Research Article

Tunnelling Mechanism in Noncommutative Space with Generalized Uncertainty Principle and Bohr-Like Black Hole

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The paper deals with nonthermal radiation spectrum by tunnelling mechanism with correction due to the generalized uncertainty principle (GUP) in the background of noncommutative geometry. Considering the reformulation of the tunnelling mechanism by Banerjee and Majhi, the Hawking radiation spectrum is evaluated through the density matrix for the outgoing modes. The GUP corrected effective temperature and the corresponding GUP corrected effective metric in noncommutative geometry are determined using Hawking’s periodicity arguments. Thus, we obtain further corrections to the nonstrictly thermal black hole (BH) radiation spectrum which give new final distributions. Then, we show that the GUP and the noncommutative geometry modify the Bohr-like BH recently discussed in a series of papers in the literature. In particular, we find the intriguing result that the famous law of Bekenstein on the area quantization is affected neither by noncommutative geometry nor by the GUP. This is a clear indication of the universality of Bekenstein’s result. In addition, we find that both the Bekenstein-Hawking entropy and the total BH entropy to third-order approximation are still functions of the BH quantum level.

1. Introduction

Hawking radiation [1] in the tunnelling mechanism [2–11] is an elegant way to approach the particle creation caused by the vacuum fluctuations near the BH horizon. The virtual particle pair can be created either just inside the horizon or just outside the horizon. For both the possibilities, the negative energy particle is absorbed by the BH, resulting in a loss of the mass of the BH, and the positive energy particle moves toward infinity, causing the subsequent emission of Hawking radiation. Considering contributions beyond the semiclassical approximation in the tunnelling process, Parikh and Wilczek [2, 3] formulated the nonthermal spectrum of the radiation from BH which leads to interesting approaches [12, 13] to resolve the information loss paradox of BH evaporation [14]. Also, subsequently, by a novel formulation of the tunnelling formalism, Banerjee and Majhi [7] directly derived the black body spectrum for both bosons and fermions from a BH with standard Hawking temperature. The analysis in [7] was improved by one of us, Christian Corda [15], who found as final result a nonstrictly black body spectrum in agreement with the emission probability in [2, 3]. This nonthermal spectrum is deeply interrelated to the underlying quantum gravity theory. As a result, the particle emission can be interpreted as a quantum transition of frequency ω between two discrete states [13, 16, 17]. Thus, the particle itself generates a tunnel through the BH horizon [3, 15–17] having finite size. This solves a problem of the thermal approximation; namely, in that case, the tunnelling points have zero separation and hence there is no clear trajectory as there is no barrier [3, 15–17]. Other aspects of tunnelling mechanism have been discussed in [18, 19].

In this paper, we analyse the corrections to the nonthermal spectrum of Parikh and Wilczek due to the noncommutative geometry and the GUP. It is shown that such corrections modify the Bohr-like BH recently discussed in a series of papers in the literature [13, 20–24]. An important result will be that the famous law of Bekenstein on the area quantization [25] is affected neither by noncommutative geometry nor by the GUP. We can consider this as a clear
indication of the universality of Bekenstein’s result. We also find that both the Bekenstein-Hawking entropy and the total BH entropy to third-order approximation are still functions of the BH quantum level.

The physical reasons that motivate this work are two. On one hand, the background of noncommutative geometry is important because it can cure, in principle, some of the usual problems encountered in the description of the terminal phase of BH evaporation [26]. On the other hand, the GUP suggests a fundamental and discrete granular structure of space with important implications in quantum gravity [27–29]. For the sake of completeness, we take the chance to signal some recent important approach where GUP importance in BH physics is emphasized [30–35].

2. Basic Equations for Noncommutative Space

In noncommutative space, the usual definition of mass density in the form of Dirac delta function does not hold due to position-position uncertainty relation. A particle mass, instead of being exactly localized at the point, is diffused to position-position uncertainty relation. A particle mass, instead of being exactly localized at the point, is diffused instead of being exactly localized at the point, is diffused. Thus, the usual Schwarzschild metric, that is,

\[ ds^2 = - \left( 1 - \frac{2m}{r} \right) dt^2 + \left( 1 - \frac{2m}{r} \right)^{-1} dr^2 + r^2 d\Omega^2, \]  

(5)

takes the noncommutative form as [37]

\[ ds^2 = - \left( 1 - \frac{2m}{r} \right) dt^2 + \left( 1 - \frac{2m}{r} \right)^{-1} dr^2 + r^2 d\Omega^2. \]

Note that line element (5) can also be obtained as the solution of the Einstein field equations with (2) being the matter content. Thus, the event horizon can be obtained by solving \( g^{rr}(r_h) = 0 \) as

\[ r_h = 2m \sqrt{\frac{3}{2} \frac{r_h^2}{4\delta}}, \]

(6)

which clearly shows that \( r_h \) cannot be solved in closed form.

