

Research Article

Solution of the κ -Deformed Dirac Equation with Vector and Scalar Interactions in the Context of Spin and Pseudospin Symmetries

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The deformed Dirac equation invariant under the κ -Poincaré-Hopf quantum algebra in the context of minimal and scalar couplings under spin and pseudospin symmetry limits is considered. The κ -deformed Pauli-Dirac Hamiltonian allows us to study effects of quantum deformation in a class of physical systems, such as a Zeeman-like effect, Aharonov-Bohm effect, and an anomalous-like contribution to the electron magnetic moment, between others. In our analysis, we consider the motion of an electron in a uniform magnetic field and interacting with (i) a planar harmonic oscillator and (ii) a linear potential. We verify that the particular choice of a linear potential induces a Coulomb-type term in the equation of motion. Expressions for the energy eigenvalues and wave functions are determined taking into account both symmetry limits. We verify that the energies and wave functions of the particle are modified by the deformation parameter as well as by the element of spin.

1. Introduction

Quantum deformations based on the κ -Poincaré-Hopf algebra constitute an important branch of research that enables us to address problems in condensed matter and high energy physics through field equations. A pioneer work presenting these equations can be viewed in Ref. [1] (see also Refs. [2–6]), where a new real quantum Poincaré algebra with a standard real structure is obtained by contraction of $U_q(O(3,2))$. The resulting algebra of this contraction is a standard real Hopf algebra that depends on a dimension-full parameter κ instead of the real deformation parameter q . This algebra is defined by the following commutation relations:

$$\begin{aligned}
 [p_\nu, p_\mu] &= 0, \quad \mu, \nu = 0, 1, 2, 3, \\
 [\tilde{M}_i, p_\mu] &= (1 - \delta_{0\mu}) i\varepsilon_{ijk} p_k, \quad i, j = 1, 2, 3, \\
 [L_i, p_\mu] &= i[p_i]^{\delta_{0\mu}} [\delta_{ij} \varepsilon^{-1} \sinh(\varepsilon p_0)]^{1-\delta_{0\mu}}, \\
 [\tilde{M}_i, \tilde{M}_j] &= i\varepsilon_{ijk} \tilde{M}_k, \\
 [\tilde{M}_i, L_j] &= i\varepsilon_{ijk} L_k, \\
 [L_i, L_j] &= -i\varepsilon_{ijk} \left[\tilde{M}_k \cosh(\varepsilon p_0) - \frac{\varepsilon^2}{4} p_k p_l \tilde{M}_l \right],
 \end{aligned} \tag{1}$$

where ε is defined by

$$\varepsilon = \kappa^{-1} = \lim_{R \rightarrow \infty} (R \ln q), \quad (2)$$

with R is being the de Sitter curvature and $p_\mu = (p_0, \mathbf{p})$ are the κ -deformed generators for energy and momenta. In the above commutation relations, \tilde{M}_i and L_i represent the spatial rotations and deformed boost generators, respectively. The coalgebra and antipode for the κ -deformed Poincaré algebra were established in Ref. [7] and have the following representation:

$$\begin{aligned} \Delta \tilde{M}_i &= \tilde{M}_i \otimes I + I \otimes \tilde{M}_i, \\ \Delta p_0 &= p_0 \otimes I + I \otimes p_0, \\ \Delta p_i &= p_i \otimes e^{(1/2)\varepsilon p_0} + e^{-(1/2)\varepsilon p_0} \otimes p_i, \\ \Delta L_i &= L_i \otimes e^{(1/2)\varepsilon p_0} + e^{-(1/2)\varepsilon p_0} \otimes L_i \\ &\quad + \frac{1}{2} \varepsilon \varepsilon_{ijk} \left[p_j \otimes \tilde{M}_k e^{(1/2)\varepsilon p_0} + e_j^{-(1/2)\varepsilon p_0} \tilde{M} \otimes p_k \right], \\ S(p_\mu) &= -p_\mu, \\ S(\tilde{M}_i) &= -\tilde{M}_i, \\ S(L_i) &= -L_i + \frac{3\varepsilon i}{2} p_i, \end{aligned} \quad (3)$$

where I is the identity operator. The deformed Casimir operators for the κ -Poincaré algebra have the following representation:

$$\begin{aligned} C_1 &= \left[\frac{2}{\varepsilon} \sinh(\varepsilon p_0) \right]^2 - p_i p_i, \\ C_2 &= \left[\cosh(\varepsilon p_0) - \frac{\varepsilon^2}{4} p_i p_i \right] W_0^2 - W_i W_i, \end{aligned} \quad (4)$$

where

$$\begin{aligned} W_0 &= p_i \tilde{M}_i, \\ W_i &= \frac{1}{\varepsilon} \sinh(\varepsilon p_0) \tilde{M}_i + \varepsilon_{ijk} p_j p_k. \end{aligned} \quad (5)$$

With these definitions in hand, we write

$$\begin{aligned} \mathcal{P}_\mu &= p_\mu \\ \mathcal{M}_i &= \tilde{M}_i + m_i, \\ \mathcal{L}_i &= L_i + e^{-(1/2)\varepsilon p_0} l_i - \frac{1}{2} \varepsilon \varepsilon_{ijk} m_j p_k, \\ m_i &= \frac{1}{4} i \varepsilon_{ijk} \gamma_j \gamma_k, \\ l_i &= -\frac{1}{2} i \gamma_0 \gamma_i. \end{aligned} \quad (6)$$

