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Research Article

Klein–Gordon Oscillator in a Topologically Nontrivial Space-Time

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In this study, we analyze solutions of the wave equation for scalar particles in a space-time with nontrivial topology. Solutions for the Klein–Gordon oscillator are found considering two configurations of this space-time. In the first one, the $S^1 \times R^3$ space is assumed where the metric is written in the usual inertial frame of reference. In the second case, we consider a rotating reference frame adapted to the circle S^1. We obtained compact expressions for the energy spectrum and for the particles wave functions in both configurations. Additionally, we show that the energy spectrum of the solution associated with the rotating system has an additional term that breaks the symmetry around $E = 0$.

1. Introduction

The local and the global structures of space-time play an important role in the behavior of quantum systems. In this aspect, it is believed that global features of space-time may be directly related to the shift of energy levels of quantum particles. In the particular case of the $S^1 \times R^1(\text{time}) \times R^2(\text{space})$ space-time, which is locally flat but which has a nontrivial topology, one may consider the effect of periodic boundary conditions in one spatial direction. In this space-time we have one compactified spacelike dimension; thus it is expected that despite the flat geometry, measurable effects occur in observable quantities. Nontrivial space-times have been studied extensively in literature; interesting applications are found in the study of atomic Bose-Einstein condensates [1] with toroidal optical dipole traps. In the context of quantum field theory, vacuum polarization in a nonsimply connected space-time with the topology of $S^1 \times R^3$ is considered in [2]. It was found that the vacuum energy for a free spinor field in twisted and untwisted configurations is different in $S^1 \times R^3$ space.

On the other hand, in quantum mechanics the harmonic oscillator is one of the most significant systems to be studied. In recent years, the relativistic version of the harmonic oscillator has been considered in several studies [3–16]. This important potential has been introduced as a linear interaction in the Klein–Gordon equation [17]. In the case of the Dirac equation, the so-called Dirac oscillator has been introduced as an instance of a relativistic potential such that its nonrelativistic limit leads to the harmonic oscillator plus a strong spin-orbit coupling [18]; this result is similar to the one that is obtained for the Klein–Gordon oscillator when the spin-orbit is absent. Most recently, the relativistic harmonic oscillator has been studied in the context of the Kaluza–Klein theory [19], where the Klein–Gordon oscillator coupled to a series of cosmic strings in five dimensions has been considered. In [20, 21] the author considers the effect of such kind of topological defect on scalar bosons described by the Duffin–Kemmer–Petiau (DKP) formalism. The Klein–Gordon oscillator in a noncommutative phase space under a uniform magnetic field has been studied in [22]. In this paper the authors conclude that the Klein–Gordon oscillator in a noncommutative space with a uniform magnetic field has behavior similar to the Landau problem in the usual space-time.

Another aspect of interest in our work is the influence of noninertial effects on quantum systems. As in classical physics, quantum mechanics is sensitive to the use of noninertial reference systems. These effects can be taken into

account through an appropriate coordinate transformation. Previous research reported in literature [23, 24] shows that rotating frames in the Minkowski space-time can play the role of a hard-wall potential. Recently these ideas have been applied to the case of spaces with nontrivial topology. In particular, a rotating system was proposed recently in [25] where a scalar field on a circle (topology $S^1 \times R^1$) with a Dirichlet cut has been considered. In [26], a similar study was carried out in the case of a five-dimensional space-time.

Therefore, in this contribution, we will study bosons in the $S^1 \times R^3$ space-time by considering the scalar wave equation for the Klein–Gordon oscillator. In fact, solutions of wave equations in curved spaces and nontrivial topology have been explored in various contexts [16, 26–37]. We will examine the combination of the Klein–Gordon oscillator and a space with nontrivial topology. Afterwards, a rotating frame in the $S^1 \times R^3$ space-time will be considered. We will show that the oscillator potential can form bound states for the Klein–Gordon equation in this space-time, and beyond that the momentum associated with the nontrivial topology is discrete. This is an expected result, since the topology of $S^1 \times R^3$ space is associated with the periodicity of the boundary conditions. In the case of a rotating frame in the $S^1 \times R^3$ space-time, we will see interesting results associated with noninertial effects: the energy levels are shifted and the region of the space-time where the particle can be placed is restricted.

This work is organized as follows: In Section 2, we will study the space-time metric with a nontrivial topology and define a coordinate transformation that connects it to a rotating frame. In Section 3, we will derive the Klein–Gordon (KG) equation with a potential of the harmonic oscillator type and solve the associated differential equation. Similarly, we will solve again the KG equation in Section 4 but we will consider a noninertial frame. Finally, we will present our conclusions in Section 5.