For the sake of completeness, we stress that noncommutative corrections to different thermodynamical quantities have been discussed in [38].

3. Tunnelling Approach to Radiation Spectrum: GUP Corrections

In strictly thermal approach, the probability of emission of Hawking quanta is [1]

\[ \Gamma \sim \exp \left( -\frac{\omega}{T_H} \right), \]  

(7)

where \( \omega \) is the energy frequency of the emitted radiation and \( T_H = 1/8\pi M \) is the usual Hawking temperature. This probability of emission has been modified by Parikh and Wilezek in the tunnelling framework, considering contributions beyond the semiclassical approximation as [2, 3]

\[ \Gamma \sim \exp \left[ -\frac{\omega}{T_H} \left( 1 - \frac{\omega}{2M} \right) \right] = \alpha \exp \left[ -\frac{\omega}{T_H} \left( 1 - \frac{\omega}{2M} \right) \right], \]

(8)

with \( \alpha \sim 1 \). For the sake of completeness, we stress that, in [39], nonthermal corrections to the Hawking effect were discussed through a different approach. In that case, the structure of \( \Gamma \) is retained as \( \exp[-\beta_{(corr)} \omega] \), where \( \beta_{(corr)} \) is given by [39]

\[ \beta_{(corr)} \equiv \beta_H \left( 1 + \sum \frac{\beta_i}{M^s} \right). \]

(9)

The nonleading terms are the corrections to the temperature due to quantum effects and \( \beta_H \equiv T_H^{-1} \) (see [39] for details).

Now, comparing the above two probabilities in (7) and (8), one can introduce the notion of the effective BH temperature \( T_E \) as [13, 15–17]

\[ T_E \equiv \frac{2M}{2M - 1} \frac{T_H}{T_H} = \frac{1}{4\pi (2M - \omega)}. \]

(10)

Analogously, one can define the effective mass and effective horizon radius as [13, 15–17]

\[ M_E = M - \frac{\omega}{2}, \]

(11)

\[ r_E = 2M_E = 2M - \omega. \]
Further, these effective quantities can be interpreted as average values of the corresponding quantities before (i.e., initial) and after (i.e., final) the particle emission \[13, 15–17\]. Accordingly, the effective temperature \(T_E\) is the inverse of the average value of the inverses of the initial and final Hawking temperatures \[13, 15–17\]. As a result, the effective Schwarzschild line element is \[15\]

\[
ds^2 = -\left(1 - \frac{2M_E}{r}\right)dt^2 + \left(1 - \frac{2M_E}{r}\right)^{-1} dr^2 + r^2d\Omega_2^2.
\]  

(12)

One can describe this line element as the BH dynamical geometry during the emission of the particle. Further, proceeding along the line of approach of one of us (Christian Corda) \[15\], the final nonstrictly thermal distributions considering the BH dynamical geometry take the form \[15\]

\[
\langle n \rangle_b = \frac{1}{\exp[4\pi(2M - \omega)\omega] - 1},
\]

(13)

\[
\langle n \rangle_f = \frac{1}{\exp[4\pi(2M - \omega)\omega] + 1},
\]

where the suffices \(b\) and \(f\) represent boson and fermion particles, respectively. This tunnelling approach has been recently finalized in \[20\].

On the other hand, by taking into account the noncommutative geometry discussed in Section 1 of this paper, starting from the noncommutative form of the Schwarzschild metric \[5\], the equations from \(7\) to \(13\) must be replaced by their noncommutative counterparts as

\[
\Gamma \sim \exp\left(-\frac{\omega}{T_{BH}}\right),
\]

(14)

where \(T_{BH} = 1/8\pi m_\gamma\) is the Hawking temperature in the noncommutative geometry:

\[
\Gamma \sim \exp\left[-\frac{\omega}{T_{BH}}\left(1 - \frac{\omega}{2m_\gamma}\right)\right],
\]

\[
T_{BH} = \frac{2m_\gamma}{2m_\gamma - 1} \frac{1}{4\pi(2m_\gamma - \omega)},
\]

\[
m_{E BH} = m_\gamma - \frac{\omega}{2},
\]

\[
r_E = 2m_{E BH} = 2m_\gamma - \omega,
\]

\[
ds^2 = -\left(1 - \frac{2m_{E BH}}{r}\right)dt^2 + \left(1 - \frac{2m_{E BH}}{r}\right)^{-1} dr^2 + r^2d\Omega_2^2,
\]

