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Research Article

$\mathcal{N} = 2$ Supersymmetry with Central Charge: A Twofold Implementation

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In this work, we analyze an extended $\mathcal{N} = 2$ supersymmetry with central charge and develop its superspace formulation under two distinct viewpoints. Initially, in the context of classical mechanics, we discuss the introduction of deformed supersymmetric derivatives and their consequence on the deformation of one-dimensional nonlinear sigma model. After that, considering a field-theoretical framework, we present an implementation of this superalgebra in two dimensions, such that one of the coordinates is related to the central charge. As an application, in this two-dimensional scenario, we consider topological (bosonic) configurations of a special self-coupled matter model and present a nontrivial fermionic solution.

1. Introduction

Back in 1975, the paper by Haag, Lopuszanski, and Sohnius [1] established the most general supersymmetric algebra in four dimensions, which respects the Poincaré invariance and requirements of S–matrix. In addition to the Poincaré operators and usual Majorana supercharges, it is possible to include new operators, known as central charges, which have all vanishing commutation relations.

Essentially, the investigations associated with central charges in field theories can be divided into three parts. In the first one, the main subject was the classification of the multiplets related to massless and massive representations. In this part, the papers by Salam and Strathdee [2, 3] initially analyzed the case without central charges and a series of works clarified the general one (e.g., see the review [4] and references therein). These works have shown that, in the presence of central charges, some states are avoided in the massive representation and, in particular situations, central charges may be connected to internal symmetries.

The second part corresponds to investigations at the classical level. One of the remarkable contributions was the work by Witten and Olive [5], where the authors considered some specific models in two and four dimensions with topological configurations and obtained a connection between central charges and topological numbers. After that, other situations involving topological defects have been explored, such as complex projective space CP^{n-1} and others nonlinear sigma models [6–10]. For a more detailed discussion of central charges and topological defects, we indicate [11].

Another interesting classical point of view showed up after the dimensional reduction of supersymmetric models. Some works have discussed that central charges can be seen as an inheritance of dimensional reduction [12–15]. In particular cases, it is related to the momentum operator of the extra-dimension [16]. We highlight an exotic situation in [17], in which the dimensional reduction of super-Yang-Mills model in four dimensions with $O(N)$ symmetry group leads to Georgi-Glashow-like model in three dimensions, where the central charge is associated with the electrical charge of the abelian subgroup, i.e., other nontopological explanation to central charge. We also indicate some supersymmetric

models with central charges in higher-dimensional theories and brane-world scenarios [18–23].

The third part concerns the quantum aspects. Specifically, by means of quantum effects, the central charge has appeared as a quantum anomaly in superalgebra. From this perspective, many situations have been analyzed, such as nonlinear sigma models, kink, monopoles, domain walls, and vortices [24–31].

In the context of supersymmetric mechanics with central charge, some investigations of field-theoretical models may correspond to (quantum) mechanical systems, namely, in the study of superconducting cosmic strings, localized fermions on domain walls, and gapped and superconducting graphene [32–35]. Moreover, in some well-known mechanical situations, such as Coulomb, Aharonov–Bohm–Coulomb, and Aharonov–Casher systems, it is possible to accommodate extended supersymmetries with central charge [36], which explain the degeneracy of the energy spectra. In the presence of an external electromagnetic background, the Poincaré algebra leads to a residual symmetry algebra with a central charge, deformed translations, and Lorentz generators [37, 38].

At this point, it is worthy to emphasize that we are using the term "central charge" based on the generators with all vanishing commutation relations. However, there are situations in which new bosonic generators have been added to superalgebra with some nonvanishing commutators. For example, we address the cases of weak supersymmetry [39] and other generalizations known as centrally extensions involving the $su(2 \mid 1)$ and $su(2 \mid 2)$ algebras [40–42]. Finally, we highlight some results in supersymmetric mechanics in which the introduction of central charges can be related to duality and mirror symmetries [43–46]. We shall return to this point in Section 2.2.

In this work, some investigations of $\mathcal{N} = 2$ supersymmetry with one (real) central charge are carried out in mechanics and two-dimensional field theory. We consider a similar superalgebra adopted in [45, 46] and we propose an alternative implementation of this supersymmetry through superspace approach. Here, we shall present a prescription to implement this extended supersymmetry by means of deformed covariant derivatives.

The paper is organized as follows: in Section 2, we discuss the $\mathcal{N} = 2$ supersymmetric mechanics with central charge. We focus on the construction of the superspace formulation and central charge transformation and propose deformed derivatives. In this context, we present a deformation of one-dimensional nonlinear sigma model and revise the particular case discussed in [45]. Then, a comparison between the two supersymmetric implementations is established. After that, in Section 3, we turn our attention to supersymmetric field theories. Here, we develop a possible implementation of this superalgebra in two dimensions, where a new coordinate is related to central charge. In this scenario, we present a model with topological (bosonic) configurations and obtain a nontrivial fermionic solution. Finally, in Section 4, we display our concluding comments.

2. Supersymmetric Mechanics with Central Charge

In $\mathcal{N} = 2$ supersymmetric one-dimensional systems (mechanical case), we have two (real) supercharges, Q_1 and Q_2, which satisfy the following superalgebra with the Hamiltonian (H): $Q_1^2 = Q_2^2 = H$ and $[Q_1, H] = [Q_2, H] = 0$. In this work, we are going to deal with a possible extension of this case by including one real central charge (Z) in the anticommutator $\{Q_1, Q_2\} = 2Z$. We shall adopt a complex notation for the supercharges, namely, $Q = (1/\sqrt{2})(Q_1 + iQ_2)$ and $\overline{Q} = (1/\sqrt{2})(Q_1 - iQ_2)$, which leads to

$$\left\{Q, \overline{Q}\right\} = 2H, \tag{1}$$

$$[Q, H] = \left[\overline{Q}, H\right] = 0, \tag{2}$$

$$\begin{aligned} Q^2 &= iZ, \\ \overline{Q}^2 &= -iZ. \end{aligned} \tag{3}$$

With the aforementioned relations, one can check that $[Q, Z] = [\overline{Q}, Z] = [H, Z] = 0$; i.e., Z has the properties of a central charge. We also highlight that a similar superalgebra was considered in. [45, 46], with only different conventions in (3).