The κ -deformed Dirac operator derived from the above algebra is found to be

$$\mathcal{D} = -e^{-(1/2)\varepsilon p_0} \gamma_i p_i + \frac{1}{\varepsilon} \gamma_0 \sinh(\varepsilon p_0) - \frac{1}{2} \varepsilon \gamma_0 p_i p_i \quad (7)$$

and satisfies the following relations:

$$[\mathcal{D}, \mathcal{P}_\mu] = [\mathcal{D}, \mathcal{M}_i] = [\mathcal{D}, \mathcal{L}_i] = 0. \quad (8)$$

The square of operator (7) can be written in terms of the invariants C_1 and C_2 as

$$\mathcal{D}^2 = C_1 \left(1 + \frac{1}{4} \varepsilon^2 C_1 \right), \quad (9)$$

or

$$\mathcal{D} = \sqrt{C_1 \left(1 + \frac{1}{4} \varepsilon^2 C_1 \right)} = -\frac{4}{3} C_2. \quad (10)$$

Since $C_1 = M^2$, we can write

$$\mathcal{D} = M \sqrt{\left(1 + \frac{1}{4} \varepsilon^2 M^2 \right)}. \quad (11)$$

In this way, the Dirac equation can be written as

$$\mathcal{D}\psi - M \sqrt{1 + \frac{1}{4} \varepsilon^2 M^2} \psi = 0. \quad (12)$$

For more details on the derivation of equation (12), see [3], for example. The algebraic structure of the κ -deformed Poincaré algebra has been widely investigated intensively and has become a theoretical field of increasing interest [5, 6, 8–25]. Through the field equations from the κ -Poincaré algebra (κ -Dirac equation [3, 4, 26]), we can study the physical implications of the quantum deformation parameter κ in relativistic and nonrelativistic quantum systems. In this context, we highlight the study of relativistic Landau levels [22], the Aharonov-Bohm effect taking into account spin effects [21], the Dirac oscillator [27, 28], and the integer quantum Hall effect [29].

When we want to study the relativistic quantum dynamics of particles with spin, it is desirable to consider the presence of external fields, which include the vector and scalar fields. The inclusion of vector and scalar potentials in the Dirac equation reveals interesting properties of symmetries in nuclear theory. The first contributions in this subject revealed the existence of SU(2) symmetries, which are known in the literature as pseudospin and spin symmetries [30, 31]. Some investigations have been made in this scenario in order to give a meaning to these symmetries. However, it was only in a work by Ginocchio that pseudospin symmetry was revealed. He verified that pseudospin symmetry in nuclei could arise from nucleons moving in a relativistic mean field, which has an attractive scalar and repulsive vector potential nearly equal in magnitude [32] (for a more detailed description, see Ref. [33]). Spin and pseudospin symmetries in the Dirac equation have been studied under different aspects in

recent years (see Refs. [34, 35]). Some studies have been developed taking into account the exact spin and pseudospin symmetry limits to study the relativistic dynamics of physical systems interacting with a class of potentials [36–43].

The present work is proposed to investigate the κ -deformed Dirac equation derived in Ref. [4] in the context of minimal and scalar couplings under spin and pseudospin symmetry limits. The structure of the paper is as follows: In Section 2, we present the κ -deformed Dirac equation with couplings from which we derive the κ -deformed Pauli-Dirac equation, by using the usual procedure that consists of squaring the κ -deformed Dirac equation. In Section 3, we consider the equation of Pauli and establish the spin and pseudospin symmetry limits. As an application, we consider the particle interacting with a uniform magnetic field in the z -direction in two different physical situations: (i) particle interacting with a harmonic oscillator and (ii) particle interacting with a linear potential. We obtain expressions for the energy eigenvalues and wave functions in both limits. In Section 4, we present our comments and conclusions.

2. The κ -Deformed Dirac Equation with Couplings

In this section, we introduce the minimal and scalar substitutions into κ -deformed Dirac equation (12). Before doing this, let us write it in a more convenient way to be treated. This can be accomplished by multiplying the left of equation (12) by $e^{-(1/2)\varepsilon p_0}$ and retaining terms up to $\mathcal{O}(\varepsilon)$. The resulting equation is found to be [3, 4]

$$\left\{ (\gamma_0 p_0 - \gamma_i p_i) + \frac{1}{2}\varepsilon [\gamma_0 (p_0^2 - p_i p_i) - M p_0] \right\} \psi = M \psi. \quad (13)$$

Deformed Dirac equation (13) is invariant under the κ -deformed Poincaré quantum algebra. Manipulating this equation and using the properties of the γ Dirac matrices and then rearranging the terms, it is possible to write it in the Hamiltonian form, $(H - E)\psi = 0$, where H is the κ -deformed Dirac Hamiltonian and E are the energy eigenvalues. After identifying $p_0 = H_0 = E$, with $H_0 = \gamma_0 \gamma_i p_i + \gamma_0 M$, we can iterate equation (13). This development yields to

$$E\psi = \gamma_0 \gamma_i p_i + \gamma_0 M - \frac{\varepsilon}{2} [(\gamma_0 \gamma_i p_i + \gamma_0 M)^2 - p_i p_i] + \frac{\varepsilon}{2} [\gamma_0 M (\gamma_0 \gamma_i p_i + \gamma_0 M)] \psi, \quad (14)$$

where we have discarded the terms of order equal and superior to ε^2 on the iterative replacements of the energy p_0 . The interactions can be introduced through the following prescriptions [44]:

$$p_i \longrightarrow p_i - eA_i, \quad (15)$$

$$E \longrightarrow E - V(r), \quad (16)$$

$$M \longrightarrow M + S(r), \quad (17)$$

where $V(r)$ is the electric potential, A_i is the vector potential, and $S(r)$ is the scalar potential. It is known from relativistic quantum mechanics that for vector coupling the potential couples to the charge (which is different for particles and antiparticles) while for scalar coupling the potential couples to the mass (which is equal for both particles and antiparticles).