(15)

\[
\langle n \rangle_b = \frac{1}{\exp[4\pi(2m_\gamma - \omega)\omega] - 1},
\]

\[
\langle n \rangle_f = \frac{1}{\exp[4\pi(2m_\gamma - \omega)\omega] + 1}.
\]

Now, let us consider the modified Hawking temperature due to GUP corrections \[27, 28\]. It can be expressed as \[27, 28\]

\[
T_{H}^{(\text{GUP})} = \frac{1}{8\pi M}\left[1 - \frac{\alpha}{8\pi M} + 5\left(\frac{\alpha}{8\pi M}\right)^2\right].
\]

(16)

As a consequence, we can introduce a GUP modified BH mass and a GUP modified horizon radius as \[27, 28\]

\[
M^{(\text{GUP})} = \frac{M}{1 - \alpha/8\pi M + 5(\alpha/8\pi M)^2},
\]

(17)

\[
r^{(\text{GUP})} = 2M^{(\text{GUP})},
\]

respectively. Thus, \(16\) reads

\[
T_{H}^{(\text{GUP})} = \frac{1}{8\pi M^{(\text{GUP})}}.
\]

(18)

Using Hawking’s periodicity argument \[15, 27, 39–41\], one obtains the modified GUP Schwarzschild-like line element \[27\]:

\[
\left[ds^{(\text{GUP})}\right]^2 = -\left(1 - \frac{2M^{(\text{GUP})}}{r}\right)dt^2 + \frac{dr^2}{1 - 2M^{(\text{GUP})}/r} + r^2\left(\sin^2\theta d\varphi^2 + d\theta^2\right)
\]

(19)

and \[27\]

\[
\kappa^{(\text{GUP})} = \frac{1}{4M^{(\text{GUP})}}
\]

(20)

as the GUP modified surface gravity. The modified Schwarzschild solution \(19\) is obtained using the GUP in the background of noncommutative geometry. Thus, it is clear that the GUP Schwarzschild solution is not a solution of the Einstein field equation. It will be interesting to find modification of Einstein gravity for which the GUP Schwarzschild solution is a solution. This could be the subject of future works.

Now, combining this GUP correction with the notion of effective temperature, one can introduce the GUP corrected effective temperature as \[27\]

\[
T_{E}^{(\text{GUP})}(\omega) = \frac{2M^{(\text{GUP})}}{2M^{(\text{GUP})} - \omega}T_{H}^{(\text{GUP})}
\]

(21)

\[
= \frac{1}{4\pi(2M^{(\text{GUP})} - \omega)},
\]

the GUP corrected effective Boltzmann factor as \[27\]

\[
P_{E}^{(\text{GUP})}(\omega) = \frac{1}{T_{E}^{(\text{GUP})}(\omega)},
\]

(22)

and the GUP corrected effective mass and effective horizon radius as \[27\]

\[
M_{E}^{(\text{GUP})} = M^{(\text{GUP})} - \frac{\omega}{2},
\]

\[
r_{E}^{(\text{GUP})} = 2M_{E}^{(\text{GUP})} = 2M^{(\text{GUP})} - \omega.
\]

(23)
Note that the equations from (16) to (20) represent GUP corrections using Hamilton-Jacobi method beyond the semiclassical approximation. By taking into account the noncommutative geometry, the equations from (16) to (23) become

\[ T^{(GUP)}_{EH} = \frac{1}{8\pi m_\delta} \left[ 1 - \frac{\alpha}{8\pi m_\delta} + 5 \left( \frac{\alpha}{8\pi m_\delta} \right)^2 \right], \tag{24} \]

\[ m^{(GUP)}_\delta = \frac{m_\delta}{1 - \alpha/8\pi m_\delta + 5 (\alpha/8\pi m_\delta)^2}, \tag{25} \]

\[ r^{(GUP)}_\delta = 2m^{(GUP)}_\delta, \tag{26} \]

\[ [ds^{(GUP)}_\delta]^2 = -\left( 1 - \frac{2m^{(GUP)}_\delta}{r} \right) dt^2 + \frac{dr^2}{1 - 2m^{(GUP)}_\delta/r} + r^2 \left( \sin^2 \theta d\phi^2 + d\theta^2 \right), \tag{27} \]

\[ \kappa^{(GUP)}_\delta = \frac{1}{4m^{(GUP)}_\delta}, \tag{28} \]

\[ T^{(GUP)}_{EH} (\omega) = \frac{2m^{(GUP)}_\delta T^{(GUP)}_{EH}}{2m^{(GUP)}_\delta - \omega}, \tag{29} \]

\[ \bar{E}^{(GUP)}_{EH} (\omega) = \frac{1}{4\pi \left( 2m^{(GUP)}_\delta - \omega \right)}, \tag{30} \]

\[ m^{(GUP)}_{EH} = m^{(GUP)}_\delta - \omega, \tag{31} \]

