics of systems and processes with power-law nonlocality, long-range memory, and/or fractal properties.

Recently, the spatial fractional-order derivatives have been actively used in the space-fractional quantum mechanics [6, 7], the quantum field theory and gravity for fractional space-time [8], and the fractional quantum field theory at positive temperature [9, 10].

Fractional calculus allows us to take into account fractional power-law nonlocality of continuously distributed systems and classical fields. Using the fractional calculus, we can consider space-time fractional differential equations in the quantum field theory. For the first time, the fractional-order Laplace and d'Alembert operators have been suggested by Riesz in [11]. Then, noninteger powers of d'Alembertian are considered in different works (for example, see Section 28 in [4] and [12–14]). The fractional Laplace and d'Alembert operators of the Riesz type are the basis for the fractional field theory in multidimensional spaces.

As it was shown in [15–19], the continuum equations with fractional derivatives of the Riesz type can be directly connected to lattice models with long-range properties. A connection between the dynamics of lattice system with long-range properties and the fractional continuum equations are proved by using the transform operation [15–18].

There are different definitions of fractional derivatives such as Riemann-Liouville, Caputo, Grünwald-Letnikov, Marchaud, Weyl, Sonin-Letnikov, and Riesz [4, 5]. Unfortunately, all these fractional derivatives have a lot of unusual properties. For example, the well-known Leibniz rule does not hold for differentiation of noninteger orders [20].

It should be noted that the use of the fractional derivative of noninteger order is actually equivalent to using an infinite number of derivatives of all integer orders. For example, the Riemann-Liouville derivative ${}^{\mathrm{RL}}\mathscr{D}^{\alpha}_{a+}$ can be represented in the form of the infinite series

$$\left({}^{\mathrm{RL}}\mathscr{D}^{\alpha}_{a+} f\right)(x) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} \frac{(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} \frac{d^k f(x)}{dx^k} \quad (1)$$

for analytic (expandable in a power series on the interval) functions on (a,b), (see Lemma 15.3 of [4]). Therefore, an application of fractional derivatives means that we take into account a contribution of derivatives of all integer orders with the power-law weight.

In the suggested FD regularization, we use fractional derivatives of noninteger orders α close to integer values. The use of these fractional derivatives of noninteger orders α close to integer values in the regularization procedure means that we consider a weak nonlocality, i.e., a slight deviation from the locality.

2. Fractional Laplacian for Euclidean space $\mathbb{R}^n$

2.1. Fractional Integration in the Riesz Form. The Riesz fractional integral of order α for $\mathbb{R}^n$ is defined [11] (see also Section 25 of [4]) by equation

$${}^{R}I^{\alpha} f(P) = \frac{1}{H_n(\alpha)} \int_{\mathbb{R}^n} f(Q)\, r_{PQ}^{\alpha-n} dQ, \quad (2)$$

where P and Q are points of the space $\mathbb{R}^n$, and

$$H_n(\alpha) = \frac{\pi^{n/2} 2^{\alpha} \Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}. \quad (3)$$

Let us give some properties of the Riesz fractional integral that are proved in [11].

(1) The semigroup property of the Riesz fractional integration is

$${}^{R}I^{\alpha_1}\left({}^{R}I^{\alpha_2} f(P)\right) = {}^{R}I^{\alpha_1+\alpha_2} f(P), \quad (4)$$

where $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_1 + \alpha_2 < n$.

(2) The action of the Laplace operator on the Riesz fractional integral is

$$\Delta\, {}^{R}I^{\alpha+2} f(P) = -{}^{R}I^{\alpha} f(P), \quad (5)$$

and we also have

$$\Delta^{k}\, {}^{R}I^{\alpha+2k} f(P) = (-1)^k\, {}^{R}I^{\alpha} f(P), \quad (k \in \mathbb{N}). \quad (6)$$

(3) The Riesz fractional integration of exponential function does not change this function:

$$\begin{aligned} {}^{R}I^{\alpha} e^{ix_j} &= e^{ix_j}, \\ {}^{R}I^{\alpha} e^{i\sum_{j=1}^{n} a_j x_j} &= \left(\sum_{j=1}^{n} a_j x_j\right)^{-\alpha/2} e^{i\sum_{j=1}^{n} a_j x_j}. \end{aligned} \quad (7)$$

There is the following important property:

$${}^{R}I^{\alpha} f(P) = (-1)^m\, {}^{R}I^{\alpha+2m} \Delta^m f(P). \quad (8)$$

This property allows us to use an analytic continuation of ${}^{R}I^{\alpha} f(P)$ for negative values of $\alpha > -2m$, where $m \in \mathbb{R}$ (see [11]). In this case, the semigroup property (4) can be used for $\alpha_1 > -2m$, $\alpha_2 > -2m$, $\alpha_1 + \alpha_2 > -2m$, and $m \in \mathbb{R}$. Analogously, we can consider property (6) in the form

$$\Delta^{k}\, {}^{R}I^{\alpha} f(P) = (-1)^k\, {}^{R}I^{\alpha-2k} f(P), \quad (k \in \mathbb{N}). \quad (9)$$