In order to implement this extended supersymmetry through a superfield approach, we first introduce two (complex) Grassmann parameters, θ and $\overline{\theta}$, such that the superspace can be described by $(t; \theta, \overline{\theta})$, where t is the time parameter. Throughout this work, the Grassmann derivatives are understood in the sense of acting from left to the right; e.g., $(\partial/\partial\theta)(\overline{\theta}\theta) = -\overline{\theta}$. Furthermore, we draw the attention to the procedure we adopt to realize the representation of central charge. Our proposal consists in implementing the central charge by means of a deformation of the supersymmetric generators. Now, to read the explicit form of this deformation, we act with the supersymmetric transformation on a supermultiplet and, by imposing a number of conditions that we are going to present later on, we get the final form of the deformation.

The superalgebra above can be realized in a differential representation. To achieve this, we define $\delta^H = i\partial_t$ and the following (deformed) supercharge operators:

$$\delta^Q = \partial_\theta + i\overline{\theta}\partial_t + i\theta\delta^z, \tag{4}$$

$$\delta^{\overline{Q}} = \partial_{\overline{\theta}} + i\theta\partial_t - i\overline{\theta}\delta^z, \tag{5}$$

where we have used the notations $\partial_t = \partial/\partial t$, $\partial_\theta = \partial/\partial\theta$, and $\partial_{\overline{\theta}} = \partial/\partial\overline{\theta}$.

Moreover, we point out an important comment: it is worthy to introduce a deformation δ^z in the (supersymmetric) covariant derivatives. Similarly to the case of the supercharges, (4) and (5), we define

$$\mathfrak{D} = \partial_\theta - i\overline{\theta}\partial_t - i\theta\delta^z, \tag{6}$$

$$\overline{\mathfrak{D}} = \partial_{\overline{\theta}} - i\theta\partial_t + i\overline{\theta}\delta^z, \tag{7}$$

which satisfy $\{\mathfrak{D}, \overline{\mathfrak{D}}\} = -2i\partial_t$ and have vanishing anticommutation relations with the supercharges, $\{\delta^{Q}, \mathfrak{D}\} = \{\delta^{Q}, \overline{\mathfrak{D}}\} = \{\delta^{\overline{Q}}, \mathfrak{D}\} = \{\delta^{\overline{Q}}, \overline{\mathfrak{D}}\} = 0$.

By using these (deformed) covariant derivatives instead of the usual $D = \partial_\theta - i\overline{\theta}\partial_t$ and $\overline{D} = \partial_{\overline{\theta}} - i\theta\partial_t$, we automatically take into account the contribution of central charge and ensure the extended supersymmetric invariance of the action formulated in terms of the covariant derivatives and superfields. For a trivial central charge transformation, we recover the usual $\mathcal{N} = 2$ supersymmetry. On the other hand, one could use the derivatives D and $\overline{D}$, as done in [45, 46]. However, in this situation, one should perform a careful analysis of the supersymmetric transformation of the Lagrangian (in components) and add some counter-terms in order to maintain the invariance. Here, we claim that these additional terms are exactly the ones generated by the derivatives $\mathfrak{D}$ and $\overline{\mathfrak{D}}$. We shall return to this point in more detail in Section 2.2, where a particular case will be analyzed.

At this moment, it is advisable to point out that, in this section, the introduction of central charge is not associated with an extra coordinate in superspace. In other words, the superspace is parametrized by $(t, \theta, \overline{\theta})$ and the central charge is implemented through deformation of supercharges and covariant derivatives.

The introduction of supercharges and deformed derivatives in connection with central charge is not exclusive to this work. For instance, in [47], the authors investigated a central charge (Σ) in the relation $\{Q, \overline{Q}\} = 2(H - \Sigma)$ and the invariance of the action is also implemented in a more simple way by using the superfields and the correspondent deformed derivatives.

As already mentioned, one of the goals in this work is to study in what multiplet we can introduce a nontrivial central charge transformation related to the superalgebra (1)-(3). Here, we consider the multiplet $(1, 2, 1)$, described by one bosonic, two fermionic (Grassmann), and one auxiliary bosonic coordinates. The correspondent superfield is given by

$$X \equiv X\left[t; \theta, \overline{\theta}\right] = x(t) + i\theta\xi(t) + i\overline{\theta}\,\overline{\xi}(t) + \theta\overline{\theta}\mathscr{W}(t). \quad (8)$$

Let us first establish the supersymmetric transformation of these components. By using $\delta = \epsilon\delta^{Q} + \overline{\epsilon}\delta^{\overline{Q}}$, with complex Grassmann parameters ϵ and $\overline{\epsilon}$, one can show that

$$\begin{aligned}
\delta x &= i\epsilon\xi + i\overline{\epsilon}\overline{\xi},\\
\delta\xi &= -\overline{\epsilon}\dot{x} - i\overline{\epsilon}\mathscr{W} - \epsilon\delta^{z}x,\\
\delta\overline{\xi} &= -\epsilon\dot{x} + i\epsilon\mathscr{W} + \overline{\epsilon}\delta^{z}x,\\
\delta\mathscr{W} &= \frac{d}{dt}\left(\epsilon\xi - \overline{\epsilon}\overline{\xi}\right) - \epsilon\delta^{z}\overline{\xi} - \overline{\epsilon}\delta^{z}\xi.
\end{aligned} \quad (9)$$

Now we can proceed to fix the central charge transformation. In the case of real superfield, we note that $\delta\xi$ is the complex conjugation of $\delta\overline{\xi}$; therefore $\delta^{z}x$ must be bosonic and pure imaginary. Moreover, $\delta\mathscr{W}$ is bosonic; then $\delta^{z}\xi$ and $\delta^{z}\overline{\xi}$ should be fermionic. At this point, we suggest that $\delta^{z}x = i\mu$, where μ is a constant (real) parameter. Bearing this in mind and applying the relation $[\delta^{z}, \delta] = 0$ in all superfield components, one can arrive at the conditions $\delta^{z}\xi = 0 = \delta^{z}\overline{\xi}$ and $\delta^{z}\mathscr{W} = 0$. Actually, the main reason to fix this particular transformation is that $\delta\mathscr{W} = (d/dt)(\cdots)$, which guarantees the invariance of the action in the superfield approach. For example, if we do not consider a constant, namely, $\delta^{z}x = if(x)$, with $f(x)$ being an arbitrary function of x, we do not obtain a total derivative in $\delta\mathscr{W}$.

Finally, we remember that, in $\mathcal{N} = 2$ supersymmetry (without Z), the quiral (ϕ) and antiquiral $(\overline{\phi})$ superfields have been defined in a manner that satisfies the conditions $\overline{D}\phi = 0$ and $D\overline{\phi} = 0$. By using a Kähler (pre)potential, $K(\phi, \overline{\phi})$, these superfields were applied in the study of supersymmetric models associated with eletromagnetic interaction of a point particle [48, 49]. Here, we emphasize that the imposition of the aforementioned conditions with the deformed derivatives leads to trivial central charge transformations.