Further, by Hawking's periodicity argument [15, 27, 40, 41], one easily obtains the modified GUP noncommutative effective Schwarzschild-like line element as

\[ [ds^{(GUP)}_{EH}]^2 = -\left( 1 - \frac{2m^{(GUP)}_{EH}}{r} \right) dt^2 + \frac{dr^2}{1 - 2m^{(GUP)}_{EH}/r} + r^2 \left( \sin^2 \theta d\phi^2 + d\theta^2 \right). \tag{32} \]

Now, if one follows step by step the analysis in [15, 27], then at the end one obtains the correct physical states for boson and fermions as

\[ |\Psi\rangle_{boson} = \left( 1 - \exp \left( -8\pi m^{(GUP)}_{EH} \omega \right) \right)^{1/2} \]

\[ \cdot \sum_n \exp \left( -4\pi m^{(GUP)}_{EH} \omega \right) |n^{(L)}_{out}\rangle \otimes |n^{(R)}_{out}\rangle, \tag{33} \]

\[ |\Psi\rangle_{fermion} = \left( 1 + \exp \left( -8\pi m^{(GUP)}_{EH} \omega \right) \right)^{-1/2} \]

\[ \cdot \sum_n \exp \left( -4\pi m^{(GUP)}_{EH} \omega \right) |n^{(L)}_{out}\rangle \otimes |n^{(R)}_{out}\rangle. \]

and the correct distributions as

\[ \langle n\rangle_{boson} = \frac{1}{\exp \left( 8\pi m^{(GUP)}_{EH} \omega \right) - 1} = \frac{1}{\exp \left[ 4\pi \left( 2m^{(GUP)}_{EH} - \omega \right) \right] - 1}, \tag{34} \]

\[ \langle n\rangle_{fermion} = \frac{1}{\exp \left( 8\pi m^{(GUP)}_{EH} \omega \right) + 1} = \frac{1}{\exp \left[ 4\pi \left( 2m^{(GUP)}_{EH} - \omega \right) \right] + 1}. \tag{35} \]

The above expressions of the distributions clearly show that they are not thermal in nature as the BH dynamical geometry during the emission of the particle, the GUP corrections to the semiclassical Hawking temperature, and the noncommutative geometry are taken into account.

4. Corrections to the Bohr-Like Black Hole

The general conviction that BHs should be highly excited states representing the fundamental bricks of quantum gravity [42] has been shown to be correct in the recent works [13, 20–24]. In such papers, one of us (Christian Corda) has indeed shown that the Schwarzschild BH is the gravitational analog of the historical semiclassical Bohr's hydrogen atom [43, 44]. The Bohr-like approach to BH quantum physics started with the pioneering works [16, 17]. It works through the natural correspondence between Hawking radiation and BH quasi-normal modes (QNMs) [13, 20–24]. Considering an isolated BH (in the same way that Bohr considered an isolated hydrogen atom), the emissions of Hawking quanta and the absorptions of external particles "trigger" the BH QNMs [13, 20–24]. In this analogy, BH QNMs represent the "electron" jumping from a quantum level to another one. Hence, their absolute values are the energy "shells" of the "gravitational hydrogen atom" [13, 20–24]. Remarkably, the time evolution of the system permits to solve the BH information puzzle [13, 22]. The results in [13, 20–24] are also consistent with previous results in the literature, including the historic result of Bekenstein on the area quantization [25]. The Bohr-like framework for BH quantum physics also finalizes the famous tunnelling approach of Parikh and Wilczek. One indeed finds the correct value of the prefactor...
of the Parikh and Wilczek probability of emission, that is, \( \alpha \) in (8), as [20]

\[
\alpha = \alpha_m = \frac{1 - \exp[-2\pi]}{1 - \exp[-2\pi (n_{\text{max}} - m + 1)]}.
\]

In this equation, \( n_{\text{max}} \) represents the maximum value of the principal quantum number \( n \) which can be found in (17) in [20], while \( m \) is the BH excited level. Thus, \( \alpha \) depends on the BH quantum level (see [20] for details). This result permits to write down the probability of emission between two generic BH quantum levels \( m \) and \( n \) in the intriguing form [20]:

\[
\Gamma_{m \rightarrow n} = \alpha_m \exp\left\{ \frac{\Delta E_{m \rightarrow n}}{T_E(\omega)} \right\} \\
= \alpha_m \exp[-2\pi (n - m)] \\
= \left\{ \frac{1 - \exp[-2\pi]}{1 - \exp[-2\pi (n_{\text{max}} - m + 1)]} \right\} \cdot \exp[-2\pi (n - m)].
\]

In a quantum mechanical framework, Hawking radiation can be physically interpreted in terms of quantum jumps among unperturbed levels [13, 16, 17, 20–24].