2.2. Fractional Laplacian in the Riesz Form. For the first time, fractional Laplace operators have been suggested by Riesz in [11] (see also Section 25 of [4]). The fractional Laplacian $(-\Delta)^{\alpha/2}$ in the Riesz form for n-dimensional Euclidean space $\mathbb{R}^n$ can be considered [4] as an inverse Fourier's integral transform $\mathscr{F}^{-1}$ of $|\mathbf{k}|^{\alpha}$ by

$$\left((-\Delta)^{\alpha/2} \varphi\right)(x) = \mathscr{F}^{-1}\left(|\mathbf{k}|^{\alpha} (\mathscr{F}\varphi)(\mathbf{k})\right), \quad (10)$$

where $\alpha > 0$ and $x \in \mathbb{R}^n$.

For $\alpha > 0$, the fractional Laplacian of the Riesz form can be defined [4] as the hypersingular integral

$$\left((-\Delta)^{\alpha/2} \varphi\right)(x) = \frac{1}{d_n(m,\alpha)} \int_{\mathbb{R}^n} \frac{1}{|z|^{\alpha+n}} \left(\Delta^m_z \varphi\right)(z)\, d^4 z, \quad (11)$$

where $m > \alpha$ and $(\Delta^m_z \varphi)(z)$ is a finite difference of order m of a field $\varphi(x)$ with a vector step $z \in \mathbb{R}^n$ centered at the point $x \in \mathbb{R}^n$:

$$\left(\Delta^m_z \varphi\right)(z) = \sum_{j=0}^{m} (-1)^j \frac{m!}{j!(m-j)!} \varphi(x - jz). \quad (12)$$

The constant $d_n(m,\alpha)$ is defined by

$$d_n(m,\alpha) = \frac{\pi^{1+n/2} A_m(\alpha)}{2^{\alpha}\Gamma(1+\alpha/2)\Gamma((n+\alpha)/2)\sin(\pi\alpha/2)}, \quad (13)$$

where

$$A_m(\alpha) = \sum_{j=0}^{m} (-1)^{j-1} \frac{m!}{j!(m-j)!} j^{\alpha}. \quad (14)$$

Note that the hypersingular integral (11) does not depend on the choice of $m > \alpha$. The Fourier transform $\mathscr{F}$ of the fractional Laplacian is given by $\mathscr{F}\{(-\Delta)^{\alpha/2}\varphi\}(\mathbf{k}) = |\mathbf{k}|^{\alpha}(\mathscr{F}\varphi)(\mathbf{k})$. This equation is valid for the Lizorkin space [4] and the space $C^{\infty}(\mathbb{R}^n)$ of infinitely differentiable functions on $\mathbb{R}^n$ with compact support.

2.3. Fractional Laplacian in the Riesz-Trujillo Form. Using property (8), we can define the fractional Laplace operator by the equation

$${}^{RT}\Delta^{\alpha/2} f(P) = (-1)^m\, {}^{R}I^{2m-\alpha} \Delta^m f(P), \quad (15)$$

where $m > \alpha/2$ and $m \in \mathbb{R}$. This form of definition of the fractional Laplacian has been suggested by Trujillo in 2012 (see also [21]).

It is important that the semigroup property holds for these operators

$$ {}^{RT}\Delta^{\alpha_1/2}\,{}^{RT}\Delta^{\alpha_2/2} f(P) = {}^{RT}\Delta^{(\alpha_1+\alpha_2)/2} f(P), \tag{16} $$

where $0 < \alpha_1 < 2m$, $0 < \alpha_2 < 2m$, and $\alpha_1 + \alpha_2 < 2m$.

We can see that

$$ {}^{RT}\Delta^{k} f(P) = (-1)^k \Delta^k f(P), \tag{17} $$

where $k \in \mathbb{N}$. In Riesz's notation [11], this equation means that

$$ {}^{R}I^{-2k} f(P) = (-1)^k \Delta^k f(P). \tag{18} $$

We also have a generalization in the form

$$ {}^{R}I^{\alpha-2k} f(P) = (-1)^k \Delta^k\,{}^{R}I^{\alpha} f(P). \tag{19} $$

Note that the value of m can be chosen sufficiently large to fulfill all the conditions on the fractional order α in property (16).

3. Fractional d'Alembertian for Pseudo-Euclidean Space-Time $\mathbb{R}^n_{1,n-1}$

For the first time, the fractional d'Alembertian has been suggested by Riesz in [11] (see also [21]). Note that, in Riesz's paper [11], the d'Alembertian $\square$ is denoted by Δ and it is called for the Laplace operator for $\mathbb{R}^n_{1,n-1}$. We will use the generally accepted notation.