2.1. Deformed Nonlinear Sigma Model. The nonlinear sigma models in connection with extended supersymmetric mechanics have been a subject of intense investigation. In [50–55], the authors discussed some cases with $\mathcal{N} > 2$ supersymmetries. The inclusion of central charges was done in [45, 46] and [56] for $\mathcal{N} = 2$ and $\mathcal{N} = 4$, respectively. We also highlight some deformations involving 2− and 4−forms, which are related to torsion-like contributions [57] and applications to Black-Hole [58]. For a review of one-dimensional nonlinear sigma model and some aspects of geometry and topology, we point out the works [59, 60]. In the study of superparticles, one can introduce the so-called tensorial central charges (see [61–64] and references therein), which are responsible to fix some constraints in the equations of motion and have found applications in higher-spin models.

Having established the superspace formulation, let us investigate a possible application. Here, we propose what we refer to as our deformed nonlinear sigma model in terms of the deformed derivatives, a multiplet of real superfields X^{a}, and an arbitrary metric $g_{ab}(x)$. Then, the supersymmetric model is described by the following action

$$S = \int dt d\overline{\theta} d\theta \left[\frac{1}{2} g_{ab}(X)\, \mathfrak{D}X^{a}\overline{\mathfrak{D}}X^{b}\right], \quad (10)$$

which can be written in components as

$$\begin{aligned}
S = \int dt \Bigg\{ & \frac{g_{ab}}{2}\Big[\dot{x}^{a}\dot{x}^{b} + \mathscr{W}^{a}\mathscr{W}^{b} + i\xi^{a}\dot{\overline{\xi}}^{b} - i\dot{\xi}^{a}\overline{\xi}^{b}\\
& + i\xi^{a}\delta^{z}\xi^{b} + i\left(\delta^{z}\overline{\xi}^{a}\right)\overline{\xi}^{b} + \left(\delta^{z}x^{a}\right)\left(\delta^{z}x^{b}\right)\Big]\\
& - \frac{i\partial_{j}g_{ab}}{2}\Big[\left(\dot{x}^{b}\overline{\xi}^{j}\xi^{a} - i\mathscr{W}^{b}\overline{\xi}^{j}\xi^{a} - \overline{\xi}^{j}\overline{\xi}^{b}\delta^{z}x^{a}\right)\\
& + \left(\dot{x}^{a}\xi^{j}\overline{\xi}^{b} + i\mathscr{W}^{a}\xi^{j}\overline{\xi}^{b} + \xi^{j}\xi^{a}\delta^{z}x^{b}\right) - i\mathscr{W}^{j}\xi^{a}\overline{\xi}^{b}\Big]\\
& - \frac{1}{2}\left(\partial_{j}\partial_{k}g_{ab}\right)\xi^{j}\overline{\xi}^{k}\xi^{a}\overline{\xi}^{b}\Bigg\}.
\end{aligned} \quad (11)$$

By taking $\delta^z x^a = i\mu^a$ and $\delta^z \xi^a = 0$, we obtain two new contributions: one coupled to the metric and other with its first derivative. Notice that, for an euclidean metric ($g_{ab} = \delta_{ab}$), we obtain a trivial result; namely, the Lagrangian (without central charge) is shifted by a constant. Therefore, only in specific curved spaces, one can introduce a nontrivial deformation.

2.2. A Particular Nonlinear Sigma Model. In this subsection, let us revise a particular one-dimensional nonlinear sigma model described in [45] under a distinct point of view. The model consists of a point-particle restricted to the cylinder-like topology with a variable radius. More specifically, we consider a particular metric $ds^2 = g_{ab}(x)dx^a dx^b$ $(a, b = 1, 2)$, where $g_{ab} = \mathrm{diag}(1, h(x^1))$ and $h(x^1)$ denotes an arbitrary nonnegative function. Hence, x^1 is the axial coordinate with $\sqrt{h(x^1)}$ being the cylinder radius and x^2 corresponds to the angular coordinate.

In the supersymmetric description of this model, we have two real superfields X^1 and X^2 with the following central charge transformations: $\delta^Z x^1 = i\mu$ and $\delta^Z x^2 = 0$. After considering these particular metric and transformations in action (11), one arrives at the correspondent Lagrangian

$$\begin{aligned} L = {} & \frac{1}{2}\left[\left(\dot{x}^1\right)^2 + i\left(\xi^1\dot{\overline{\xi}}^1 - \dot{\xi}^1\overline{\xi}^1\right) + \left(\mathscr{W}^1\right)^2\right] + \frac{1}{2} \\ & \cdot h\left[\left(\dot{x}^2\right)^2 + i\left(\xi^2\dot{\overline{\xi}}^2 - \dot{\xi}^2\overline{\xi}^2\right) + \left(\mathscr{W}^2\right)^2\right] + \frac{1}{2} \\ & \cdot h'\left[-i\dot{x}^2\left(\xi^1\overline{\xi}^2 + \overline{\xi}^1\xi^2\right) + \mathscr{W}^2\left(\xi^1\overline{\xi}^2 - \overline{\xi}^1\xi^2\right)\right. \\ & \left. - \mathscr{W}^1\xi^2\overline{\xi}^2\right] - \frac{1}{2}h''\xi^1\overline{\xi}^1\xi^2\overline{\xi}^2 + \frac{1}{2}\mu h'\left(\xi^1\xi^2 - \overline{\xi}^1\overline{\xi}^2\right) \\ & - \frac{1}{2}\mu^2 h, \end{aligned} \tag{12}$$

where h' and h'' denote the first and second derivatives of $h(x^1)$, respectively.

According to [45], after carrying out the canonical quantization and imposing the superalgebra on the operators, it is possible to recognize a connection involving the central charge parameter (μ) and the quantum number (ν) of the angular momentum, given by the map duality $\mu \longleftrightarrow \nu$. In addition, this duality also maps the radius $R = \sqrt{h(x^1)}$ and x^1 with the following simultaneous transformations: $R \longleftrightarrow 1/R$ and $x^1 \longleftrightarrow -x^1$.