As mentioned in Ref. [4], the couplings ((15), (16), and (17)) are quite satisfactory from the point of view of the κ -Poincaré-Hopf algebra. Indeed, there are no operator ordering problems after the gauging that would require symmetrization in equation (13). Using these couplings, we obtain the following expression:

$$\begin{aligned} & [\gamma_0 \gamma_i (p_i - eA_i) + \gamma_0 (M + S(r))] \psi \\ & - \frac{1}{2} \varepsilon [[\gamma_0 \gamma_i (p_i - eA_i) + \gamma_0 (M + S(r))]^2] \psi \\ & + \frac{1}{2} \varepsilon (M + S(r)) [\gamma_i (p_i - eA_i) + (M + S(r))] \psi \\ & + \frac{1}{2} \varepsilon (p_i - eA_i)^2 \psi = (E - V(r)) \psi. \end{aligned} \quad (18)$$

Since we are interested in planar dynamics, i.e., when both p_z and z as well as the third directions of the fields involved are zero, we choose the following representation for the gamma matrices in terms of the Pauli's matrices [45]:

$$\begin{aligned} \gamma_0 &= \sigma_3, \\ \alpha_1 &= \gamma_0 \gamma_1 = \sigma_1, \\ \alpha_2 &= \gamma_0 \gamma_2 = s \sigma_2, \end{aligned} \quad (19)$$

where parameter s , which has a value of twice the spin value, can be introduced to characterizing the two spin states, with $s = +1$ for spin *up* and $s = -1$ for spin *down*. The matrices σ_i ($i = 1, 2, 3$) satisfy the properties

$$\begin{aligned} \sigma_i^2 &= I, \\ [\sigma_i, \sigma_j] &= 2i\varepsilon_{ijk} \sigma_k, \\ \{\sigma_i, \sigma_j\} &= 2\delta_{ij}, \\ \sigma_i \sigma_j &= i\varepsilon_{ijk} \sigma_k + \delta_{ij}, \\ \sigma_i \sigma_k \sigma_l &= i\varepsilon_{ikl} I + \delta_{ik} \sigma_l + \delta_{kl} \sigma_i. \end{aligned} \quad (20)$$

Using the representation and the properties of the Pauli's matrices above, the κ -deformed Dirac equation including the interactions can be written as

$$\begin{aligned} & [\boldsymbol{\alpha} \cdot (\mathbf{p} - e\mathbf{A}) + \gamma_0 (M + S(r))] \psi - [E - V(r)] \psi \\ & + \frac{\varepsilon}{2} [es(\boldsymbol{\sigma} \cdot \mathbf{B}) \psi + \gamma_0 ((\boldsymbol{\alpha} \cdot \mathbf{p}) S(r))] \psi \\ & + \frac{\varepsilon}{2} [M \gamma_0 (\boldsymbol{\alpha} \cdot \mathbf{p}) + \gamma_0 S(r) (\boldsymbol{\alpha} \cdot \mathbf{p}) \psi - \gamma_0 e (\boldsymbol{\alpha} \cdot \mathbf{A}) M \psi] \\ & - \frac{\varepsilon}{2} [\gamma_0 e (\boldsymbol{\alpha} \cdot \mathbf{A}) S(r) \psi] = 0, \end{aligned} \quad (21)$$

where the magnetic field B emerges when we use the relation

$$i\varepsilon_{ijk}\sigma_k(p_i - eA_i)(p_j - eA_j)\psi = -s\sigma_k B_k \psi. \quad (22)$$

Let us now determine the Dirac equation in its quadratic form. This can be accomplished by applying the matrix operator [46, 47]

$$\begin{aligned} & \alpha \cdot (\mathbf{p} - e\mathbf{A}) + \gamma_0(M + S(r)) + E - V(r) \\ & + \frac{\varepsilon}{2}[\gamma_0 S(r)(\alpha \cdot \mathbf{p}) + es(\sigma \cdot \mathbf{B}) + \gamma_0((\alpha \cdot \mathbf{p})S(r))] \\ & + \frac{\varepsilon}{2}[M\gamma_0(\alpha \cdot \mathbf{p}) - \gamma_0 e(\alpha \cdot \mathbf{A})M - \gamma_0 e(\alpha \cdot \mathbf{A})S(r)] \end{aligned} \quad (23)$$

in equation (21). In this procedure, we follow a similar algebra and the same mathematical properties of the parameters and variables involved on the deduction of equation (21). The result is the κ -deformed Dirac-Pauli equation

$$\begin{aligned} & (\mathbf{p} - e\mathbf{A})^2 \psi + \alpha \cdot [\mathbf{p}V(r)]\psi - \gamma_0 \alpha \cdot [\mathbf{p}S(r)]\psi \\ & + [M + S(r)]^2 \psi - [E - V(r)]^2 \psi - es\sigma_z B \psi \\ & - \frac{\varepsilon}{2} \{ \gamma_0 [\mathbf{p}^2 S(r)] + \gamma_0 [(\alpha \cdot \mathbf{p})S(r)][(\alpha \cdot \mathbf{p})] \} \psi \\ & + \frac{\varepsilon}{2} \{ 2is\gamma_0 [\sigma \cdot ((\mathbf{p}S(r)) \times \mathbf{p}) - 2ise\sigma \cdot (\mathbf{p}S(r)) \times \mathbf{A}] \} \psi \\ & + \frac{\varepsilon}{2} \{ e\gamma_0 [(\alpha \cdot \mathbf{p})S(r)][(\alpha \cdot \mathbf{A})] - S(r)[(\alpha \cdot \mathbf{p})S(r)] \} \psi \\ & + \frac{\varepsilon}{2} \{ 2MesB + 2S(r)esB - M[(\alpha \cdot \mathbf{p})S(r)] \} \psi \\ & + \frac{\varepsilon}{2} \{ M\gamma_0 [(\alpha \cdot \mathbf{p})V(r)] + \gamma_0 S(r)[(\alpha \cdot \mathbf{p})V(r)] \} \psi = 0. \end{aligned} \quad (24)$$

It can be easily verified in equation (24) that after considering $\varepsilon = 0$, the resulting equation is the one well known in the literature (see, for example, Ref. [48]). In order to apply equation (24) to a given physical system, we need to choose a representation for the vector potential \mathbf{A} and the scalar potentials $S(r)$ and $V(r)$. For some particular choices of these quantities, we can study the physical implications of quantum deformation on the properties of various physical systems of interest.