For the pseudo-Euclidean space-time $\mathbb{R}^n_{1,n-1}$, we use

$$ r^2_{QP} = r^2_{PQ} = (x_1 - y_1)^2 - \sum_{j=2}^{n} (x_j - y_j)^2, \tag{20} $$

and the operator

$$ \square = \frac{\partial^2}{\partial x_1^2} - \sum_{j=2}^{n} \frac{\partial^2}{\partial x_j^2}. \tag{21} $$

The Riesz fractional integral is defined as

$$ {}^{R}I^{\alpha} f(P) = \frac{1}{H_n(\alpha)} \int_{\mathbb{R}^n_{1,n-1}} f(Q)\, r^{\alpha-n}_{PQ} dQ, \tag{22} $$

where $H_n(\alpha)$ is defined by the equations (see Eqs. 20 and 88 of [11])

$$ \begin{aligned} H_n(\alpha) &= \pi^{(n-2)/2} 2^{\alpha-1} \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{(\alpha+2-n)}{2}\right) \\ &= \pi^{(n-1)/2} \frac{\Gamma((\alpha+2-n)/2)\,\Gamma(\alpha)}{\Gamma((\alpha+1)/2)}. \end{aligned} \tag{23} $$

This expression of $H_n(\alpha)$ does not coincide with the Euclidean case (3). To ensure the convergence of the integral, we should assume that the parameter α satisfies the condition $\alpha - n > -2$, i.e., $\alpha > n - 2$.

There is the following important property (see Eq. 67 of Chapter 3 in [11]):

$$ {}^{R}I^{\alpha} f(P) = {}^{R}I^{\alpha+2m} \square^m f(P), \tag{24} $$

which allows us to use an analytic continuation of ${}^{R}I^{\alpha} f(P)$ for negative values of α if

$$ \alpha > n - 2 - 2m. \tag{25} $$

Using property (24), we can define the fractional d'Alembertian for n-dimensional pseudo-Euclidean space-time $\mathbb{R}^n_{1,n-1}$ by the equation

$$ \square^{\alpha/2} f(P) = {}^{R}I^{2m-\alpha} \square^m f(P), \tag{26} $$

where $m > (\alpha + n - 2)/2$ and $m \in \mathbb{R}$.

It is important that the semigroup property,

$$ \square^{\alpha_1/2} \square^{\alpha_2/2} f(P) = \square^{(\alpha_1+\alpha_2)/2} f(P), \tag{27} $$

holds for operators (26) if $0 < \alpha_1 < 2m + 2 - n$, $0 < \alpha_2 < 2m + 2 - n$, and $\alpha_1 + \alpha_2 < 2m + 2 - n$.

The fractional d'Alembertian $\square^{\alpha/2}$ for n-dimensional pseudo-Euclidean space-time $\mathbb{R}^n_{1,n-1}$ can be considered as an inverse Fourier's integral transform $\mathcal{F}^{-1}$ of $|\mathbf{k}|^{\alpha}$ by

$$ \left(\square^{\alpha/2} \varphi\right)(x) = \mathcal{F}^{-1}\left(|\mathbf{k}|^{\alpha} (\mathcal{F}\varphi)(\mathbf{k})\right), \tag{28} $$

where $\alpha > 0$ and $x \in \mathbb{R}^n_{1,n-1}$.

4. Scalar Field in Pseudo-Euclidean Space-Time

Let us consider a scalar field $\varphi(x)$ in the n-dimensional pseudo-Euclidean space-time $\mathbb{R}^n_{1,n-1}$ that is described by the field equation

$$ \left(\square + m^2\right) \varphi(x) = J(x), \tag{29} $$

where $\square$ is the d'Alembert operator, $\varphi(x)$ is a real field, and $x \in \mathbb{R}^n_{1,n-1}$ is the space-time vector with components x_μ, where $\mu = 0, 1, 2, \dots, n$. Suppose that the scalar field $\varphi(x)$ has a source $J(x)$, and we have put $\hbar = 1$. Field equation (29) follows from stationary action principle, $\delta S[\varphi] = 0$, where the action $S[\varphi]$ has the form

$$ S[\varphi] = \int_{\mathbb{R}^n_{1,n-1}} d^n x \mathcal{L}[\varphi(x)] \tag{30} $$

with the Lagrangian

$$ \mathcal{L}[\varphi(x)] = -\frac{1}{2} \varphi(x) \left(\square + m^2\right) \varphi(x). \tag{31} $$

The solution of (31) can be represented in the form

$$ \varphi(x) = -\int \Sigma_F(x - y) J(y) d^n y, \tag{32} $$

where $\Sigma_F(x-y)$ is the so-called Feynman propagator, obeying

$$\left(\Box + m^2 - i\varepsilon\right)\Sigma_F\left(x-y\right)(x) = -\delta\left(x\right). \tag{33}$$

Here, $\delta(x)$ is the Dirac delta function. It is easy to see that $\Sigma_F(x-y)$ has the Fourier representation

$$\Sigma_F\left(x-y\right)(x) = \int \frac{d^n k}{(2\pi)^n}\frac{e^{-ikx}}{k^2-m^2+i\varepsilon}. \tag{34}$$