Finally, we notice that the Lagrangian (12) is equivalent to the one obtained in [45]. It only differs from some signs and i–factors which are due to our different conventions. At this point, it is interesting to compare both methodologies. In [45], the authors considered the following action

$$S_0 = \int dt d\overline{\theta} d\theta \left[\frac{1}{2} g_{ab}(X) DX^a \overline{D} X^b\right], \tag{13}$$

where $D = \partial_\theta - i\overline{\theta}\partial_t$ and $\overline{D} = \partial_{\overline{\theta}} - i\theta\partial_t$ correspond to derivatives without central charge. Then, (13) is decomposed in components and an analysis of supersymmetric transformation (with central charge) leads to a noninvariant action. In order to restore the invariance, some corrections terms $(c - t)$ are needed; i.e., a new contribution S_{c-t} is added to S_0 such that the complete action $S = S_0 + S_{c-t}$ remains invariant. Here, we claim that this complete action coincides with (10). Therefore, the deformed derivatives automatically include the correspondent L_{c-t}, namely, the last two terms in (12) with parameters μ and μ^2.

3. Supersymmetric Field Theory with Central Charge

In this section, we present another point of view to investigate this extended supersymmetry. Our proposal is to introduce a new coordinate ν related to central charge Z. It is worthy to comment that the introduction of extra bosonic coordinates – and, consequently, an extended superspace and deformed derivatives – is not exclusive to this work. We highlight some works [65–68] in which this approach has been adopted in other contexts. For instance, in [69], the authors developed an N–extended Poincaré superalgebra with $N(N-1)/2$ tensorial central charges and applied this formulation in higher-dimensional superparticle.

Here, we shall generalize the procedure described in [48] for $\mathcal{N} = 2$ supersymmetric mechanics (without Z). Initially, let us define a group element

$$G\left(t, \nu, \theta, \overline{\theta}\right) = e^{itH + i\nu Z + i\theta Q + i\overline{Q}\overline{\theta}}. \tag{14}$$

By using the superalgebra, (1)-(3), and the well-known Baker-Campbell-Hausdorff identity, $e^A e^B = e^{A+B} e^{(1/2)[A,B]}$, where $[[A, B], A] = 0 = [[A, B], B]$, one may verify the group structure with the multiplicative rule

$$\begin{aligned} G\left(t', \nu', \theta', \overline{\theta}'\right) G\left(t, \nu, \theta, \overline{\theta}\right) = {} & G\left(t + t'\right. \\ & + i\left(\theta'\overline{\theta} + \overline{\theta}'\theta\right), \nu + \nu' + \left(\theta'\theta - \overline{\theta}'\overline{\theta}\right), \theta + \theta', \overline{\theta} \\ & \left. + \overline{\theta}'\right), \end{aligned} \tag{15}$$

which leads to the following translation in the new superspace

$$\begin{aligned} \left(t, \nu, \theta, \overline{\theta}\right) \longrightarrow {} & \left(t + t' + i\left(\theta'\overline{\theta} + \overline{\theta}'\theta\right), \nu + \nu'\right. \\ & \left. + \left(\theta'\theta - \overline{\theta}'\overline{\theta}\right), \theta + \theta', \overline{\theta} + \overline{\theta}'\right). \end{aligned} \tag{16}$$

Notice that a real translation acts on ν in a similar way to time, namely, in both cases appear bilinear of the Grassmann parameters $(\theta, \overline{\theta})$.

Starting with the group element $G(t, \nu, \theta, \overline{\theta})$ described previously, we establish the superfield as

$$\begin{aligned} & X\left(t, \nu, \theta, \overline{\theta}\right) \\ & \quad = G\left(t, \nu, \theta, \overline{\theta}\right) X(0, 0, 0, 0) G^{-1}\left(t, \nu, \theta, \overline{\theta}\right). \end{aligned} \tag{17}$$

Now we consider an infinitesimal transformation and obtain the differential representation to supercharges and central charge. By using infinitesimal parameters $(t', v', \theta', \overline{\theta}') \longrightarrow (\varepsilon, \zeta, \epsilon, \overline{\epsilon})$ and the multiplicative rule (15), we have

$$G(\varepsilon,\zeta,\epsilon,\overline{\epsilon})\, X\left(t,v,\theta,\overline{\theta}\right) G^{-1}(\varepsilon,\zeta,\epsilon,\overline{\epsilon}) = X\left(t+\varepsilon + i\left(\epsilon\overline{\theta}+\overline{\epsilon}\theta\right), v+\zeta+\left(\epsilon\theta-\overline{\epsilon}\overline{\theta}\right), \theta+\epsilon, \overline{\theta}+\overline{\epsilon}\right). \tag{18}$$

A general operator $\mathcal{O}$ satisfies $i[\mathcal{O}, X] \sim \delta^{\mathcal{O}} X$. Thus, from (18), one may expand both sides and obtain the following differential representation:

$$\delta^{H} = i\partial_t, \qquad \delta^{Z} = -i\partial_v, \tag{19}$$

$$\delta^{Q} = \partial_\theta + i\overline{\theta}\partial_t + \theta\partial_v, \qquad \delta^{\overline{Q}} = \partial_{\overline{\theta}} + i\theta\partial_t - \overline{\theta}\partial_v. \tag{20}$$

At this point, we realize that the central charge behaves like a momentum operator of the "extra dimension" v. In comparison with the supercharges in mechanical case, (4) and (5), we have an analogous structure, but now central charge is completely fixed, $\delta^Z = -i\partial_v$, independently of the superfield components. These differential operators satisfy the same superalgebra: $\{\delta^Q, \delta^{\overline{Q}}\} = 2\delta^H$, $(\delta^Q)^2 = i\delta^Z$ and $(\delta^{\overline{Q}})^2 = -i\delta^Z$, with δ^H and δ^Z commuting with all operators.

In order to obtain the covariant derivatives, we consider an alternative multiplication to the right of the group elements, $G(t,v,\theta,\overline{\theta})G(t',v',\theta',\overline{\theta}')$, which leads to

$$\mathfrak{D} = \partial_\theta - i\overline{\theta}\partial_t - \theta\partial_v, \qquad \overline{\mathfrak{D}} = \partial_{\overline{\theta}} - i\theta\partial_t + \overline{\theta}\partial_v. \tag{21}$$

These covariant derivatives exhibit the same representation of the deformed derivatives in the mechanical case (now with $\delta^Z = -i\partial_v$) and anticommutation rules with the supercharges δ^Q and $\delta^{\overline{Q}}$. In both mechanical and field theory situations, the main role of the deformed derivatives is to provide a description with a manifested (extended) supersymmetry. In other words, by using the superfields and deformed derivatives, the invariance of the action is guaranteed. Moreover, these deformed derivatives are intrinsically related to the central charge by the following algebra: $\overline{\mathfrak{D}}^2 = i\delta^Z$ and $\mathfrak{D}^2 = -i\delta^Z$. Thus, for a vanishing central charge, the derivatives shall be nilpotent.