For the field configuration, we consider a constant magnetic field along the z -direction (in cylindrical coordinates), $\mathbf{B} = B\hat{\mathbf{z}}$, which is obtained from the vector potential (in the Landau gauge) [49],

$$\mathbf{A} = \frac{Br}{2} \hat{\boldsymbol{\phi}}. \quad (25)$$

In this configuration, equation (24) reads

$$X + \frac{\varepsilon}{2} Y = 0, \quad (26)$$

with

$$\begin{aligned} X = & -\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \varphi^2} + ieB \frac{\partial \psi}{\partial \varphi} + \frac{1}{4} e^2 B^2 r^2 \psi \\ & + [M + S(r)]^2 \psi - [E - V(r)]^2 \psi - es\sigma_z B \psi \\ & + i \left[\frac{\partial S(r)}{\partial r} \right] \gamma_0 \alpha_r \psi - i \left[\frac{\partial V(r)}{\partial r} \right] \alpha_r \psi, \\ Y = & \gamma_0 \left[\frac{\partial^2 S(r)}{\partial r^2} + \frac{1}{r} \frac{\partial S(r)}{\partial r} \right] \psi - \gamma_0 \left[\frac{\partial S(r)}{\partial r} \right] \frac{\partial \psi}{\partial r} \\ & - is \left[\frac{1}{r} \frac{\partial S(r)}{\partial r} \right] \frac{\partial \psi}{\partial \varphi} - 2is \left[\frac{1}{r} \frac{\partial S(r)}{\partial r} \right] \frac{\partial \psi}{\partial \varphi} \\ & - iS(r) \gamma_r \left[\frac{\partial V(r)}{\partial r} \right] \psi + iS(r) \alpha_r \left[\frac{\partial S(r)}{\partial r} \right] \psi \\ & + es \left[\frac{\partial S(r)}{\partial r} \right] \frac{Br}{2} \psi - 2es \left[\frac{\partial S(r)}{\partial r} \right] \frac{Br}{2} \psi \\ & - iM \gamma_r \left[\frac{\partial V(r)}{\partial r} \right] \psi + iM \alpha_r \left[\frac{\partial S(r)}{\partial r} \right] \psi \\ & + 2MesB \psi + 2S(r)esB \psi, \end{aligned} \quad (27)$$

where the matrices (19) are now given in cylindrical coordinates, $\gamma_r = i\sigma_\varphi$, $\gamma_\varphi = -is\sigma_r$, with [48]

$$\begin{aligned} \alpha_r = \gamma_0 \gamma_r &= \begin{pmatrix} 0 & e^{-is\varphi} \\ e^{is\varphi} & 0 \end{pmatrix}, \\ \alpha_\varphi = \gamma_0 \gamma_\varphi &= \begin{pmatrix} 0 & -ie^{-is\varphi} \\ ie^{is\varphi} & 0 \end{pmatrix}, \\ \gamma_0 = \sigma_z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \quad (28)$$

We will attribute expressions to functions $V(r)$ and $S(r)$ in equation (26) in the next section, when we treat the analysis of spin and pseudospin symmetries. We will argue after that only some particular choices for these functions will lead to a differential equation that admits an exact solution.

3. Symmetry Limits

To implement the spin and pseudospin symmetry limits, we make in equation (26) the requirement that $S(r) = \pm V(r)$, where the plus (minus) signal refers to spin (pseudospin) symmetry, respectively [32]. Next, by using $\psi = (\psi_+, \psi_-)^T$, the first and second lines in equation (26) can be written in a simple form, which allows us to solve them separately. Furthermore, as mentioned above, we need to choose a representation for the radial function $V(r)$. We give a representation in terms of cylindrically symmetric scalar potentials which lead to results well-known in the literature.

3.1. Particle Interacting with a Harmonic Oscillator. It is well known that the harmonic oscillator can be used to describe a large number of physical systems. The interest in this issue appears in physical systems at an atomic scale and extends to high energy physics. It appears, for instance, as the interaction between atoms in the elastic crystal [50–52], as the effective potential acting on electrons moving in a uniform magnetic field (Landau levels) [49, 53–57] and in the quantization of the electromagnetic field [58]. The potential of the harmonic oscillator is included into equation (26) by making $V(r) = \pm S(r) = ar^2$, where a is a constant. One can check that the spin symmetry is sufficient to decouple the radial equation that comes from the upper spinor component while the pseudospin symmetry decouples the radial equation that comes from the lower component of the spinor. Thus, by adopting solutions of the form

$$\Psi_{\pm} = \begin{pmatrix} \sum_m f_+(r) e^{im\varphi} \\ i \sum_m f_-(r) e^{i(m+s)\varphi} \end{pmatrix}, \quad (29)$$

we arrive at radial equations

$$\frac{d^2 f_{\pm}(r)}{dr^2} + \left(\frac{1}{r} + \varepsilon ar \right) \frac{df_{\pm}(r)}{dr} - \frac{(m^{\pm})^2}{r^2} f_{\pm}(r) - (\Omega^{\pm})^2 r^2 f_{\pm}(r) + k^{\pm} f_{\pm}(r) = 0, \quad (30)$$