We can consider φ^4 theory with an interaction constant g. It is known that $\Sigma_F(0)$ is a divergent quantity, which modifies the free particle propagator and contributes to the self-energy. In momentum space, it corresponds to the loop integral

$$g\Sigma_F\left(0\right) = g\int \frac{d^n q}{(2\pi)^n}\frac{1}{q^2-m^2}. \tag{35}$$

There are nth powers of q in the numerator and two in the denominator, so the integral diverges quadratically at large q; i.e., it is ultraviolet divergent. Expression (35) corresponds to a one-loop Feynman diagram that has the order g. Another divergent diagram is the $O(g^2)$ graph. The corresponding expression of the Feynman integral is

$$\begin{aligned} &g^2\int \frac{d^n q_1}{(2\pi)^n}\frac{d^n q_2}{(2\pi)^n}\frac{\delta\left(q_1+q_2-p_1-p_2\right)}{\left(q_1^2-m^2\right)\left(q_2^2-m^2\right)} \\ &\quad = g^2\int\frac{d^n q}{(2\pi)^n}\frac{1}{\left(q^2-m^2\right)\left(\left(p_1+p_2-q\right)^2-m^2\right)}. \end{aligned} \tag{36}$$

Here, there are nth powers of q in numerator and the four powers in denominator. For $n=4$, we have four powers of q in both numerator and denominator, so we get a logarithmic divergence.

5. Regularization

Regularization is a method of isolating the divergences. There are several techniques of regularization. The most intuitive one is to introduce a lattice regularization [22].

(1) The Pauli-Villars regularization modifies the quadratic terms by subtraction of the same quadratic terms with a much larger mass:

$$\frac{1}{2}\partial_\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 - \left(\frac{1}{2}\partial_\mu\phi\partial_\mu\phi - \frac{1}{2}M^2\phi^2\right). \tag{37}$$

The Pauli-Villars regularization is a contribution of another field (the Pauli-Villars field ϕ) with the same quantum numbers as the original field, but having the opposite statistics. For example, in the φ^4-theory, the Pauli-Villars field ϕ is used, and the Pauli-Villars regularization changes the propagator as

$$\frac{1}{p^2+m^2} \longrightarrow \frac{1}{p^2+m^2} - \frac{1}{p^2+m^2}\frac{e^{-p^2/\Lambda^2}}{p^2+M^2}. \tag{38}$$

(2) The Gaussian cut-off regularization is implemented by the modified kinetic term in the form

$$\frac{1}{2}\partial_\mu\varphi e^{\Box/\Lambda^2}\partial_\mu\varphi, \tag{39}$$

which changes the propagator

$$\frac{1}{p^2+m^2} \longrightarrow \frac{e^{-p^2/\Lambda^2}}{p^2+m^2}. \tag{40}$$

(3) The higher derivative regularization also modifies the quadratic terms such as

$$\frac{1}{2}\partial_\mu\varphi\left(1-\frac{\Box}{\Lambda^2}\right)\partial_\mu\varphi + \frac{1}{2}m^2\varphi\left(1-\frac{\Box}{\Lambda^2}\right)\varphi. \tag{41}$$

Then, the propagator is modified:

$$\frac{1}{p^2+m^2} \longrightarrow \left(1+\frac{p^2}{\Lambda^2}\right)\frac{1}{p^2+m^2}. \tag{42}$$

6. Regularization by Fractional Derivatives

We propose a new regularization, where integer-order differential operators are replaced by fractional-order operators. For example, we can replace $\Box$, which is the d'Alembert operator in the field equations, by the fractional d'Alembert operator $\Box^\alpha$:

$$\Box \longrightarrow \Box^{\alpha/2}. \tag{43}$$

Here, we use the dimensionless coordinates.

The fractional derivative regularization modifies the quadratic terms such as

$$\begin{aligned} &-\frac{1}{2}\varphi\left(x\right)\left(\Box + m^2\right)\varphi\left(x\right) \\ &\quad\longrightarrow -\frac{1}{2}\varphi\left(x\right)\left(\Box^{\alpha/2}+m^2\right)\varphi\left(x\right). \end{aligned} \tag{44}$$

Using the fractional d'Alembertian for pseudo-Euclidean space-time $\mathbb{R}^n_{1,n-1}$, we can replace field equation (29) by the equation

$$\left(\Box^{\alpha/2}+m^2\right)\varphi\left(x\right) = J\left(x\right), \tag{45}$$

where $\Box^\alpha$ is the fractional d'Alembert operator, $\varphi(x)$ is a real field, and $x \in \mathbb{R}^n_{1,n-1}$ is the space-time vector with components x_μ, where $\mu = 0,1,2,\ldots,n$.