Having established the supercharges and covariant (deformed) derivatives, we turn our attention to the discussion of superfield and supersymmetric transformation. We define a real (bosonic) superfield as

$$X = f_1(t,v) + i\theta\psi(t,v) + i\overline{\theta}\,\overline{\psi}(t,v) + f_2(t,v)\,\theta\overline{\theta}. \tag{22}$$

It is important to mention that, in this formulation, we do not have a classical mechanics description, because now we deal with the component fields f_1, f_2, ψ, and $\overline{\psi}$, which in general depend on (t, v). However, by taking a dimensional reduction, one can arrive at mechanical case. For example, in the trivial reduction, ∂_v (all fields) $= 0$, we recover the usual $\mathcal{N} = 2$ supersymmetry (without central charge) with the identification $(f_1, \psi, \overline{\psi}, f_2) \longrightarrow (x, \xi, \overline{\xi}, \mathcal{W})$.

Let us obtain the component transformations with this kind of supersymmetry. By taking the Taylor expansion of the right-hand side of (18) and comparing with $\delta X \equiv \delta f_1 + i\theta\delta\psi + i\overline{\theta}\delta\overline{\psi} + \theta\overline{\theta}\delta f_2$, we have the following variations

$$\delta f_1 = i\epsilon\psi + i\overline{\epsilon}\overline{\psi} + \varepsilon\dot{f}_1 + \zeta\partial_v f_1, \tag{23}$$

$$\delta\psi = i\epsilon\partial_v f_1 - i\overline{\epsilon} f_2 - \overline{\epsilon}\dot{f}_1 + \varepsilon\dot{\psi} + \zeta\partial_v\psi, \tag{24}$$

$$\delta f_2 = \frac{\partial}{\partial t}\left(\epsilon\psi - \overline{\epsilon}\overline{\psi} + \varepsilon f_2\right) + \frac{\partial}{\partial v}\left(i\epsilon\overline{\psi} + i\overline{\epsilon}\psi + \zeta f_2\right). \tag{25}$$

Once f_2 transforms as partial derivatives of v and t, one may describe an invariant action in terms of real superfield and covariant derivatives. For supersymmetric transformations, we just need to fix $\zeta = \varepsilon = 0$ in the last variations, such that $\delta^{SUSY} = \epsilon\delta^Q + \overline{\epsilon}\delta^{\overline{Q}}$, where δ^Q and $\delta^{\overline{Q}}$ are given by (20). In this case, the component transformations are very similar to the mechanical case, (9).

Finally, we emphasize that in our formalism the chiral and antichiral superfields do not depend on v and have the same structure of the supersymmetric model without central charge.

3.1. Topological Configurations in (1+1)D. In this subsection, we discuss an application of the previous formalism. We investigate a particular model in two dimensions. Using the covariant derivatives (21) and real superfield (22), we propose the following action

$$S = \int dt\,dv\,d\overline{\theta}\,d\theta\left[\frac{1}{2}\mathfrak{D}X\overline{\mathfrak{D}}X + U(X)\right], \tag{26}$$

where $U(X)$ denotes an arbitrary superpotential.

This action leads to the following Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left(\dot{f}_1\right)^2 - \frac{1}{2}\left(\partial_v f_1\right)^2 + \frac{(f_2)^2}{2} + f_2\frac{\partial U}{\partial f_1} + \psi\overline{\psi}\frac{\partial^2 U}{\partial f_1^2} + \frac{i}{2}\left(\psi\dot{\overline{\psi}} - \dot{\psi}\overline{\psi}\right) + \frac{1}{2}\left(\left(\partial_v\overline{\psi}\right)\overline{\psi} + \psi\left(\partial_v\psi\right)\right). \tag{27}$$

Hence, we obtain a Klein-Gordon profile (first two terms), so we again note that v and Z can be interpreted as spatial coordinate and momentum operator ($\delta^Z = -i\partial_v$), respectively. In this case, we concluded that the role of central charge is to accommodate field theory in $(1+1)D$.

Since the auxiliary field $f_2(t, v)$ does not have dynamics, it can be eliminated by its equation of motion, $f_2(t, v) = -\partial U/\partial f_1$, resulting in the on-shell Lagrangian density

$$\mathscr{L}_{\text{on-shell}} = \frac{1}{2}\left(\dot{f}_1\right)^2 - \frac{1}{2}\left(\partial_v f_1\right)^2 - \frac{1}{2}\left(\frac{\partial U}{\partial f_1}\right)^2 + \psi\overline{\psi}\frac{\partial^2 U}{\partial f_1^2} + +\frac{i}{2}\left(\psi\dot{\overline{\psi}} - \dot{\psi}\overline{\psi}\right) + \frac{1}{2}\left(\left(\partial_v\overline{\psi}\right)\overline{\psi} + \psi\left(\partial_v\psi\right)\right). \tag{28}$$

The equations of motion are given by

$$\ddot{f}_1 - \partial_v^2 f_1 + U'U'' - \psi\overline{\psi}U''' = 0, \tag{29}$$

$$i\dot{\psi} - \partial_v\overline{\psi} - \psi U'' = 0, \tag{30}$$

where we used $U' \equiv \partial U/\partial f_1$ and similarly for higher derivatives.

Here, we notice that these equations of motion can accommodate topological configurations. Let us initially consider a trivial fermionic sector, $\psi(t,v) = \psi^{(1)}(t,v) = 0$. Then, for a particular case $f_1(t,v) = f_1^{(1)}(v)$, satisfying

$$\partial_v f_1^{(1)} = \pm U'\left(f_1^{(1)}\right), \tag{31}$$

we obtain a possible solution. This first-order equation can describe some topological configurations. We only need to specify the potential $U(f_1)$. Before doing that, we would like to point out some comments about the fermionic sector.

It is interesting to highlight that we can take advantage of the supersymmetric structure to get a set of solutions with nontrivial fermionic sector. We shall adopt a similar procedure discussed in [70], where oscillinos (fermionic) solutions were obtained by means of supersymmetric transformations on the oscillons (bosonic configurations).

First, one may verify that, by applying supersymmetric transformations, (29) and (30) are mapped into each other. That is exactly the sense of supersymmetry. In order to obtain a nontrivial fermionic solution with topological structure in the bosonic sector, we use the on-shell supersymmetric transformations, namely, (23) and (24) with $\varepsilon = \zeta = 0$ and $f_2 = -U'$. In other words, we need to consider a supersymmetric perturbation of the previous solution,

$$f_1^{(2)} = f_1^{(1)} + \delta f_1^{(1)} \equiv f_1^{(1)}, \tag{32}$$

$$\psi^{(2)} = \psi^{(1)} + \delta\psi^{(1)} \equiv \delta\psi^{(1)} = i\epsilon\partial_v f_1^{(1)} + i\overline{\epsilon}U', \tag{33}$$

which are also solutions of the equations of motion (29) and (30).