where $k^+ = E^2 - M^2 + 2(m+s)\omega - \varepsilon(2a + 3sma + 2M\omega s)$, $k^- = E^2 - M^2 + 2\omega(m+s) + 2\omega s - \varepsilon[2a - 3sa(m+s) + 2M\omega s]$, $[\Omega^+]^2 = \omega^2 + 2(M+E)a - \varepsilon\omega sa$, $[\Omega^-]^2 = \omega^2 + 2(E-M)a - \varepsilon\omega sa$, $\omega = eB/2$, $m^+ = m$, and $m^- = m + s$. It is convenient to write equation (30) in a known canonical form. This can be accomplished using $f_{\pm}(r)$ as

$$f_{\pm}(\rho) = e^{-(1/2)(\kappa^{\pm}+1)\rho} \rho^{(1/2)|m^{\pm}|} F_{\pm}(\rho), \quad \rho = \Omega^{\pm} r^2, \quad (31)$$

where $\kappa^{\pm} = \varepsilon a / 2\Omega^{\pm}$, which leads to the equation

$$\rho \frac{d^2 F_{\pm}}{d\rho^2} + (1 + |m^{\pm}| - \rho) \frac{dF_{\pm}}{d\rho} - \left[\frac{1}{2} (1 + |m^{\pm}| + \kappa^{\pm}) - \frac{k^{\pm}}{4\Omega^{\pm}} \right] F_{\pm} = 0. \quad (32)$$

Equation (32) is of the confluent hypergeometric equation type, and its solution is given in terms of the Kummer functions. In this manner, the general solution for equation (32) is given by [59]

$$\begin{aligned} f_{\pm}(\rho) = & c_1 e^{-(1/2)(1+\kappa^{\pm})\rho} \rho^{(1/2)|m^{\pm}|} \\ & \times M\left(\frac{1}{2}(1 + |m^{\pm}| + \kappa^{\pm}) - \frac{k^{\pm}}{4\Omega^{\pm}}, 1 + |m^{\pm}|, \rho\right) \\ & + c_2 e^{-(1/2)(1+\kappa^{\pm})\rho} \rho^{-(1/2)|m^{\pm}|} \\ & \times M\left(\frac{1}{2}(1 - |m^{\pm}| + \kappa^{\pm}) - \frac{k^{\pm}}{4\Omega^{\pm}}, 1 - |m^{\pm}|, \rho\right), \end{aligned} \quad (33)$$

where M are the Kummer functions. In particular, when $(1 + |m^{\pm}| + \kappa^{\pm})/2 - k^{\pm}/4\Omega^{\pm} = -n$, with $n = 0, 1, 2, \dots$, the function M becomes a polynomial in ρ of degree not exceeding n . From this condition, we extract the energies for the spin and pseudospin symmetry limits, given, respectively, by

$$E^2 - M^2 = 2\sqrt{\omega^2 + 2(M+E)a - \varepsilon\omega sa(2n + |m| + 1)} + \varepsilon(3a + 3sma + 2\omega Ms) - 2\omega(m + s), \quad (34)$$

$$E^2 - M^2 = 2\sqrt{\omega^2 + 2(E-M)a + \varepsilon\omega sa(2n + 1 + |m + s|)} - 2\omega(m + s) - 2\omega s - \varepsilon[a - 3sa(m + s) + 2M\omega s]. \quad (35)$$

Equations (34) and (35) are, respectively, the particle and antiparticle energies in the context of quantum deformation, and they can be read as a relativistic generalization of the Landau levels. It must be emphasized that, since ε and a are positive, the quantum deformation affects the separation of the energy levels of the system. This feature, however, should not depend on the value of the spin projection parameter s . Figure 1 shows energy profile (34) as a function of the frequency ω for some values of the quantum number m . In Figure 1(a), we plotted for $s = 1$ and in Figure 1(b) for $s = -1$. In this analysis as well as for the others, we use $\varepsilon = 10^{-7}$ [18, 29]. For this value, the effects of quantum deformation become more evident. We clearly see that both particle and antiparticle belong to the same energy spectrum. However, in Figure 1(b), we find that the antiparticle energy is not defined in the frequency ranges $0 < \omega < 1.37$ (for $m = 0$) and $0 < \omega < 2$ (for $m = -1$). These same characteristics are also present in the energy profile of equation (35) as shown in Figure 2. However, in Figure 2(b), the energies are not defined in the frequency ranges $0 < \omega < 3.35$ (for $m = -1$), $0 < \omega < 3.1$ (for $m = -1$), and $0 < \omega < 2.65$ (for $m = 1$). The appearance of such regions characterizing the absence of energy eigenvalues is due to the quantum deformation effects present in the model. On the other hand, when we compare the energy (34) (spin symmetry case) with the energy (7) of Ref. [28], we observe the absence of the interaction term between the Dirac oscillator and the angular momentum. Here, the frequency ω comes from the gauge field (25), which couples with the spin of the particle. Although these spectra are similar, physically, they are different. In fact, when $\varepsilon = 0$, we obtain