For large values of \( n \), that is, for excited BHs, the QNMs' expression of the Schwarzschild BH is independent of the angular momentum quantum number [13, 16, 17, 20–24]. In order to take into account the nonstrict therality of the radiation spectrum, one replaces the Hawking temperature with the effective temperature in the standard (thermal) QNMs equation obtaining [13, 16, 17, 20–24]:

\[
\omega_n = a + ib + 2\pi in \times T_E(\omega_n) = 2\pi in \times T_E(\omega_n) \\
= \frac{in}{4M - 2|\omega_n|}.
\]

Here \( a \) and \( b \) are real numbers with \( a = (\ln 3) \times T_E(\omega_n) \) and \( b = \pi \times T_E(\omega_n) \) for \( j = 0, 2 \) (scalar and gravitational perturbations), \( a = 0 \) and \( b = 0 \) for \( j = 1 \) (vector perturbations), and \( a = 0 \) and \( b = \pi \times T_E(\omega_n) \) for half-integer values of \( j \).

Now, let us see how the corrections due to the GUP and to noncommutative geometry change the model. As we use the modified GUP in noncommutative effective Schwarzschild line element (32) instead of the effective Schwarzschild line element (12), (38) must be replaced by

\[
\omega_n = a + ib + 2\pi in \times T_{E_{\text{GUP}}}^{(\text{GUP})}(\omega_n) \\
= 2\pi in \times T_{E_{\text{GUP}}}^{(\text{GUP})}(\omega_n) = \frac{in}{4m_{\text{GUP}}^2 - 2|\omega_n|} \\
= \frac{in}{4\left[m_{\delta}^2 \left[1 - \alpha/8\pi m_{\delta} + 5(\alpha/8\pi m_{\delta})^2\right]\right] - 2|\omega_n|}.
\]

The solution of (39) in terms of \( |\omega_n| \) reads

\[
|\omega_n| = \frac{m_{\delta}^2}{1 - \alpha/8\pi m_{\delta} + 5(\alpha/8\pi m_{\delta})^2} \pm \sqrt{\left[\frac{m_{\delta}^2}{1 - \alpha/8\pi m_{\delta} + 5(\alpha/8\pi m_{\delta})^2}\right]^2 - \frac{n^2}{2}}.
\]

Clearly, a BH does not emit more energy than its total mass [13, 16, 17, 20–24]. In this case, we must take into account the corrections to the mass arising from the GUP and noncommutative geometry. Thus, the physical solution is the one obeying

\[
|\omega_n| < \frac{m_{\delta}^2}{1 - \alpha/8\pi m_{\delta} + 5(\alpha/8\pi m_{\delta})^2};
\]

that is,

\[
E_n = |\omega_n| \\
= \frac{m_{\delta}^2}{1 - \alpha/8\pi m_{\delta} + 5(\alpha/8\pi m_{\delta})^2} \\
- \sqrt{\left[\frac{m_{\delta}^2}{1 - \alpha/8\pi m_{\delta} + 5(\alpha/8\pi m_{\delta})^2}\right]^2 - \frac{n^2}{2}}.
\]

\( E_n \) is the value of total energy emitted by the BH when it is excited at a level \( n \) [13, 16, 17, 20–24].

Let us consider an emission from the ground state (i.e., a BH that is not excited) to a state with large \( n = n_1 \). Then, by using (42), the GUP corrected mass of the analogous BH changes from \( m_{\delta}^{(\text{GUP})} \) to

\[
m_{\delta}^{(\text{GUP})}_{n_1} = m_{\delta}^{(\text{GUP})} - E_{n_1} = \sqrt{m_{\delta}^{(\text{GUP})^2 - \frac{n_1}{2}}} \\
= \sqrt{\left[\frac{m_{\delta}^2}{1 - \alpha/8\pi m_{\delta} + 5(\alpha/8\pi m_{\delta})^2}\right]^2 - \frac{n_1}{2}}.
\]