2. Nontrivial Space-Time Topology and Noninertial Reference Frame

In this section, we define the line element that describes the space-time geometry in agreement with the proposal of this work. We want to study the behavior of massive scalar fields (zero spin particles) under the influence of a gravitational field generated by a space-time with the nontrivial topology $S^1 \times R^3$. In this geometry, R^3 represents the usual uncompactified space-time directions, and S^1 is an compactified dimension. We discuss the relationship between $S^1 \times R^3$ and the effects for a rotational frame inserted in that scenario. Figure 1 shows a representation of this space-time where the temporal coordinate is absent.

The metric in polar coordinates describing the space-time under consideration is described by the expression

$$ds'^2 = -dt'^2 + dr'^2 + r^2 d\phi'^2 + R^2 d\theta'^2, \tag{1}$$

where $-\infty < t' < \infty$ and $0 < r' < \infty$ represent the temporal and radial coordinate range respectively. The

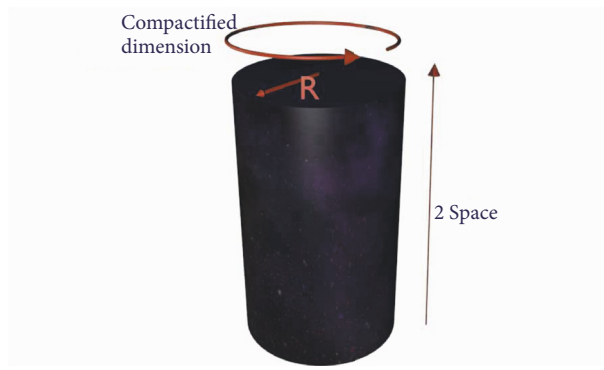

Figure 1: Representation of the topologically nontrivial space-time $S^1 \times R^3$. In this background, we discuss a frame with a constant angular velocity adapted to the circle S^1.

parameter R is the radius of the circle S^1, θ' and ϕ' are angular coordinates defined in the range $0 \leq \theta', \phi' \leq 2\pi$. Also in this coordinate system, (1) can be rewritten for a S^1 rotating system of reference with constant angular velocity Ω, by means of the transformation

$$\begin{aligned} t' &= t, \\ r' &= r, \\ \phi' &= \phi, \\ \theta' &= \theta + \Omega t, \end{aligned} \tag{2}$$

and inserting the coordinate transformation (2) into (1), we were able to get the line element

$$\begin{aligned} ds^2 &= -dt^2 + dr^2 + r^2 d\phi^2 + R^2 (d\theta + \Omega dt)^2 \\ &= -\left(1 - R^2\Omega^2\right) dt^2 + dr^2 + r^2 d\phi^2 + R^2 d\theta^2 \\ &\quad + 2R^2 \Omega d\theta dt, \end{aligned} \tag{3}$$

where we write the components of the covariant metric tensor related to (3) in the matrix form

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - R^2\Omega^2\right) & 0 & 0 & R^2\Omega \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ R^2\Omega & 0 & 0 & R^2 \end{pmatrix}, \tag{4}$$

together with its contravariant version

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & \Omega \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{1}{r^2} & 0 \\ \Omega & 0 & 0 & \dfrac{1}{R^2} - \Omega^2 \end{pmatrix}. \tag{5}$$

We can see that both $g_{\mu\nu}$ and $g^{\mu\nu}$ are nondiagonal. The non-null components outside the diagonal are effects

originated from the rotational frame inserted in this scenario by means of the coordinate transformation (2). Note that for $\Omega = 0$ we retrieve the line element described in (1). From (3), we can observe that R is defined in the interval $0 < R < R_0$ where $R_0 = 1/\Omega$. That is, the rotating reference system can only be defined for distances where $R < 1/\Omega$. In fact, since for $R > 1/\Omega$, the time component g_{00} of the metric becomes negative, so that such a condition is not physically acceptable. The noncompatibility of the rotational reference system with real particles at great distances is related to the fact that at that location the velocity of the frame becomes greater than the velocity of light in the vacuum and therefore such a system can not be defined from real bodies [26]. Thus, it is convenient to restrict a range to R, where of course this condition imposes a size limit for the compact dimension in a noninertial frame. The direction of the movement occurs counterclockwise; that is, $\Omega \geq 0$.