Finally, let us present a particular case. If we fix the arbitrary function $U(f_1)$ such that

$$\frac{1}{2}\left[U'\left(f_1\right)\right]^2 = A\left[1 - \cos\left(\frac{2\pi}{F}f_1\right)\right], \tag{34}$$

with A and F being constant parameters, then we obtain a new supersymmetric version of the sine-Gordon model [71]. In this case, the bosonic solution is given by

$$f_1^{(2)}(v) = \frac{2F}{\pi}\arctan\left[e^{m(v-v_0)}\right], \tag{35}$$

where v_0 is an arbitrary constant and $m \equiv 2\pi\sqrt{A}/F$.

By using the supersymmetric pertubation, (33), one can arrive at the following fermionic (static) solution

$$\psi^{(2)}(v) = i\epsilon 4\sqrt{A}\frac{e^{m(v-v_0)}}{1+e^{2m(v-v_0)}} \pm i\overline{\epsilon}\left\{2A\left[1-\cos\left(\frac{2\pi}{F}f_1^{(2)}(v)\right)\right]\right\}^{1/2}. \tag{36}$$

Finally, it is worthy to comment that, by construction, this fermionic solution has a trivial condensate, $\overline{\psi}^{(2)}\psi^{(2)} = 0$. This could indicate a possible relation between the extended superalgebra (with central charge and coordinate v) and other two-dimensional supersymmetries. We shall return to this point in our conclusions.

4. Concluding Comments

Initially, we have discussed the $\mathcal{N} = 2$ supersymmetric mechanics with one (real) central charge for the multiplet $(1,2,1)$. A prescription to obtain deformed $\mathcal{N} = 2$ models by central charge was developed. To establish this in a superfield approach, we have introduced deformed covariant derivatives, (6) and (7), which take into account the new terms related to the central charge. As an application, we have obtained a deformation of one-dimensional nonlinear sigma model. Also, we have recast the particular nonlinear sigma model of [45] and shown an equivalence between the two prescriptions for the specific transformations given by (9). However, we have noticed that an introduction of deformed derivatives allows us to implement this extended supersymmetry in a simplest way, once we maintain the superfields, and it is not necessary to decompose the Lagrangian in components and add counter-terms to recover the supersymmetry. Bearing this in mind, we would like to point out some possible subjects of investigations. In the context of general nonlinear sigma model (with Riemann curvature), one may add the torsion and generalized torsion terms [57], namely, some couplings involving 2– and 4–form with DX^iDX^j, $DX^iDX^jDX^kDX^l$ and its complex conjugations. A deformation of these models could be obtained through the following prescription proposed here: $D \longrightarrow \mathfrak{D}$ and $\overline{D} \longrightarrow \overline{\mathfrak{D}}$. It would be interesting to investigate the quantization of these deformed models and possible new connections between central charge and symmetries (e.g., duality and mirror) or some restrictions to the manifold.

In the second part, we have considered an implementation of superalgebra (1)-(3) in two-dimensional field theory. Among the lines discussed in the Introduction, a particular point of view has been adopted here; namely, we have introduced a new coordinate v and interpreted the central charge as a momentum operator. With these assumptions, the supersymmetric transformations are fully fixed, given by (23)-(25), where $\delta^Z = -i\partial_v$. This interpretation allowed us to obtain a nontrivial fermionic solution through supersymmetric tranformations rather than solving directly the fermionic equation of motion.

Finally, we point out some possible investigations related to this supersymmetry in two dimensions. First, in order to

accommodate charged matter and gauge fields, one could analyze the introduction of other multiplets, such as complex bosonic (fermionic) scalars and vectors superfields. Moreover, the connection between this superalgebra and other two-dimensional (Poincaré) supersymmetries remains a subject of further investigation. In particular, one could study the usual supersymmetry in two dimensions and redefine or drop out some Lorentz (boost) generators. Remember that, in two dimensions, it is possible to have a Majorana-Weyl fermion and implement the heterotic (p, q)–supersymmetries. This perspective is based on the fact that the Majorana-Weyl condition in two dimensions implies one degree of freedom with a trivial condensate. In our context, we have also obtained a similar situation for the fermionic solution $\psi^{(2)}(v)$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References

[1] R. Haag, J. T. Lopuszanski, and M. Sohnius, "All possible generators of supersymmetries of the *S*-matrix," *Nuclear Physics B*, vol. 88, pp. 257–274, 1975.

[2] A. Salam and J. Strathdee, "Supergauge Transformations," *Nuclear Physics B*, vol. 76, no. 3, pp. 477–482, 1974.

[3] A. Salam and J. Strathdee, "SU(6) and supersymmetry," *Nuclear Physics B*, vol. 84, p. 127, 1975.

[4] M. F. Sohnius, "Introducing supersymmetry," *Physics Reports*, vol. 128, no. 2-3, pp. 39–204, 1985.

[5] E. Witten and D. Olive, "Supersymmetry algebras that include topological charges," *Physics Letters B*, vol. 78, no. 1, pp. 97–101, 1978.

[6] S. Aoyama, "The (2+1)-dimensional supersymmetric CP^{N-1} model and the central charge," *Nuclear Physics B*, vol. 168, no. 2, pp. 354–364, 1980.

[7] S. Rouhani, "The central charge in the supersymmetric CP^{n-1} model," *Nuclear Physics B*, vol. 169, no. 5-6, pp. 430–444, 1980.

[8] V. K. Krivoshchekov and P. B. Medvedev, "On central charge dynamical generation in the d = 2 SUSY model," *Physics Letters B - Journal*, vol. 149, pp. 138–142, 1984.

[9] J. A. Gracey, "On topological central charges and solitons," *Physics Letters. B. Particle Physics, Nuclear Physics and Cosmology*, vol. 185, no. 1-2, pp. 104–106, 1987.

[10] J. A. Gracey, "Exact supersymmetric S-matrices and central charges," *Nuclear Physics B*, vol. 288, p. 475, 1987.

[11] Z. Hlousek and D. Spector, "Topological charges and central charges in (3+1)-dimensional supersymmetry," *Physics Letters. B. Particle Physics, Nuclear Physics and Cosmology*, vol. 283, no. 1-2, pp. 75–79, 1992.