If now one considers a transition from \( n = n_1 \) to a different state with \( n = n_2 \), having \( n_2 > n_1 \), the GUP corrected mass of the analogous BH changes again from \( m_{\delta}^{(\text{GUP})}_{n_1} \) to

\[
m_{\delta}^{(\text{GUP})}_{n_2} = m_{\delta}^{(\text{GUP})}_{n_1} - \Delta E_{n_1 \rightarrow n_2} = m_{\delta}^{(\text{GUP})} - E_{n_2} \\
= \sqrt{\left[\frac{m_{\delta}^2}{1 - \alpha/8\pi m_{\delta} + 5(\alpha/8\pi m_{\delta})^2}\right]^2 - \frac{n_2}{2}}.
\]
\( \Delta E_{n_1 \rightarrow n_2} \) in previous equation is given by

\[
\Delta E_{n_1 \rightarrow n_2} \equiv E_{n_2} - E_{n_1} = m_\delta^{(GUP)}_{n_2} - m_\delta^{(GUP)}_{n_1}
\]

\[
= \sqrt{\left( \frac{m_\delta}{1 - \alpha/8\pi\delta + 5(\alpha/8\pi\delta)^2} \right)^2 - \frac{n_1}{2}} - \sqrt{\left( \frac{m_\delta}{1 - \alpha/8\pi\delta + 5(\alpha/8\pi\delta)^2} \right)^2 - \frac{n_2}{2}},
\]

(45)

and it represents the jump between the two levels due to the emission of a particle having frequency \( \Delta E_{n_1 \rightarrow n_2} \). Such a discrete amount of energy corresponds to a quantum jump. The issue that for large \( n \) one finds independence on the other quantum numbers is perfectly consistent with the Correspondence Principle stated by Bohr [45]. This principle indeed claims that "transition frequencies at large quantum numbers should equal classical oscillation frequencies." We stress again the analogy with Bohr's hydrogen atom. In that model [43, 44], energy is gained and lost by electrons through quantum jumps from one allowed energy shell to another. Hence, radiation can be absorbed or emitted and the energy difference of the levels respects the Planck constant and \( f \) is the transition frequency. In the present GUP corrected analogous Bohr-like BH, QNMs (the "gravitational electrons") only gain and lose energy through jumps from one allowed energy shell to another with absorptions or emissions of Hawking quanta, but now the energy difference of the levels is governed by (45). Remarkably, one interprets (42) in terms of a particle, the "electron," which is quantized on a circle of length

\[
L = \frac{1}{T_{E \delta}^{(GUP)}} (E_n) = 4\pi \left[ \frac{m_\delta}{1 - \alpha/8\pi\delta + 5(\alpha/8\pi\delta)^2} \right]^2 + \sqrt{\left( \frac{m_\delta}{1 - \alpha/8\pi\delta + 5(\alpha/8\pi\delta)^2} \right)^2 - \frac{n}{2}}.
\]

(46)

This finalizes the cited similarity to Bohr's hydrogen atom. Equation (46) represents indeed a perfect analogy with the electron travelling around the hydrogen nucleus with circular orbits in Bohr's approach [43, 44] and is also similar to planets travelling around the Sun in our solar system. We stress that Bohr's hydrogen atom represents an approximated model with respect to the valence shell atom model of full quantum mechanics. In the same way, the present GUP corrected analogous Bohr-like BH should be a better approximated model with respect to previous results in [13, 20–23]. But it is still far from the final, currently unknown, BH model of a full unitary quantum gravity theory.

Now, let us set \( n_1 = n - 1 \) and \( n_2 = n \) in (45). We get the emitted energy for a jump between two neighboring levels as

\[
\Delta E_{n \rightarrow n-1} = \sqrt{\left( \frac{m_\delta}{1 - \alpha/8\pi\delta + 5(\alpha/8\pi\delta)^2} \right)^2 - \frac{n}{2}} - \sqrt{\left( \frac{m_\delta}{1 - \alpha/8\pi\delta + 5(\alpha/8\pi\delta)^2} \right)^2 - \frac{n-1}{2}}.
\]

(47)

Bekenstein [25] has shown that the Schwarzschild BH area quantum should be \( \Delta A = 8\pi \) (the Planck length \( \ell_p = 1.616 \times 10^{-33} \) cm is equal to one in Planck units). In Schwarzschild BHs, the horizon area is connected to the mass by the relation \( A = 16\pi M^2 \). Hence, a variation \( \Delta M \) of the mass enables the variation

\[
\Delta A = 32\pi M \Delta M
\]