3. KG Oscillator in Nontrivial Topology

In this section we study the KG oscillator in a geometry $S^1 \times R^3$. Spin-0 bosonic particles are described by a scalar field denoted as $\Psi(x)$. The dynamics of such a scalar field is determined by the so-called KG equation, to which we now turn our attention. Let us investigate, from this point on, the formulation of an equation equivalent to the scalar wave equation for the KG oscillator under the curved space-time under consideration and then to solve it. Let us first write in the Minkowski geometry. Recalling that a true real scalar field in this space-time (in units $c = \hbar = 1$) obeys the wave equation

$$\left(\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2\right)\Psi = 0, \tag{6}$$

where m is the mass of the particle and $\eta^{\mu\nu}$ are the components of the metric tensor. The generalization of the KG equation for curved space-time, that is, a region affected by a gravitational field, is done by substituting the metric tensor $\eta^{\mu\nu}$ by the metric tensor that describes the curved space-time, $g^{\mu\nu}$, and the partial derivative ∂_μ for the covariant derivative ∇_μ. Thus, the KG equation in a curved space-time is written as

$$\left[-\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\right) + m^2\right]\Psi = 0, \tag{7}$$

where g is the determinant of $g^{\mu\nu}$. Now, the quantum description of a particle takes into account elements of the space-time geometry in question.

Let us now study the KG oscillator. Such a procedure is similar to one that is carried out for the insertion of an electromagnetic interaction which is made by the introduction of a 4-vector potential external A_μ (Gauge field). The KG oscillator at $S^1 \times R^3$ is examined by the following replacements in the operator p_μ [9]:

$$p_\mu \longrightarrow p_\mu - im\omega X_\mu, \tag{8}$$

and

$$p_\mu^\dagger \longrightarrow p_\mu + im\omega X_\mu, \tag{9}$$

where m is the mass of the particle, ω is the oscillation frequency, and $X_\mu = (0, r, 0, 0)$ is the potential in polar coordinates in the radial direction r. Thus, the KG equation $[(p)_{op}^2 + m^2]\Psi = 0$ is replaced by the general form

$$\left[-\frac{1}{\sqrt{-g}}\left(\partial_\mu + m\omega X_\mu\right)g^{\mu\nu}\sqrt{-g}\left(\partial_\nu - m\omega X_\nu\right) + m^2\right]\Psi = 0. \tag{10}$$

Therefore, now we write (10) in the space-time given by the line element (1). The result is given by

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{1}{r}\left(\frac{\partial}{\partial r} + m\omega r\right)\left(r\frac{\partial}{\partial r} - m\omega r^2\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{1}{R^2}\frac{\partial^2}{\partial\theta^2} - m^2\right]\Psi = 0, \tag{11}$$

that may be written in the form

$$\left[-\frac{\partial^2}{\partial t^2} + \left(\frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial r^2} - 2m\omega - m^2\omega^2 r^2 + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{1}{R^2}\frac{\partial^2}{\partial\theta^2} - m^2\right]\Psi = 0, \tag{12}$$

that is dependent of the coordinates t, θ, and ϕ. Equation (12) can be solved by the variable separation method in which the wave function is written in the form $\Psi(t, r, \theta, \phi) = f(r)G(\theta)h(\phi)$. Let us assume that the solution is given as follows:

$$\Psi(t, r, \theta, \phi) = f(r)e^{-iEt}e^{il\phi}e^{in\theta}, \tag{13}$$

where l and n are the quantum numbers and E is the energy of the particle. The periodicity conditions for the considered variables $G(0) = G(2\pi)$ and $h(0) = h(2\pi)$ imply that n and $l \in \mathbb{Z}$. Substituting (13) into (12), we obtain

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - m^2\omega^2 r^2 - \frac{l^2}{r^2} + E^2 - m^2 - 2m\omega - \frac{n^2}{R^2}\right]f(r) = 0, \tag{14}$$

that is a second-order differential equation for the radial coordinate of the KG equation. Considering the transformation $f(r) = F(r)/\sqrt{r}$ into (14), we have the expression

$$\left[\frac{d^2}{dr^2} - m^2\omega^2 r^2 - \frac{\left(l^2 - 1/4\right)}{r^2} + \lambda^2\right]F(r) = 0, \tag{15}$$

with $\lambda^2 = E^2 - m^2 - 2m\omega - n^2/R^2$. The radial differential equation above describes the KG oscillator in a space-time with nontrivial topology. To obtain the solution of (15), we first propose a transformation in the radial coordinate of the following form:

$$\rho = m\omega r^2, \tag{16}$$

which, inserted in the differential equation (15), lead to the expression

$$\left[\rho\frac{d^2}{d\rho^2}+\frac{1}{2}\frac{d}{d\rho}-\frac{1}{4}\rho-\frac{\left(l^2-1/4\right)}{4\rho}+\frac{\lambda^2}{4m\omega}\right]F(\rho)=0. \tag{17}$$