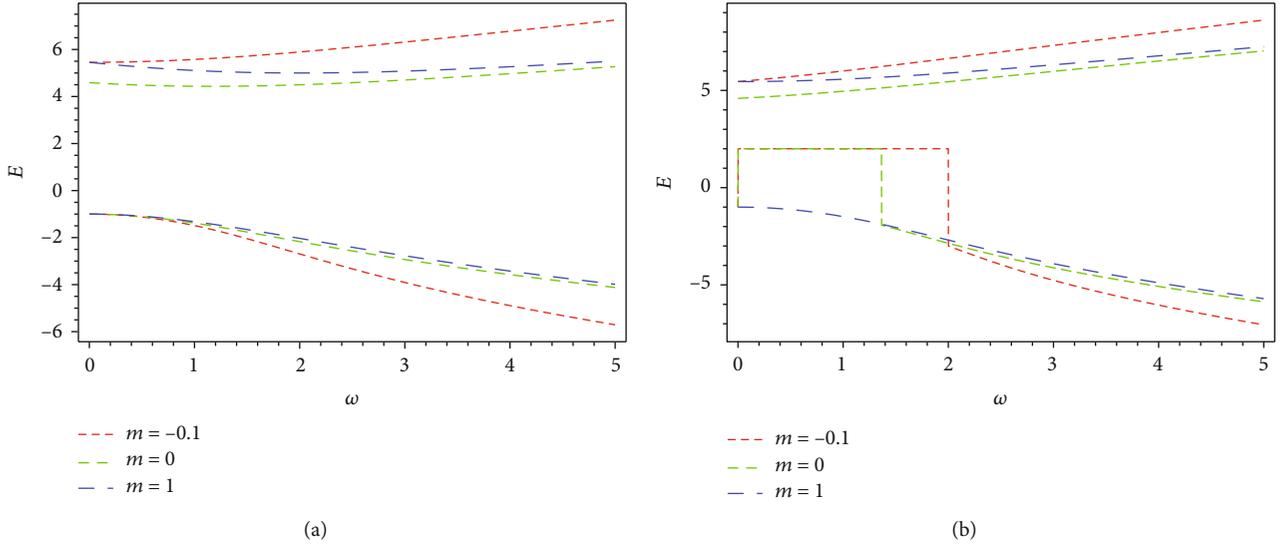


FIGURE 1: Illustration of the energy eigenvalues in the spin symmetry limit (equation (34)) as a function of the parameter ω for (a) $s = 1$ and (b) $s = -1$. We use units such as $M = 1$, $n = 1$, $a = 1$, and an upper bound $\varepsilon = 10^{-7}$.

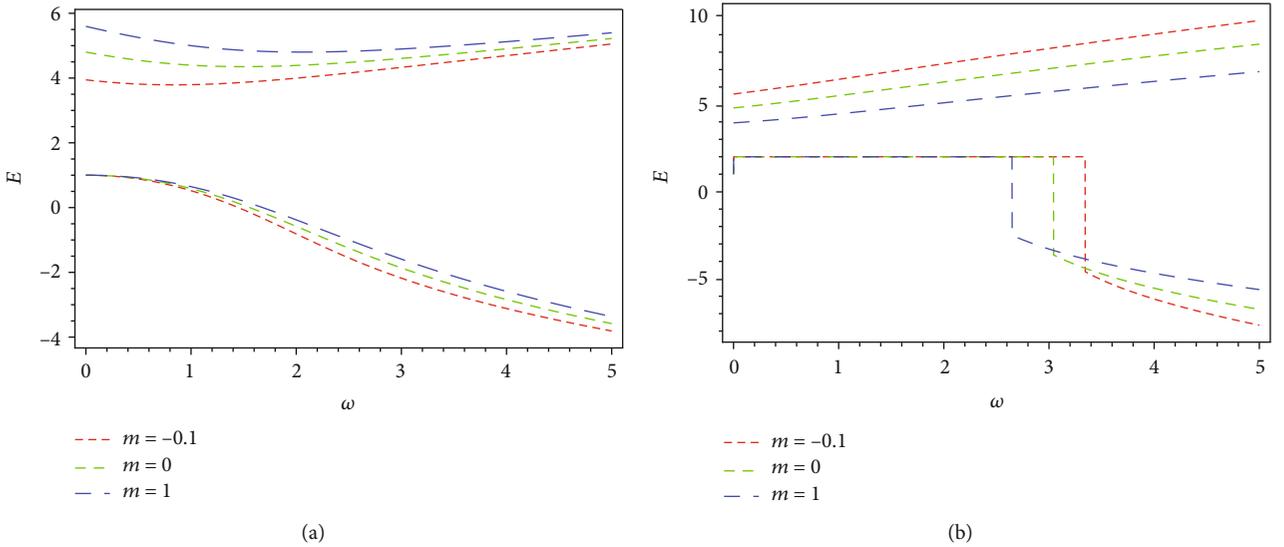


FIGURE 2: Illustration of the energy eigenvalues in the pseudospin symmetry limit (Eq. (35)) as a function of the parameter ω for (a) $s = 1$ and (b) $s = -1$. We use units such as $M = 1$, $n = 1$, $a = 1$ and an upper bound $\varepsilon = 10^{-7}$.

the energy eigenvalues of Ref. [48] (after removing the magnetic flux parameter), which shows the consistency of the model in question. In particular, when a and ε are null, we obtain

$$\begin{aligned} E^2 - M^2 &= 2\omega[2n + 1 + |m| - m - s], \\ E^2 - M^2 &= 2\omega[2n + 1 + |m + s| - (m + s) - s], \end{aligned} \quad (36)$$

which are the usual relativistic Landau levels with the inclusion of the element of spin.