The solution of (45) is

$$\varphi\left(x\right) = -\int \Sigma_F\left(x-y\right)J\left(y\right)d^n y, \tag{46}$$

where $\Sigma_F(x-y)$ is the Feynman propagator, obeying

$$\left(\Box^{\alpha/2}+m^2-i\varepsilon\right)\Sigma_F\left(x-y\right)(x) = -\delta\left(x\right). \tag{47}$$

Using (28), it is easy to see that $\Sigma_F(x-y)(x)$ has the Fourier representation

$$\Sigma_F\left(x\right) = \int\frac{d^n k}{(2\pi)^n}\frac{e^{-ikx}}{\left(k^2\right)^{\alpha/2}-m^2+i\varepsilon}. \tag{48}$$

In this case, $\Sigma_F(0)$ is a divergent quantity, which modifies the free particle propagator and contributes to the self-energy. In momentum space, it corresponds to the loop integral

$$g\Sigma_F(0) = g\int \frac{d^n q}{(2\pi)^n}\frac{1}{(q^2)^{\alpha/2} - m^2}. \tag{49}$$

There are nth powers of q in the numerator and α in the denominator. Another divergent diagram is the $O(g^2)$ graph. The corresponding expression of the Feynman integral is

$$\begin{aligned} &g^2\int \frac{d^n q_1}{(2\pi)^n}\frac{d^n q_2}{(2\pi)^n}\frac{\delta(q_1+q_2-p_1-p_2)}{\left((q_1^2)^{\alpha/2}-m^2\right)\left((q_2^2)^{\alpha/2}-m^2\right)} \\ &= g^2\int \frac{d^n q}{(2\pi)^n}\frac{1}{(q^2)^{\alpha/2}-m^2} \\ &\cdot \frac{1}{\left((p_1+p_2-q)^2\right)^{\alpha/2}-m^2}. \end{aligned} \tag{50}$$

Here, there are nth powers of q in numerator and the 2α powers in denominator.

7. FD Regularization for Massless Theory

For simplification, we will consider a massless scalar field theory. For this case, expression (36) with $m = 0$ has the form

$$\begin{aligned} &g^2\int \frac{d^n q_1}{(2\pi)^n}\frac{d^n q_2}{(2\pi)^n}\frac{\delta(q_1+q_2-p_1-p_2)}{\left((q_1^2)^{\alpha/2}\right)\left((q_2^2)^{\alpha/2}\right)} \\ &= g^2\int \frac{d^n q}{(2\pi)^n}\frac{1}{(q^2)^{\alpha/2}}\cdot\frac{1}{\left((p_1+p_2-q)^2\right)^{\alpha/2}}. \end{aligned} \tag{51}$$

The denominators in the integrand of (36) are combined by using Feynman's parametric integral formula

$$\frac{1}{A^a B^b} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1 dz\frac{z^{a-1}(1-z)^{b-1}}{[Az+B(1-z)]^{a+b}}. \tag{52}$$

Changing the variables $q \to p$ and $p_1 + p_2 \to q$, integral (51) takes the form

$$I(q) = g^2\int \frac{d^n p}{(2\pi)^n}\frac{1}{(p^2)^{\alpha/2}}\cdot\frac{1}{\left((p-q)^2\right)^{\alpha/2}}. \tag{53}$$

Using formula (52), we get

$$\begin{aligned} &\frac{1}{(p^2)^{\alpha/2}}\cdot\frac{1}{\left((p-q)^2\right)^{\alpha/2}} \\ &= \int_0^1 dz\frac{z^{\alpha/2-1}(1-z)^{\alpha/2-1}}{[p^2-2pq(1-z)+q^2(1-z)]^{\alpha}}. \end{aligned} \tag{54}$$

As a result, we have

$$\begin{aligned} I(q) &= g^2\int_0^1 z^{\alpha/2-1}(1-z)^{\alpha/2-1}dz \\ &\cdot\int \frac{d^n p}{(2\pi)^n}\frac{1}{[p^2-2pq(1-z)+q^2(1-z)]^{\alpha}}. \end{aligned} \tag{55}$$

Using $p = (p_0, \mathbf{r})$ and the polar coordinates, we get that momentum integral of (55) is

$$\begin{aligned} I[n,\alpha,q] &= \int \frac{d^n p}{(p^2-2pQ+M^2)^{\alpha}} = \frac{2\pi^{(n-1)/2}}{\Gamma((n-1)/2)} \\ &\cdot\int_{-\infty}^{+\infty}dp_0\int_0^{\infty}\frac{r^{n-2}dr}{[p_0^2-r^2-2pQ+M^2]^{\alpha}}, \end{aligned} \tag{56}$$

where

$$\begin{aligned} Q &= q(1-z), \\ M^2 &= q^2(1-z). \end{aligned} \tag{57}$$

This integral is Lorentz invariant, so we evaluate it in the frame $q_\mu = (\mu, 0)$. Then, $2pQ = 2\mu(1-z)p_0$. Changing variables to $p'_\mu = p_\mu - q_\mu(1-z)$, which implies that

$$\begin{aligned} p_0^2 - 2q(1-z)p_0 &= p_0^2 - 2\mu(1-z)p_0 \\ &= \left(p'_0\right)^2 - q^2(1-z)^2, \end{aligned} \tag{58}$$

we have

$$\begin{aligned} I[n,\alpha,q] &= \frac{2\pi^{(n-1)/2}}{\Gamma((n-1)/2)}\int_{-\infty}^{+\infty}dp'_0 \\ &\cdot\int_0^{\infty}\frac{r^{n-2}dr}{\left[\left(p'_0\right)^2-r^2+q^2z(1-z)\right]^{\alpha}}, \end{aligned} \tag{59}$$

where we use $q^2(1-z) - q^2(1-z)^2 = q^2 z(1-z)$.