Therefore, to get the normalizable eigenfunctions we can suppose a general solution in the form

$$F(\rho)=\rho^{(1/2)(|l|+1/2)}e^{-\rho/2}K(\rho), \tag{18}$$

where, from (17), by substituting expression (18) for $F(\rho)$ and remembering that the parameter l is a constant, we obtain the differential equation associated with the radial solution

$$\rho\frac{d^2K(\rho)}{d\rho^2}+(|l|+1-\rho)\frac{dK(\rho)}{d\rho}-\left(\frac{1}{2}|l|+\frac{1}{2}-\frac{\lambda^2}{4m\omega}\right)K(\rho)=0. \tag{19}$$

The general solution of the above equation is given by the expression

$$K=AM(a,b;\rho)+B\rho^{1-b}M(a-b+1,2-b;\rho), \tag{20}$$

where

$$a=\beta+\frac{\gamma}{\delta}+\frac{1}{2},\qquad b=2\beta+1, \tag{21}$$

together with the arbitrary constants A and B. The term $M(a,b;\rho)$ denotes a hypergeometric function. We can observe that the final differential equation for $K(\rho)$, a second-order linear homogeneous equation, is of the confluent hypergeometric type or the Kummer equation. Note that the second term of (20) is singular at the origin, *i.e.*,$\rho=0$. Thus, we define $B=0$. Therefore, the solution is given by the expression

$$K(a,b;\rho)=AM(a,b;\rho), \tag{22}$$

with the parameters a and b defined by

$$a=\frac{1}{2}|l|+\frac{1}{2}-\frac{\lambda^2}{4m\omega}, \tag{23}$$

$$b=|l|+1. \tag{24}$$

The wave function Ψ or in particular f, F, and M must be normalizable with respect to the product

$$(\Psi_b,\Psi_a)=i\int\sqrt{g_\Sigma}\left(\Psi_b^*\partial_t\Psi_a-\Psi_a\partial_t\Psi_b^*\right)d^3x, \tag{25}$$

i.e., they must be finite for large r (or ρ), where g_Σ is the determinant of the spatial part of the metric. In Figure 2, we can observe an oscillatory pattern of the wave function

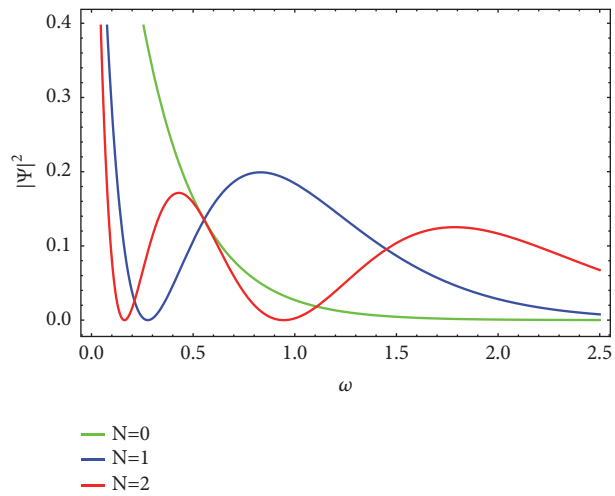

Figure 2: The plots of $|\Psi|^2$ as functions of the variable ω displayed for three different values of N with the parameters $m=1$, $r=1.9$, and $l=0$.

and its dependence with the frequency. At this stage we may obtain the energy spectrum related to the confluent hypergeometric type solution. Initially it is necessary to analyze the asymptotic behavior of the confluent hypergeometric function $M(a,b;\rho)$ for large values of ρ. If $\rho\longrightarrow\infty$, the asymptotic behavior of function is given by [38]

$$M(a,b;\rho)\simeq\frac{\Gamma(b)}{\Gamma(b-a)}e^{-i\pi a}\rho^{-a}+\frac{\Gamma(b)}{\Gamma(a)}e^{\rho}\rho^{a-b}. \tag{26}$$

In the above equation, we can observe that the factor e^ρ in the second term of this expression diverges. If parameter a is a negative integer, that term is null. So it is necessary that the confluent hypergeometric function be a polynomial function of degree N. Thus, it is certainly possible to write the expression

$$a=\frac{1}{2}|l|+\frac{1}{2}-\frac{\lambda^2}{4m\omega}=-N,\quad N=0,1,2,\ldots \tag{27}$$

Now we can solve this equation for energy; since λ depends on E, the result is given by

$$E_\pm=\pm\sqrt{m^2+\frac{n^2}{R^2}+2m\omega\left(2N'+|l|\right)},\qquad N'=N+1=1,2,3,\ldots \tag{28}$$