(48)

of the area. Setting \( \Delta M = -\Delta E_{n \rightarrow n-1} \) (the case of an emission), if one uses (43) and (47), one gets

\[
\Delta A_{n \rightarrow n-1} \equiv -32\pi m_\delta^{(GUP)}_{n-1} \Delta E_{n \rightarrow n-1}.
\]

(49)

One can think that (49) gives the area quantum of an excited GUP corrected analogous BH for a jump from the level \( n \) to the level \( n-1 \) in function of the "overtone" number \( n \) and of the initial GUP corrected analogous BH mass. But we see that one has a problem using (49). An absorption from the level \( n \) to the level \( n-1 \) is indeed possible through the absorbed energy:

\[
\Delta E_{n \rightarrow n-1} = -\Delta E_{n-1 \rightarrow n} = \sqrt{\left( \frac{m_\delta}{1 - \alpha/8\pi\delta + 5(\alpha/8\pi\delta)^2} \right)^2 - \frac{n}{2}} - \sqrt{\left( \frac{m_\delta}{1 - \alpha/8\pi\delta + 5(\alpha/8\pi\delta)^2} \right)^2 - \frac{n-1}{2}}.
\]

(50)

Hence, if one sets \( \Delta M = -\Delta E_{n \rightarrow n-1} = \Delta E_{n-1 \rightarrow n} \), one gets a quantum of area

\[
\Delta A_n \equiv -32\pi m_\delta^{(GUP)}_{n} \Delta E_{n \rightarrow n-1} = 32\pi m_\delta^{(GUP)}_{n} \Delta E_{n-1 \rightarrow n}.
\]

(51)

Thus, one finds that the absolute value of the area quantum for an absorption between two levels is not the same as the absolute value of the area quantum for an emission between the same levels. This is because \( m_\delta^{(GUP)}_{n} \neq m_\delta^{(GUP)}_{n-1} \).
Instead, we intuitively expect the area spectrum to be equal for both absorption and emission [13, 22]. One solves this problem considering the GUP corrected analogous effective mass which corresponds to the transitions between the two levels \( n \) and \( n-1 \). This latter is indeed the same for emission and absorption:

\[
m_{\delta}(GUP)_{n-1} = \frac{1}{2} \left[ \frac{m_{\delta}}{1 - \alpha/8\pi m_{\delta} + 5(\alpha/8\pi m_{\delta})^2} \right]^{n-1}_n + \left[ \frac{m_{\delta}}{1 - \alpha/8\pi m_{\delta} + 5(\alpha/8\pi m_{\delta})^2} \right]^{n}_n \]

Hence, let us replace \( m_{\delta}(GUP)_{n-1} \) with \( m_{\delta}(GUP)_{n-1} \) in (49) and \( m_{\delta}(GUP)_{n} \) again with \( M_{E(\delta(n-1))} \) in (51). We find

\[
\Delta A_{n-1} = -32\pi n m_{\delta}(GUP)_{n-1} \Delta E_{n-1 \rightarrow n} \quad \text{emission}
\]

\[
\Delta A_{n} = -32\pi n m_{\delta}(GUP)_{n-1} \Delta E_{n \rightarrow n-1} \quad \text{absorption}.
\]

Thus, now one gets \( |\Delta A_{n}| = |\Delta A_{n-1}| \). Equations (50) and (52) and some algebra give

\[
|\Delta A_{n}| = |\Delta A_{n-1}| = 8\pi.
\]

Thus, we find the very intriguing result that the famous law of Bekenstein on the area quantization [25] is affected neither by noncommutative geometry nor by the GUP. This is a clear indication of the universality of Bekenstein's result on the area quantization [25].

If one puts \( A_{n-1} = 16\pi (m_{\delta}(GUP)_{n-1})^2 \) and \( A_{n} = 16\pi (m_{\delta}(GUP)_{n})^2 \), one finds the formulas of the number of quanta of area as

\[
N_{n-1} = \frac{A_{n-1}}{|\Delta A_{n-1}|} = 16\pi \left( \frac{m_{\delta}(GUP)_{n-1}}{2m_{\delta}(GUP)_{E(\delta(n-1))} \Delta E_{n-1 \rightarrow n}} \right)^2
\]

before the emission and

\[
N_{n} = \frac{A_{n}}{|\Delta A_{n}|} = 16\pi \left( \frac{m_{\delta}(GUP)_{n}}{2m_{\delta}(GUP)_{E(\delta(n-1))} \Delta E_{n-1 \rightarrow n}} \right)^2
\]