3.2. Particle Interacting with a Linear Potential. Let us consider the case where the particle interacts with a linear potential, *br.* As discussed above, linear interactions are included in the relativistic motion equations through the quadripotential $A_{\mu\nu}$. In the case of the Dirac equation, depending on the configuration of external potentials involved, it can be solved exactly. An example of an exactly soluble model described by a linear interaction in relativistic quantum mechanics is the Dirac oscillator [60]. It is known that the Hamiltonian of the Dirac oscillator describes the interaction of a neutral particle with an electric field via an anomalous magnetic coupling [61]. In the

context of confinement of quarks in mesons and baryons, it is well established that electrostatic linear potentials lead to the occurrence of the Klein paradox and particles cannot be confined [62]. To establish such confinement, we must invoke the scalar substitution (17). In this case, we make $V(r) = \pm S(r) = br$ (where b is a constant) in equation (26) to the limits of spin and pseudospin symmetries and proceed as before. The resulting equation is given by

$$\left[\frac{d^2}{dr^2} + \left(\frac{1}{r} + \frac{\varepsilon b}{2} \right) \frac{d}{dr} - \frac{(m^\pm)^2}{r^2} - \omega^2 r^2 - \mu^\pm r \right] f_\pm(r) - \left(\frac{k^\pm}{r} - l^\pm \right) f_\pm(r) = 0, \quad (37)$$

with $m^+ = m$, $m^- = m + s$, $\mu^+ = 2(E + M)b + 3\varepsilon\omega bs/2$, $\mu^- = 2(E - M)b - 3\varepsilon\omega bs/2$, $k^+ = eb(1 + 3sm)/2$, $k^- = eb[1 - 3s(m + s)]/2$, $l^+ = E^2 - M^2 + 2\omega(m + s) - 2M\varepsilon\omega s$, and $l^- = E^2 - M^2 + 2\omega(m + s) + 2\omega s(1 - \varepsilon M)$. In equation (37), the $+$ ($-$) signs refer to spin and pseudospin symmetries, respectively. By performing the variable change, $x = \sqrt{\omega}r$, equation (37) assumes the form

$$\left[\frac{d^2}{dx^2} + \left(\frac{1}{x} + \frac{\kappa}{2} \right) \frac{d}{dx} - \frac{(m^\pm)^2}{x^2} - x^2 - a_L^\pm x \right] f_\pm(x) - \left(\frac{a_C^\pm}{x} - \frac{l^\pm}{\omega} \right) f_\pm(x) = 0, \quad (38)$$

where we have defined the parameters $\kappa = \varepsilon b/\sqrt{\omega}$, $a_L^\pm = \mu^\pm/\omega\sqrt{\omega}$, and $a_C^\pm = k^\pm/\sqrt{\omega}$. Note that the choice $V(r) = br$ induces a Coulomb-like interaction in the resulting deformed sector of the eigenvalue equation. The origin of this Coulomb potential is due purely to the quantum deformation and boundary symmetries involved.

Equation (38) is of the Heun equation type, which is a homogeneous, linear, second-order, differential equation defined in the complex plane [63]. This equation can be put into its canonical form using the solution

$$f_\pm(x) = x^{|m^\pm|} e^{-(1/2)x^2} e^{-(1/2)(a(+L)+(1/2)\kappa)x} y_\pm(x), \quad (39)$$

where y_\pm satisfies the biconfluent Heun differential equation

$$y_\pm'' + \left(\frac{\alpha^\pm + 1}{x} - 2x - \beta^\pm \right) y_\pm' + (\gamma^\pm - \alpha^\pm - 2) y_\pm - \frac{1}{2x} [\beta^\pm (\alpha^\pm + 1) + \delta^\pm] y_\pm = 0, \quad (40)$$

with $\alpha^\pm = 2|m^\pm|$, $\beta^\pm = a_L^\pm$, $\gamma^\pm = (\beta^\pm)^2/4 + l^\pm/\omega$, and $\delta^\pm = \kappa/2 + 2a_C^\pm$. Equation (40) has a regular singularity at $x = 0$ and an irregular singularity at ∞ of rank 2. Usually, the solution of this equation is given in terms of two linearly independent solutions as

$$y_\pm(x) = N(\alpha^\pm, \beta^\pm, \gamma^\pm, \delta^\pm; x) + x^{-\alpha^\pm} N(-\alpha^\pm, \beta^\pm, \gamma^\pm, \delta^\pm; x), \quad (41)$$

where (assuming that α^\pm is not a negative integer)

$$N(\alpha^\pm, \beta^\pm, \gamma^\pm, \delta^\pm; x) = \sum_{q=0}^{\infty} \frac{\mathcal{A}_q^\pm(\alpha^\pm, \beta^\pm, \gamma^\pm, \delta^\pm) x^q}{(1 + \alpha^\pm)_q q!} \quad (42)$$

are the Heun functions. After the insertion of this solution into equation (40), we find ($q \geq 0$)

$$\mathcal{A}_0^\pm = 1, \quad (43)$$

$$\mathcal{A}_1^\pm = \frac{1}{2} [\delta^\pm + \beta^\pm (1 + \alpha^\pm)], \quad (44)$$

$$\mathcal{A}_{q+2}^\pm = \left\{ (q+1)\beta^\pm + \frac{1}{2} [\delta^\pm + \beta^\pm (1 + \alpha^\pm)] \right\} \mathcal{A}_{q+1}^\pm - (q+1)(q+1 + \alpha^\pm) [\gamma^\pm - \alpha^\pm - 2 - 2q] \mathcal{A}_q^\pm, \quad (45)$$

where

$$(1 + \alpha^\pm)_q = \frac{\Gamma(q + \alpha^\pm + 1)}{\Gamma(\alpha^\pm + 1)}, \quad q = 0, 1, 2, 3, \dots \quad (46)$$