From (6) it is possible to show that the four-current

$$j^\mu=i\left(\Psi^*\frac{\partial\Psi}{\partial x_\mu}-\Psi\frac{\partial\Psi*}{\partial x_\mu}\right) \tag{29}$$

satisfies $\nabla^\mu j_\mu=0$; in this way, the temporal component of the four-current is given by

$$j_0=2E_\pm\left(\Psi^{(\pm)}\right)^2, \tag{30}$$

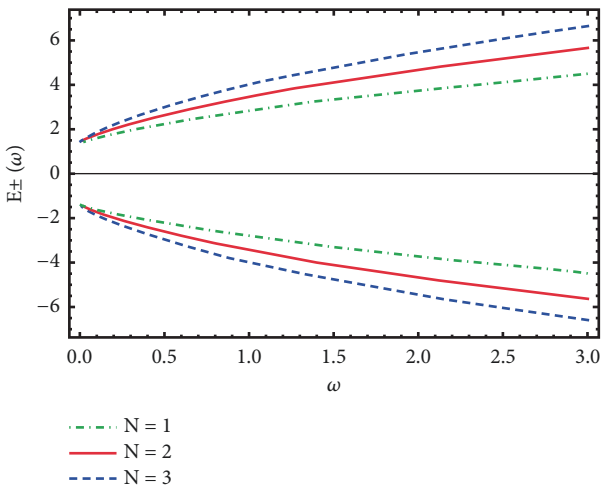

FIGURE 3: The plots of $E_{\pm}$ as functions of the variable ω displayed for three different values of N with the parameters $m = 1$, $R = 1$, and $n = 1$.

where

$$\Psi^{(\pm)}(t, r, \theta, \phi) = f(r)\, e^{-iE_{\pm}t} e^{il\phi} e^{in\theta}. \tag{31}$$

Hence, we can identify j^0 as a charge density, and thus, we have particles with charge $+e$ and particles with charge $-e$ in accordance with the existence of particles and antiparticles in the theory. We also must note that $j_0 \longrightarrow -j_0$ under charge conjugation.

It is interesting to note that the energy is symmetric around $E_{\pm} = 0$ that means that particles and antiparticles have the same energy spectrum with different signs, *i.e.*, the energy in terms of absolute values is the same. As shown in Figure 3, it is possible to observe this behavior explicitly. In relation to the frequency of the oscillator, we can see that the energy increases, in absolute values, as ω grows. The spectrum also depends on the quantum number n and on the radius R of the compactified dimension. Due to the mathematical form of expression (28), as n increases, the values of the energy also increase. It is easy to see that when the length $L = 2\pi R$ of the compact dimension tends to infinity, the term $p_n^2 = n^2/R^2$ is replaced by p_z^2; the continuous momentum in the z direction and the result can be written in the form

$$E_{\pm} = \pm\sqrt{m^2 + p_z^2 + 2m\omega\left(2N' + |l|\right)}, \qquad N' = N + 1 = 1, 2, 3, \ldots \tag{32}$$

In this case we recover the energy spectrum of KG oscillator in the usual Minkowski space-time in cylindrical coordinates.

4. KG Oscillator in Noninertial Reference Frame

In this section, we will study the influence of noninertial effects of a referential in rotation in a space-time with nontrivial topology, applied to the KG wave equation with a KG oscillator potential. Therefore, based on the procedures that were discussed in the previous section, in our computational developments that will be developed here, we will first recast the differential equation for the KG oscillator in the space-time described by the line element of (3). In this case (10) becomes

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - 2m\omega - m^2\omega^2 r^2 + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \left(\frac{1}{R^2} - \Omega^2\right)\frac{\partial^2}{\partial \theta^2} + 2\Omega\frac{\partial}{\partial t}\frac{\partial}{\partial \theta} - m^2\right]\Psi = 0. \tag{33}$$

This is the wave equation for spin-0 particles with the potential of the KG oscillator submitted to a rotating frame in a space-time with nontrivial topology. We can see that the solution of (33) will have the same form discussed above, *i.e.*, (13). Thus, by substituting (13) into (33), the expression is in effect,

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - m^2\omega^2 r^2 - \frac{l^2}{r^2} + (E + \Omega n)^2 - 2m\omega - \frac{n^2}{R^2} - m^2\right] f(r) = 0. \tag{34}$$