Using the beta function

$$2\int_0^{\infty}\frac{t^{2x-1}}{(1+t^2)^{x+y}} = B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{60}$$

we get

$$\int_0^{\infty}\frac{s^{\beta}ds}{(s^2+M^2)^{\alpha}} = \frac{\Gamma((1+\beta)/2)\Gamma(\alpha-(1+\beta)/2)}{2(M^2)^{\alpha-(1+\beta)/2}\Gamma(\alpha)}, \tag{61}$$

where we can use

$$\begin{aligned} M^2 &= -\left(p'_0\right)^2 - q^2z(1-z), \\ \beta &= n-2. \end{aligned} \tag{62}$$

Using (61) with (62), (59) gives

$$\begin{aligned} I[n,\alpha,q] &= \pi^{(n-1)/2}\frac{\Gamma(\alpha-(n-1)/2)}{\Gamma(\alpha)} \\ &\cdot\int_{-\infty}^{+\infty}\frac{dp'_0}{\left[-\left(p'_0\right)^2-q^2z(1-z)\right]^{\alpha-(n-1)/2}} \\ &= (-1)^{-\alpha+(n-1)/2}\pi^{(n-1)/2}\frac{\Gamma(\alpha-(n-1)/2)}{\Gamma(\alpha)} \\ &\cdot\int_{-\infty}^{+\infty}\frac{dp'_0}{\left[\left(p'_0\right)^2+q^2z(1-z)\right]^{\alpha-(n-1)/2}}. \end{aligned} \tag{63}$$

Using (61), we get

$$I[n,\alpha,q] = i\pi^{n/2}\frac{\Gamma(\alpha-n/2)}{\Gamma(\alpha)}\frac{1}{[-q^2z(1-z)]^{\alpha-n/2}}. \quad (64)$$

As a result, we obtain

$$\begin{aligned} I[n,\alpha,q] &= \int \frac{d^n p}{[p^2-2pq(1-z)+q^2(1-z)]^{\alpha}} \\ &= i\pi^{n/2}\frac{\Gamma(\alpha-n/2)}{\Gamma(\alpha)}\frac{1}{[-q^2z(1-z)]^{\alpha-n/2}}. \end{aligned} \quad (65)$$

Using (65), expression (55) has the form

$$I(q) = ig^2\pi^{n/2}\frac{\Gamma(\alpha-n/2)}{\Gamma(\alpha)}\int_0^1 \frac{z^{\alpha/2-1}(1-z)^{\alpha/2-1}dz}{[-q^2z(1-z)]^{\alpha-n/2}}. \quad (66)$$

Then, we have

$$\begin{aligned} I(q) &= ig^2\pi^{n/2}\frac{\Gamma(\alpha-n/2)}{\Gamma(\alpha)}\frac{1}{[-q^2]^{\alpha-n/2}} \\ &\cdot \int_0^1 z^{(n-\alpha)/2-1}(1-z)^{(n-\alpha)/2-1}dz. \end{aligned} \quad (67)$$

For $n = 4$ and $\alpha = 2+\varepsilon$, we get

$$\frac{1}{[-q^2]^{\alpha-n/2}} = (-1)^{n/2-\alpha}\left(1+\varepsilon\ln\left(q^2\right)+O\left(\varepsilon^2\right)\right). \quad (68)$$

Here, we can use

$$(-1)^{n/2-\alpha} = (-1)^{-\varepsilon} = e^{i\pi\varepsilon} = 1+i\pi\varepsilon. \quad (69)$$

We can introduce the function

$$A(n,\alpha) = \int_0^1 z^{(n-\alpha)/2-1}(1-z)^{(n-\alpha)/2-1}dz. \quad (70)$$

For $n = 4$ and $\alpha = 2+\varepsilon$, expression (70) takes the form

$$\begin{aligned} A(4,2+\varepsilon) &= \int_0^1 (z(1-z))^{-\varepsilon/2}dz \\ &= 1-\frac{\varepsilon}{2}\int_0^1 \ln(z(1-z))\,dz + O\left(\varepsilon^2\right). \end{aligned} \quad (71)$$

Using

$$\int_0^1 \ln(z(1-z))\,dz = -2, \quad (72)$$

(71) is written as

$$A(4,2+\varepsilon) = 1+\varepsilon+O\left(\varepsilon^2\right). \quad (73)$$

For 4-dimensional space-time,

$$\frac{\Gamma(\alpha-n/2)}{\Gamma(\alpha)} = \frac{1}{(\alpha-1)(\alpha-2)}. \quad (74)$$