after the emission, respectively. This implies that

\[
N_{n} - N_{n-1} = \frac{\left( m_{\delta}(GUP)_{n} \right)^2 - \left( m_{\delta}(GUP)_{n-1} \right)^2}{2m_{\delta}(GUP)_{E(\delta(n-1))} \Delta E_{n-1 \rightarrow n}} - \frac{\Delta E_{n-1 \rightarrow n} m_{\delta}(GUP)_{n} + m_{\delta}(GUP)_{n-1}}{2m_{\delta}(GUP)_{E(\delta(n-1))} \Delta E_{n-1 \rightarrow n}} = 1,
\]

as one expects. Now, one can write down the famous formula of Bekenstein-Hawking entropy [1, 46, 47] as

\[
(S_{BH})_{n} = \frac{A_{n}}{4} = 8\pi N_{n} m_{\delta}(GUP)_{n} \cdot \Delta E_{n-1 \rightarrow n} = 4\pi \left( m_{\delta}(GUP)_{n} \right)^2 - \frac{n + 1}{2}
\]

before the emission and

\[
(S_{BH})_{n} = \frac{A_{n}}{4} = 8\pi N_{n} m_{\delta}(GUP)_{n} \cdot \Delta E_{n-1 \rightarrow n} = 4\pi \left( m_{\delta}(GUP)_{n} \right)^2 - \frac{n}{2}
\]

after the emission, respectively. Hence, the Bekenstein-Hawking entropy can be written as a function of the QNMs principal quantum number, that is, of the BH quantum excited state. Equations (58) and (59) permit to generalize the results in [13, 20–24] to the current case of the GUP corrected mass of the analogous BH.

Now, we recall that the Bekenstein-Hawking entropy cannot be considered the definitive answer for a correct quantum theory of gravity [48]. It is indeed very important going beyond the Bekenstein-Hawking entropy and finding its subleading corrections [48]. By using the quantum tunnelling approach, one remarkably arrives to the subleading corrections at third-order approximation [49]:

\[
S_{\text{total}} = S_{BH} - \ln S_{BH} + \frac{3}{2A} + \frac{2}{A^2}.
\]

In this approach, the total BH entropy depends on four different parts: the standard Bekenstein-Hawking entropy, a logarithmic term, an inverse area term, and an inverse squared area term [49]. Thus, one can find the formulas of the total BH entropy taking into account the subleading corrections at third-order approximation and considering the GUP corrected mass of the analogous BH as

\[
(S_{total})_{n-1} = 4\pi \left( m_{\delta}(GUP)_{n-1} \right)^2 - \frac{n - 1}{2} - \ln 4\pi \left( m_{\delta}(GUP)_{n-1} \right)^2 + \frac{3}{2A} + \frac{2}{16\pi \left( m_{\delta}(GUP)_{n-1} \right)^2 - \frac{n - 1}{2}}.
\]
before the emission and
\[
(S_{\text{total}})_n = 4\pi \left[ \left( m_\delta^{\text{(GUP)}} \right)^2 - \frac{n}{2} \right] - \ln 4\pi \left[ \left( m_\delta^{\text{(GUP)}} \right)^2 - \frac{n}{2} \right] + \frac{3}{32\pi} \left[ \left( m_\delta^{\text{(GUP)}} \right)^2 - \frac{n}{2} \right]^2 + \frac{2}{16\pi} \left[ \left( m_\delta^{\text{(GUP)}} \right)^2 - \frac{n}{2} \right]^2
\] (62)

after the emission, respectively. Therefore, the total BH entropy at third-order approximation can be written as a function of the BH excited state \( n \). Again, here we improve the results in [13, 20–24] to the current case of the GUP corrected mass of the analogous BH.

5. Summary and Concluding Remarks

The present work considers GUP correction of nonthermal radiation spectrum in the background of noncommutative geometry using the framework of tunnelling mechanism. At first, we formulated the line element for Schwarzschild BH in the context of noncommutative geometry. Then we introduced the notion of effective temperature, effective mass, and effective horizon radius considering contributions beyond semiclassical approximation. Also, following the idea of one of us, Christian Corda, we have determined the nonstrictly thermal distributions for bosons and fermions. Subsequently, we introduce the GUP correction to the BH dynamical geometry by finding the final distributions for the GUP corrected mass of the analogous BH.

After that, it has been shown that the GUP and the noncommutative geometry modify the Bohr-like BH model, recently discussed in [13, 20–24]. In particular, we found the intriguing result that the famous law of Bekenstein on the area quantization [25] is affected neither by noncommutative geometry nor by the GUP. This is a clear indication of the universality of Bekenstein’s result. Finally, it has been shown that both the Bekenstein-Hawking entropy and the total BH entropy to third-order approximation are still functions of the BH quantum level, generalizing the results in [13, 20–24].

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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