In this stage, by considering the transformation $f(r) = F(r)/\sqrt{r}$, we can write the radial equation (34) in the form

$$\left[\frac{d^2}{dr^2} - m^2\omega^2 r^2 - \frac{\left(l^2 - 1/4\right)}{r^2} + \gamma^2\right] F(r) = 0, \tag{35}$$

where we define $\gamma^2 = (E + \Omega n)^2 - 2m\omega - n^2/R^2 - m^2$. Using expression (16), we can rewrite the radial equation (35) in the following form:

$$\left[\rho\frac{d^2}{d\rho^2} + \frac{1}{2}\frac{d}{d\rho} - \frac{1}{4}\rho - \frac{\left(l^2 - 1/4\right)}{4\rho} + \frac{\gamma^2}{4m\omega}\right] F(\rho) = 0, \tag{36}$$

and therefore, by performing a substitution of the solution described in (18) in the radial equation (36), we find the expression

$$\rho\frac{d^2K(\rho)}{d\rho^2} + \left(|l| + 1 - \rho\right)\frac{dK(\rho)}{d\rho} - \left(\frac{1}{2}|l| + \frac{1}{2} - \frac{\gamma^2}{4m\omega}\right)K(\rho) = 0. \tag{37}$$

The asymptotic behavior of hypergeometric confluent function implies that

$$\frac{1}{2}|l| + \frac{1}{2} - \frac{\gamma^2}{4m\omega} = -N, \quad N = 0, 1, 2, \ldots \tag{38}$$

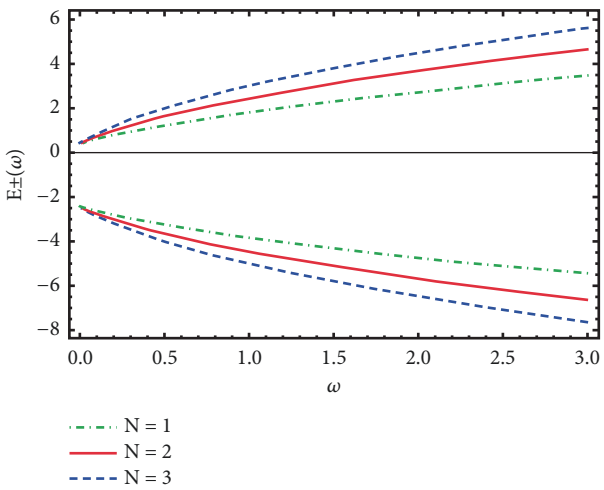

FIGURE 4: The plots of $E_{\pm}$ as functions of the variable ω displayed for three different values of N with the parameters $m = 1$, $R = 1$, $n = 1$, and $\Omega = 0.5$.

Again, we can solve this equation for E and the result obtained can be written in the form

$$E_{\pm} = -n\Omega \pm \sqrt{m^2 + \frac{n^2}{R^2} + 2m\omega\left(2N' + |l|\right)}, \qquad N' = N + 1 = 1, 2, 3, \ldots \tag{39}$$

Now the energy spectrum depends on the angular velocity of the frame. The first term is related to the noninertial effects and appears frequently in this type of physical system. The second term is the usual energy spectrum of an inertial frame. As it can be seen in Figure 4, the effect of the referential rotation breaks the symmetry around $E_{\pm} = 0$. We can calculate the charge density for the solution with rotation. The contravariant component can be written in the form

$$j^0 = -2\left(E_{\pm} + n\Omega\right)\left(\Psi^{(\pm)}\right)^2. \tag{40}$$

In contrast to (30), we can see that the density depends on the angular velocity of the noninertial frame. In addition, the rotation of the reference system modifies the wave function. Now the solution is written in the form

$$\Psi^{(\pm)}\left(t, r, \theta, \phi\right) = f\left(r\right) e^{-i(\epsilon_{\pm} - \Omega n)t} e^{il\phi} e^{in\theta}, \tag{41}$$

where $\epsilon_{\pm} = \pm\sqrt{m^2 + n^2/R^2 + 2m\omega(2N' + |l|)}$. In the same way, we recover the usual energy spectrum of the Minkowski space-time in cylindrical coordinates when Ω is equal to zero and the length $L = 2\pi R$ tends to infinity and the resulting expression coincides with (32).

5. Conclusions

In this work, we have determined solutions of the KG oscillator in a topologically nontrivial space-time. We have considered two different settings in this space. In the first case the usual $S^1 \times R^3$ space is assumed where the metric is written in the usual way and as a second case, it is considered a frame with a constant angular velocity adapted in the circle S^1 by considering a coordinate transformation. In both studied systems, we have solved the wave equations and have obtained the discrete energy spectrum associated with bound states. It was possible to see that the combined effect of a space with nontrivial topology and the KG oscillator allows the formation of bound states, as it happens in the usual Minkowski space-time with trivial topology. When noninertial effects were taken into account, we verified that the energy spectrum lost the symmetry around $E = 0$; *i.e.*, the additional term of the energy spectrum, in the case of the noninertial frame, causes a deviation from usual values. Other studies in the literature related to rotating reference systems show that the additional term that arises is associated with the coupling between the rotational angular momentum and the angular quantum number [23, 24, 39].