For $n = 4$ and $\alpha = 2+\varepsilon$, (74) gives

$$\frac{\Gamma(\alpha-n/2)}{\Gamma(\alpha)} = \frac{1}{(1+\varepsilon)\varepsilon} = \frac{1}{\varepsilon}-1. \quad (75)$$

Then, integral (67) with $n = 4$ and $\alpha = 2+\varepsilon$ has the form

$$\begin{aligned} &I(q) \\ &= ig^2\pi^2\left(\frac{1}{\varepsilon}-1\right)(1+\varepsilon)(1+i\pi\varepsilon)\left(1+\varepsilon\ln\left(q^2\right)\right). \end{aligned} \quad (76)$$

As a result, we get

$$I(q) = \frac{ig^2\pi^2}{\varepsilon}+ig^2\pi^2\ln\left(q^2\right)-\pi^3g^2+O(\varepsilon). \quad (77)$$

Remarks 1. For the special cases, we have

$$\begin{aligned} A(2,1) &= \pi, \\ A(4,1) &= \frac{\pi}{8}, \\ A(4,2) &= 1, \\ A(3,1) &= 1. \end{aligned} \quad (78)$$

The well-known asymptotic expression for the gamma function is

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon}-\gamma+\frac{1}{2}\left(\gamma^2+\frac{\pi^2}{6}\right)\varepsilon+O\left(\varepsilon^2\right) \quad (\varepsilon \longrightarrow 0). \quad (79)$$

For 2-dimensional space-time,

$$\frac{\Gamma(\alpha-n/2)}{\Gamma(\alpha)} = \frac{1}{\alpha-1}. \quad (80)$$

If $\alpha = 1+\varepsilon$, then

$$\frac{\Gamma(\alpha-n/2)}{\Gamma(\alpha)} = \frac{1}{\varepsilon}. \quad (81)$$

For 4-dimensional space-time ($n = 4$),

$$\frac{\Gamma(\alpha-n/2)}{\Gamma(\alpha)} = \frac{1}{(\alpha-1)(\alpha-2)}. \quad (82)$$

If $\alpha = 1+\varepsilon$, then

$$\frac{\Gamma(\alpha-n/2)}{\Gamma(\alpha)} = \frac{1}{\varepsilon(\varepsilon-1)} = -\frac{1}{\varepsilon}-1. \quad (83)$$

If $\alpha = 2+\varepsilon$, then

$$\frac{\Gamma(\alpha-n/2)}{\Gamma(\alpha)} = \frac{1}{(1+\varepsilon)\varepsilon} = \frac{1}{\varepsilon}-1. \quad (84)$$

Remark 2. By changing variables to $p' = p-q(1-z)$, we see that the denominator in the integrand of (54) is the square of $(p')^2-m^2+q^2z(1-z)$. But $d^n p' = d^n p$, so relabeling $p' \rightarrow p$, expression (51) becomes

$$\begin{aligned} I(q) &= g^2\int_0^1 z^{\alpha/2-1}(1-z)^{\alpha/2-1}dz \\ &\cdot \int \frac{d^n p}{(2\pi)^n}\frac{1}{[p^2+q^2z(1-z)]^{\alpha}}. \end{aligned} \quad (85)$$

8. Conclusion

In this paper, we proposed a new regularization of quantum field theories, which is based on the Riesz fractional derivatives of noninteger order $\alpha > 0$. The fractional derivatives of noninteger order are characterized by nonlocality; the proposed FD regularization is actually a regularization by space-time nonlocality.

Note that the Riesz fractional derivatives of noninteger order [11, 21] can be used not only for FD regularization. An application of fractional calculus in quantum field theory allows us to take into account space and time nonlocality [9, 10, 22–24]. It should be noted that we can use the Riesz fractional derivatives [11, 21], the fractional Laplacian, and d'Alembertian in the lattice quantum field theories [22]. Using the exact discretization of the fractional Laplacian d'Alembertian [19], we can consider exact discrete analogs [25] of the quantum field theories. The fractional derivatives of the Riesz type [11, 21] are directly connected to the long-range properties and nonlocal interactions [17, 18, 25]. Because of this, this tool can be used not only for regularization, but also for constructing new quantum theories in high-energy physics and condensed matter physics.

Appendix

For the fractional case, we should consider the integrals

$$K\left[\alpha, \beta, \gamma, M^2\right] = \int_0^\infty \frac{s^\gamma ds}{\left(s^\alpha + M^2\right)^\beta} \tag{A.1}$$

instead of the integrals of (61). Using new variable z such that $z^2 = s^\alpha$ and

$$\begin{aligned} z &= s^{\alpha/2}, \\ s &= z^{2/\alpha}, \\ ds &= \left(\frac{2}{\alpha}\right) z^{2/\alpha - 1} dz, \\ s^\gamma &= z^{2\gamma/\alpha}, \end{aligned} \tag{A.2}$$

(A.1) gives

$$\frac{2}{\alpha} K\left[\alpha, \beta, \gamma, M^2\right] = \int_0^\infty dz \frac{z^{2(\gamma+1)/\alpha - 1}}{\left(z^2 + M^2\right)^\beta}. \tag{A.3}$$

Then, using (61), we obtain

$$\begin{aligned} &K\left[\alpha, \beta, \gamma, M^2\right] \\ &\quad = \frac{2\Gamma\left(\left(\gamma+1\right)/\alpha\right)\Gamma\left(\beta - \left(\gamma+1\right)/\alpha\right)}{2\alpha\left(M^2\right)^{\beta-(\gamma+1)/\alpha}\Gamma\left(\beta\right)}. \end{aligned} \tag{A.4}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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References

[1] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields*, John Wiley & Sons, New York, NY, USA, 3rd edition, 1980.