[12] S. Ferrara, C. A. Savoy, and B. Zumino, "General massive multiplets in extended supersymmetry," *Physics Letters B*, vol. 100, no. 5, pp. 393–398, 1981.

[13] M. F. Sohnius, K. S. Stelle, and P. C. West, "Dimensional reduction by Legendre transformation generates off-shell supersymmetric Yang-Mills theories," *Nuclear Physics B*, vol. 173, no. 1, pp. 127–153, 1980.

[14] P. A. Zizzi, "Topological content of dimensional reduction," *Physical Review D: Particles, Fields, Gravitation and Cosmology*, vol. 28, no. 4, pp. 808–817, 1983.

[15] J. Gates Jr., "Superspace formulation of new nonlinear sigma models," *Nuclear Physics B*, vol. 238, no. 2, pp. 349–366, 1984.

[16] L. Alvarez-Gaumé and D. Z. Freedman, "Potentials for the supersymmetric nonlinear σ-model," *Communications in Mathematical Physics*, vol. 91, no. 1, pp. 87–101, 1983.

[17] S. Aoyama and P. Zizzi, "The supersymmetric Georgi-Glashow-like model in 2+1 dimensions and the central charge," *Nuovo Cimento A*, vol. 61, no. 4, pp. 237–244, 1981.

[18] S. Ferrara and B. Zumino, "The mass matrix of N = 8 supergravity," *Physics Letters B*, vol. 86, no. 3-4, pp. 279–282, 1979.

[19] J. Lukierski and L. Rytel, "Extended supersymmetry in five dimensions with central charges," *Physical Review D: Particles, Fields, Gravitation and Cosmology*, vol. 27, no. 10, pp. 2354–2358, 1983.

[20] J. A. de Azcárraga, J. P. Gauntlett, J. M. Izquierdo, and P. K. Townsend, "Topological extensions of the supersymmetry algebra for extended objects," *Physical Review Letters*, vol. 63, no. 22, pp. 2443–2446, 1989.

[21] I. Bars, "Supersymmetry, *p*-brane duality, and hidden spacetime dimensions," *Physical Review D: Particles, Fields, Gravitation and Cosmology*, vol. 54, no. 8, pp. 5203–5210, 1996.

[22] S. Ferrara and M. Porrati, "Central extensions of supersymmetry in four and three dimensions," *Physics Letters. B. Particle Physics, Nuclear Physics and Cosmology*, vol. 423, no. 3-4, pp. 255–260, 1998.

[23] A. Gorsky and M. Shifman, "More on the tensorial central charges in $N = 1$ supersymmetric gauge theories: BPS wall junctions and strings," *Physical Review D: Particles, Fields, Gravitation and Cosmology*, vol. 61, no. 8, Article ID 085001, 14 pages, 2000.

[24] A. Losev and M. Shifman, "$N = 2$ sigma model with a twisted mass and superpotential: Central charges and solitons," *Physical Review D: Particles, Fields, Gravitation and Cosmology*, vol. 68, no. 4, Article ID 045006, 11 pages, 2003.

[25] K. Shizuya, "Superfield formulation of central charge anomalies in two-dimensional supersymmetric theories with solitons," *Physical Review D: Particles, Fields, Gravitation and Cosmology*, vol. 69, no. 6, Article ID 065021, 10 pages, 2004.

[26] K. Shizuya, "Topological-charge anomalies in supersymmetric theories with domain walls," *Physical Review D*, vol. 70, Article ID 065003, 2004.

[27] K. Shizuya, "Central charge and renormalization in supersymmetric theories with vortices," *Physical Review D*, vol. 71, Article ID 065006, 2005.

[28] K. Shizuya, "Effect of quantum fluctuations on topological excitations and central charge in supersymmetric theories," *Physical Review D*, vol. 74, Article ID 025013, 2006.

[29] A. Rebhan, P. van Nieuwenhuizen, and R. Wimmer, "Nonvanishing quantum corrections to the mass and central charge of the *N=2* vortex and BPS saturation," *Nuclear Physics B*, vol. 679, no. 1-2, pp. 382–394, 2004.

[30] A. Rebhan, P. van Nieuwenhuizen, and R. Wimmer, "Quantum mass and central charge of supersymmetric monopoles: anomalies, current renormalization, and surface terms," *Journal of High Energy Physics*, vol. 2006, article 056, 2006.

[31] M. Shifman, A. Vainshtein, and R. Zwicky, "Central charge anomalies in 2D sigma models with twisted mass," *Journal of Physics A: Mathematical and General*, vol. 39, no. 41, pp. 13005–13024, 2006.

[32] V. K. Oikonomou, "Superconducting cosmic strings and one dimensional extended supersymmetric algebras," *Annals of Physics*, vol. 350, pp. 179–197, 2014.

[33] V. K. Oikonomou, "Localized Fermions on Domain Walls and Extended Supersymmetric Quantum Mechanics," *Classical and Quantum Gravity*, vol. 31, Article ID 025018, 2014.

[34] V. K. Oikonomou, "Localized Fermions on Superconducting Domain Walls and Extended Supersymmetry with Non-trivial Topological Charges," *Foundations of Physics*, vol. 45, no. 1, pp. 44–64, 2015.

[35] V. K. Oikonomou, "Extended Supersymmetric Structures in Gapped and Superconducting Graphene," *International Journal of Geometric Methods in Modern Physics*, vol. 12, Article ID 1550114, 2015.

[36] J. Niederle and A. G. Nikitin, "Extended SUSY with central charges in quantum mechanics," *Proceedings of Institute of Mathematics of NAS of Ukraine*, vol. 43, p. 497, 2002.

[37] E. Karat, "An example of Poincaré symmetry with a central charge," *Physics Letters B - Journal*, vol. 445, no. 3-4, pp. 337–343, 1999.

[38] H. L. Carrion, M. Rojas, and F. Toppan, "Residual symmetries in the presence of an EM background," *Modern Physics Letters A*, vol. 18, no. 9, pp. 629–641, 2003.

[39] A. V. Smilga, "Weak supersymmetry," *Physics Letters. B. Particle Physics, Nuclear Physics and Cosmology*, vol. 585, no. 1-2, pp. 173–179, 2004.

[40] E. Ivanov and S. Sidorov, "Deformed supersymmetric mechanics," *Classical and Quantum Gravity*, vol. 31, no. 7, Article ID 075013, 31 pages, 2014.

[41] E. Ivanov and S. Sidorov, "SU(2|1) mechanics and harmonic superspace," *Classical and Quantum Gravity*, vol. 33, Article ID 055001, 2016.