Additionally, we have shown that the space-time topology modifies the energy spectrum. In fact, the $S^1 \times R^3$ space is associated with the periodicity of the boundary condition of the wave function. Consequently, the quantum number associated with the S^1 circle is discrete. In future works it may be interesting to extend the results obtained in this paper to spaces with different topologies and potentials. Effects of rotation on noncompact spatial coordinates are another type of configuration that we can study. In this way, the combination of inertial effects on different spatial coordinates can be useful in understanding the effect of the choice of noninertial reference frames in quantum mechanics.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] E. M. Wright, J. Arlt, and K. Dholakia, "Toroidal optical dipole traps for atomic Bose-Einstein condensates using Laguerre-Gaussian beams," *Physical Review A: Atomic, Molecular and Optical Physics*, vol. 63, no. 1, 2000.

[2] L. H. Ford, "Vacuum polarization in a nonsimply connected spacetime," *Physical Review D: Particles, Fields, Gravitation and Cosmology*, vol. 21, no. 4, pp. 933–948, 1980.

[3] W. Jian-Hua, L. Kang, and D. Sayipjamal, "Klein-Gordon oscillators in noncommutative phase space," *Chinese Physics C*, vol. 32, no. 10, p. 803, 2008.

[4] W.-C. Qiang, "Bound states of the Klein-Gordon and Dirac equations for potential $V(r) = Ar^{-2}$-Br^{-1}," *Chinese Physics*, vol. 12, no. 10, pp. 1054–1057, 2003.

[5] K. Bakke and C. Furtado, "On the Klein–Gordon oscillator subject to a Coulomb-type potential," *Annals of Physics*, vol. 355, pp. 48–54, 2015.

[6] R. V. Maluf, "Noncommutative space corrections on the klein–gordon and dirac oscillators spectra," *International Journal of Modern Physics A*, vol. 26, no. 29, pp. 4991–5003, 2011.

[7] R. L. L. Vitória, C. Furtado, and K. Bakke, "On a relativistic particle and a relativistic position-dependent mass particle subject to the klein–gordon oscillator and the coulomb potential," *Annals of Physics*, vol. 370, no. 128, pp. 128–136, 2016.

[8] M.-L. Liang and R.-L. Yang, "Three-dimensional Klein-Gordon oscillator in a background magnetic field in noncommutative phase space," *International Journal of Modern Physics A*, vol. 27, no. 10, 1250047, 12 pages, 2012.

[9] N. A. Rao and B. Kagali, "Energy profile of the one-dimensional Klein–Gordon oscillator," *Physica Scripta*, vol. 77, article no. 015003, 2007.

[10] B. P. Mandal and S. Verma, "Dirac oscillator in an external magnetic field," *Physics Letters A*, vol. 374, no. 8, pp. 1021–1023, 2010.

[11] R. Martinez-Y-Romero, H. Nunez-Yepez, and A. Salas-Brito, "Relativistic quantum mechanics of a Dirac oscillator," *European Journal of Physics*, vol. 16, no. 3, p. 135, 1995.

[12] M. H. Pacheco, R. R. Landim, and C. A. S. Almeida, "One-dimensional Dirac oscillator in a thermal bath," *Physics Letters A*, vol. 311, no. 2-3, pp. 93–96, 2003.

[13] C. Quesne and M. Moshinsky, "Symmetry Lie algebra of the Dirac oscillator," *Journal of Physics A: Mathematical and General*, vol. 23, no. 12, pp. 2263–2272, 1990.

[14] P. Rozmej and R. Arvieu, "The Dirac oscillator. A relativistic version of the Jaynes-Cummings model," *Journal of Physics A: Mathematical and General*, vol. 32, no. 28, pp. 5367–5382, 1999.

[15] V. M. Villalba, "Exact solution of the two-dimensional Dirac oscillator," *Physical Review A: Atomic, Molecular and Optical Physics*, vol. 49, no. 1, 586 pages, 1994.

[16] L. C. N. Santos and C. C. Barros, "Relativistic quantum motion of spin-0 particles under the influence of noninertial effects in the cosmic string spacetime," *The European Physical Journal C*, vol. 78, p. 13, 2018.

[17] S. Bruce and P. Minning, "The klein-gordon oscillator," *Il Nuovo Cimento A*, vol. 106, no. 5, pp. 711–713, 1993.

[18] M. Moshinsky and A. Szczepaniak, "The Dirac oscillator," *Journal of Physics A: Mathematical and General*, vol. 22, no. 17, pp. L817–L819, 1989.