[2] L. H. Ryder, *Quantum Field Theory*, Cambridge University Press, Cambridge, UK, 1985.

[3] J. C. Collins, *Renormalization: An Intro duction to Renormalization, the Renormaliza tion Group and the Operator-Product Expansion*, Cambridge University Press, Cambridge, UK, 1984.

[4] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, New York, NY, USA, 1993.

[5] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, New York, NY, USA, Elsevier, 2006.

[6] V. E. Tarasov, "Weyl quantization of fractional derivatives," *Journal of Mathematical Physics*, vol. 49, no. 10, Article ID 102112, 6 pages, 2008.

[7] V. E. Tarasov, *Quantum Mechanics of Non-Hamiltonian and Dissipative Systems*, vol. 7, Elsevier, New York, NY, USA, 2008.

[8] G. Calcagni, "Quantum field theory, gravity and cosmology in a fractal universe," *Journal of High Energy Physics*, vol. 2010, article 120, 2010.

[9] S. C. Lim, "Fractional derivative quantum fields at positive temperature," *Physica A: Statistical Mechanics and its Applications*, vol. 363, no. 2, pp. 269–281, 2006.

[10] S. C. Lim and L. P. Teo, "Casimir effect associated with fractional Klein-Gordon field," in *Fractional Dynamics. Recent Advances*, J. Klafter, S. C. Lim, and R. Metzler, Eds., pp. 483–506, World Science Publisher, Singapore, Singapore, 2011.

[11] M. Riesz, "L'intégrale de riemann-liouville et le problème de cauchy," *Acta Mathematica*, vol. 81, no. 1, pp. 1–222, 1949 (French).

[12] C. G. Bollini and J. J. Giambiagi, "Arbitrary powers of d'Alembertians and the Huygens principle," *Journal of Mathematical Physics*, vol. 34, no. 2, pp. 610–621, 1993.

[13] D. G. Barci, C. G. Bollini, L. E. Oxman, and M. . Rocca, "Lorentz-invariant pseudo-differential wave equations," *International Journal of Theoretical Physics*, vol. 37, no. 12, pp. 3015–3030, 1998.

[14] R. L. P. G. do Amaral and E. C. Marino, "Canonical quantization of theories containing fractional powers of the d'Alembertian operator," *Journal of Physics A: Mathematical and General*, vol. 25, no. 19, pp. 5183–5200, 1992.

[15] V. E. Tarasov, "Continuous limit of discrete systems with long-range interaction," *Journal of Physics A: Mathematical and General*, vol. 39, no. 48, pp. 14895–14910, 2006.

[16] V. E. Tarasov, "Map of discrete system into continuous," *Journal of Mathematical Physics*, vol. 47, no. 9, Article ID 092901, 24 pages, 2006.

[17] V. E. Tarasov, "Toward lattice fractional vector calculus," *Journal of Physics A: Mathematical and General*, vol. 47, no. 35, Article ID 355204, 51 pages, 2014.

[18] V. E. Tarasov, "Lattice fractional calculus," *Applied Mathematics and Computation*, vol. 257, pp. 12–33, 2015.

[19] V. E. Tarasov, "Exact discretization of fractional Laplacian," *Computers & Mathematics with Applications*, vol. 73, no. 5, pp. 855–863, 2017.

[20] V. E. Tarasov, "No violation of the Leibniz rule. No fractional derivative," *Communications in Nonlinear Science and Numerical Simulation*, vol. 18, no. 11, pp. 2945–2948, 2013.

[21] H. Prado, M. Rivero, J. J. Trujillo, and M. P. Velasco, "New results from old investigation: a note on fractional *m*-dimensional differential operators. The fractional Laplacian," *Fractional Calculus and Applied Analysis*, vol. 18, no. 2, pp. 290–306, 2015.

[22] V. E. Tarasov, "Fractional quantum field theory: from lattice to continuum," *Advances in High Energy Physics*, Article ID 957863, pp. 1–14, 2014.

[23] G. V. Efimov, "Non-local quantum theory of the scalar field," *Communications in Mathematical Physics*, vol. 5, pp. 42–56, 1967.

[24] G. V. Efimov, *The Problems of The Quantum Theory of Nonlocal Interactions*, Nauka, Moscow, Russia, 1985.

[25] V. E. Tarasov, "Exact discretization by Fourier transforms," *Communications in Nonlinear Science and Numerical Simulation*, vol. 37, pp. 31–61, 2016.