[42] E. Ivanov, O. Lechtenfeld, and S. Sidorov, "SU(2|2) supersymmetric mechanics," *Journal of High Energy Physics*, vol. 2016, no. 11, article 031, 2016.

[43] E. A. Ivanov, S. O. Krivonos, and A. I. Pashnev, "Partial supersymmetry breaking in N=4 supersymmetric quantum mechanics," *Classical and Quantum Gravity*, vol. 8, no. 1, pp. 19–39, 1991.

[44] A. Losev and M. Shifman, "Reduced N=2 Quantum Mechanics: Descendants of the Kahler Geometries," *Modern Physics Letters A*, vol. 16, no. 39, pp. 2529–2543, 2001.

[45] M. Faux and D. Spector, "Duality and central charges in supersymmetric quantum mechanics," *Physical Review D: Particles, Fields, Gravitation and Cosmology*, vol. 70, no. 8, Article ID 085014, 5 pages, 2004.

[46] M. Faux, D. Kagan, and D. Spector, *Central Charges and Extra Dimensions in Supersymmetric Quantum Mechanics*, 2004, https://arxiv.org/abs/hep-th/0406152.

[47] E. Ivanov and S. Sidorov, "Long multiplets in supersymmetric mechanics," *Physical Review D: Particles, Fields, Gravitation and Cosmology*, vol. 93, no. 6, Article ID 065052, 11 pages, 2016.

[48] T. E. Clark, S. T. Love, and S. R. Nowling, "The supercharge and superconformal symmetry for N=1 supersymmetric quantum mechanics," *Nuclear Physics. B*, vol. 632, no. 1-3, pp. 3–50, 2002.

[49] R. C. Paschoal, J. A. Helayël-Neto, and L. P. G. de Assis, "Planar supersymmetric quantum mechanics of a charged particle in an external electromagnetic field," *Physics Letters A*, vol. 349, pp. 67–74, 2006.

[50] R. A. Coles and G. Papadopoulos, "The geometry of the one-dimensional supersymmetric nonlinear sigma models," *Classical and Quantum Gravity*, vol. 7, no. 3, pp. 427–438, 1990.

[51] M. Gonzales, M. Rojas, and F. Toppan, "One-dimensional σ-models with N= 5, 6, 7, 8 off-shell supersymmetries," *International Journal of Modern Physics A*, vol. 24, no. 23, pp. 4317–4326, 2009.

[52] M. Gonzales, S. Khodaee, and F. Toppan, "On non-minimal N=4 supermultiplets in 1D and their associated sigma-models," *Journal of Mathematical Physics*, vol. 52, no. 1, 013514, 18 pages, 2011.

[53] M. Gonzales, K. Iga, S. Khodaee, and F. Toppan, "Pure and entangled N=4 linear supermultiplets and their one-dimensional sigma-models," *Journal of Mathematical Physics*, vol. 53, no. 10, Article ID 103513, 17 pages, 2012.

[54] S. Krivonos and A. Shcherbakov, "N= 4, d= 1 tensor multiplet and hyper-Kähler σ-models," *Physics Letters. B. Particle Physics, Nuclear Physics and Cosmology*, vol. 637, no. 1-2, pp. 119–122, 2006.

[55] F. Delduc and E. Ivanov, "N=4 mechanics of general (4,4,0) multiplets," *Nuclear Physics B*, vol. 855, no. 3, pp. 815–853, 2012.

[56] S. Bellucci and A. Nersessian, "Note on N=4 supersymmetric mechanics on Kähler manifolds," *Physical Review D: Particles, Fields, Gravitation and Cosmology*, vol. 64, no. 2, Article ID 021702, 5 pages, 2001.

[57] S. A. Fedoruk, E. A. Ivanov, and A. V. Smilga, "Real and complex supersymmetric d=1 sigma models with torsions," *International Journal of Modern Physics A*, vol. 27, no. 25, Article ID 1250146, 25 pages, 2012.

[58] G. W. Gibbons, G. Papadopoulos, and K. S. Stelle, "HKT and OKT geometries on soliton black hole moduli spaces," *Nuclear Physics. B*, vol. 508, no. 3, pp. 623–658, 1997.

[59] C. Hull, *The Geometry of Supersymmetric Quantum Mechanics*, 1999, https://arxiv.org/abs/hep-th/9910028.

[60] M. Wasay, "Supersymmetric quantum mechanics and topology," *Advances in High Energy Physics*, vol. 2016, Article ID 3906746, 10 pages, 2016.

[61] I. Bandos and J. Lukierski, "Tensorial central charges and new superparticle models with fundamental spinor coordinates," *Modern Physics Letters A*, vol. 14, no. 19, pp. 1257–1272, 1999.

[62] S. Fedoruk and V. G. Zima, "Massive superparticle with tensorial central charges," *Modern Physics Letters A*, vol. 15, no. 37, pp. 2281–2296, 2000.

[63] I. Bandos, J. Lukierski, and D. Sorokin, "Superparticle models with tensorial central charges," *Physical Review D: Particles, Fields, Gravitation and Cosmology*, vol. 61, no. 4, Article ID 045002, 20 pages, 2000.

[64] Z. Kuznetsova and F. Toppan, "Constrained generalized supersymmetries and superparticles with tensorial central charges: A Classification," *Journal of High Energy Physics*, vol. 2005, no. 5, article 060, 2005.

[65] M. F. Sohnius, "Supersymmetry and central charges," *Nuclear Physics B*, vol. 138, no. 1, pp. 109–121, 1978.

[66] H. Osborn, "Topological charges for N = 4 supersymmetric gauge theories and monopoles of spin 1," *Physics Letters B*, vol. 83, no. 3-4, pp. 321–326, 1979.

[67] D. Olive, "The electric and magnetic charges as extra components of four-momentum," *Nuclear Physics B*, vol. 153, no. 1-2, pp. 1–12, 1979.

[68] J. Lukierski and L. Rytel, "Geometric origin of central charges," *Journal of Physics A: Mathematical and General*, vol. 15, no. 5, pp. L215–L220, 1982.

[69] J. A. de Azcárraga and J. Lukierski, "Supersymmetric particle model with additional bosonic coordinates," *Zeitschrift für Physik C*, vol. 30, no. 2, pp. 221–227, 1986.

[70] R. A. C. Correa, L. P. R. Ospedal, W. de Paula, and J. A. Helayël-Neto, "Supersymmetry and fermionic modes in an oscillon background," *Physics Letters B*, vol. 780, pp. 159–165, 2018.

[71] G. Hooft and F. Bruckmann, *Monopoles, Instantons and Confinement*, 2000, https://arxiv.org/abs/hep-th/0010225.