[19] J. Carvalho, A. M. M. de Carvalho, E. Cavalcante, and C. Furtado, "Klein–Gordon oscillator in Kaluza–Klein theory," *The European Physical Journal C*, vol. 76, p. 365, 2016.

[20] L. B. Castro, "Noninertial effects on the quantum dynamics of scalar bosons," *The European Physical Journal C*, vol. 76, p. 61, 2016.

[21] L. B. Castro, "Quantum dynamics of scalar bosons in a cosmic string background," *European Physical Journal C*, vol. 75, p. 287, 2015.

[22] Y. Xiao, Z. Long, and S. Cai, "Klein-Gordon oscillator in noncommutative phase space under a uniform magnetic field," *International Journal of Theoretical Physics*, vol. 50, no. 10, pp. 3105–3111, 2011.

[23] K. Bakke, "Relativistic bounds states for a neutral particle confined to a parabolic potential induced by noninertial effects," *Physics Letters A*, vol. 374, no. 46, pp. 4642–4646, 2010.

[24] K. Bakke, "Confinement of a dirac particle to a hard-wall confining potential induced by noninertial effects," *Modern Physics Letters B*, vol. 27, article no. 1350018, 2013.

[25] M. N. Chernodub, "Rotating Casimir systems: magnetic-field-enhanced perpetual motion, possible realization in doped nanotubes, and laws of thermodynamics," *Physical Review D*, vol. 87, article no. 025021, 2013.

[26] L. C. Santos and J. Barros, "Rotational effects on the Casimir energy in the space-time with one extra compactified dimension," *International Journal of Modern Physics A*, vol. 33, no. 20, 1850122, 13 pages, 2018.

[27] L. C. N. Santos and C. C. Barros, "Dirac equation and the melvin metric," *The European Physical Journal C*, vol. 76, p. 560, 2016.

[28] L. C. N. Santos and C. C. Barros, "Scalar bosons under the influence of noninertial effects in the cosmic string spacetime," *The European Physical Journal C*, vol. 77, article no. 186, 2017.

[29] F. M. Andrade, C. Filgueiras, and E. O. Silva, "Scattering and bound states of a spin-1/2 neutral particle in the cosmic string spacetime," *Advances in High Energy Physics*, vol. 2017, Article ID 8934691, 7 pages, 2017.

[30] R. L. L. Vitória and K. Bakke, "Rotating effects on the scalar field in the cosmic string spacetime, in the spacetime with space-like dislocation and in the spacetime with a spiral dislocation," *The European Physical Journal C*, vol. 78, p. 175, 2018.

[31] F. Ahmed, "The energy–momentum distributions and relativistic quantum effects on scalar and spin-half particles in a Gödel-type space–time," *The European Physical Journal C*, vol. 78, p. 598, 2018.

[32] R. L. L. Vitória and K. Bakke, "On the interaction of the scalar field with a Coulomb-type potential in a spacetime with a screw dislocation and the Aharonov-Bohm effect for bound states," *The European Physical Journal Plus*, vol. 133, p. 490, 2018.

[33] B.-Q. Wang, Z.-W. Long, C.-Y. Long, and S.-R. Wu, "The study of a half-spin relativistic particle in the rotating cosmic string space-time," *International Journal of Modern Physics A*, vol. 33, no. 27, 1850158, 16 pages, 2018.

[34] J. Carvalho, A. M. de M. Carvalho, and C. Furtado, "Quantum influence of topological defects in Gödel-type space-times," *The European Physical Journal C*, vol. 74, no. 6, 2014.

[35] A. L. Cavalcanti de Oliveira and E. R. Bezerra de Mello, "Exact solutions of the Klein-Gordon equation in the presence of a dyon, magnetic flux and scalar potential in the spacetime of gravitational defects," *Classical and Quantum Gravity*, vol. 23, no. 17, pp. 5249–5263, 2006.

[36] E. R. F. Medeiros and E. R. B. de Mello, "Relativistic quantum dynamics of a charged particle in cosmic string spacetime in the presence of magnetic field and scalar potential," *The European Physical Journal C*, vol. 72, p. 2051, 2012.

[37] M. Hosseinpour, F. M. Andrade, E. O. Silva, and H. Hassanabadi, "Erratum to: Scattering and bound states for the Hulthén potential in a cosmic string background," *The European Physical Journal C*, vol. 77, no. 270, 2017.

[38] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions: with Formulas, Graphs, And Mathematical Tables, Applied Mathematics Series*, Dover Publications, New York, NY, USA, 1964.

[39] K. Bakke and C. Furtado, "Geometric phase for a neutral particle in rotating frames in a cosmic string spacetime," *Physical Review D*, vol. 80, article no. 024033, 2009.