From the recursion relation (45), the function $N(\alpha^\pm, \beta^\pm, \gamma^\pm, \delta^\pm; x)$ becomes a polynomial of degree n if and only if the two following conditions are imposed [63]:

$$\gamma^\pm - \alpha^\pm - 2 = 2n, \quad n = 0, 1, 2, \dots, \quad (47)$$

$$\mathcal{A}_{n+1}^\pm = 0, \quad (48)$$

where n is a positive integer. In this case, the $(n+1)$ th coefficient in the series expansion is a polynomial of degree n in δ^\pm . When δ^\pm is a root of this polynomial, the $(n+1)$ th and subsequent coefficients cancel and the series truncates, resulting in a polynomial form of degree n for $N(\alpha^\pm, \beta^\pm, \gamma^\pm, \delta^\pm; x)$. From condition (47), we extract the energies at the spin and pseudosymmetry limits, given, respectively, by

$$\begin{aligned} E_{nm}^2 - M^2 &= 2\omega(n + |m| + 1) - \frac{b^2}{\omega^2} (E_{nm} + M)^2 \\ &\quad - \frac{3b^2}{2\omega} (E_{nm} + M)\varepsilon s + 2\omega[\varepsilon Ms - (m + s)], \\ E_{nm}^2 - M^2 &= 2\omega(n + |m + s| + 1) - \frac{b^2}{\omega^2} (E_{nm} - M)^2 \\ &\quad + \frac{3b^2}{2\omega} (E_{nm} - M)\varepsilon s - 2\omega[s(1 - \varepsilon M) + (m + s)]. \end{aligned} \quad (49)$$

The energy of a physical system must be a function involving all the parameters present in the equation of motion. In equation (49), the parameter a_C^\pm is absent. However, it can be restored using condition (48) from which we

obtain a relation between such parameter and the frequency of the system. For each value of n fixed, we have a self-energy and its corresponding wave function. Let us consider solution (42) up to second order in x of the expansion,

$$N(\alpha^\pm, \beta^\pm, \gamma^\pm, \delta^\pm; x) = \frac{\mathcal{A}_0}{(1 + \alpha^\pm)_0} + \frac{\mathcal{A}_1^\pm}{(1 + \alpha^\pm)_1} x + \frac{\mathcal{A}_2^\pm}{(1 + \alpha^\pm)_2} \frac{x^2}{2!} + \dots \quad (50)$$

If we truncate the series in a term of order x^n , the resulting finite series is related, via solution (39), to the energy level E_n . Thus, the physical quantity that we can associate most closely with the series (50) truncated in the term x^n is the energy E_n of the particle with the wave function (39). In fact, it is necessary that the series (50) becomes an n -degree polynomial for the system to admit bound states. Thus, by using relation (45) and equations (43) and (44), the coefficient above \mathcal{A}_2^\pm can be determined. If we want to truncate solution (50) in x , we must impose that $A_1^\pm = 0$ through condition (48); when we truncate in x^2 , we make $A_2^\pm = 0$, and so on. For each of these cases, we can establish appropriate constraints between conditions (47) and (48). Since equation (49) is of the relativistic Landau level type, we prefer to fix the frequency ω in order to obtain an expression for the energies corresponding to each value of n [41, 64–70]. Since it is done for both spin and pseudospin symmetries, we also label the frequency ω as ω_n^\pm , with the subscript n characterizing the values to be fixed. After obtaining ω_n^\pm and replacing them in equation (49), we will have expressions for the energies involving the quantities α^\pm , β^\pm , γ^\pm , and δ^\pm , which contains the coupling constant a_C^\pm , the mass of the particle M , and the quantum angular momentum number m^\pm . Thus, for $A_1^\pm = 0$, it means that we are investigating the particular solution for $n=0$. In this case, from equation (44), we have

$$\frac{1}{2}(\delta^\pm + \beta^\pm \tilde{m}^\pm) = 0, \quad (51)$$

where $\tilde{m}^\pm = 1 + \alpha^\pm$. Solving (51) for ω_0^\pm , we get the relations

$$\omega_0^+ = -\frac{2(E_{0m}^+ + M)(1 + 2|m|)}{\varepsilon[3s(|m| + m) + 3/2(s + 1)]}, \quad (52)$$

$$\omega_0^- = \frac{2(E_{0m}^- - M)(1 + 2|m + s|)}{\varepsilon[3s(|m + s| + m + s) + (3/2)(s - 1)]}.$$

Proceeding in a similar way to $\mathcal{A}_2^\pm = 0$, we find the following third-degree polynomial in ω_1^\pm ($n = 1$):

$$\mathcal{A}^\pm [\omega_1^\pm]^3 + \mathcal{B}^\pm [\omega_1^\pm]^2 + \mathcal{C}^\pm [\omega_1^\pm] + \mathcal{D}^\pm = 0, \quad (53)$$

with

$$\begin{aligned} \mathcal{A}^+ &= 2(1 + 2|m|), \\ \mathcal{A}^- &= 2(1 + 2|m + s|), \\ \mathcal{B}^+ &= \mathcal{B}^- = 0, \\ \mathcal{C}^+ &= -\frac{3}{2}\varepsilon b^2(E_{1m}^+ + M)[(2|m| + 1)s + C^+], \\ \mathcal{C}^- &= \frac{3}{2}\varepsilon b^2(E_{1m}^- - M)[s(2|m + s| + 1) + C^-], \\ \mathcal{D}^+ &= -b^2(E_{1m}^+ + M)^2(1 + 2|m|)(3 + 2|m|), \\ \mathcal{D}^- &= -b^2(E_{1m}^- - M)^2(1 + 2|m + s|)(3 + 2|m + s|), \end{aligned} \quad (54)$$

where $C^+ = 2(1 + |m|)(2s|m| + 2ms + s + 1)$ and $C^- = 2(|m + s| + 1)(2s(|m + s| + m) + s + 1)$. Equation (53) has only one real root. For each specific frequency, ω_0^\pm and ω_1^\pm , we can determine the energies E_{0m}^\pm and E_{1m}^\pm by the following expressions:

$$\begin{aligned} (E_{0m}^+)^2 - M^2 &= 2\omega_0^+(|m| + 1) - \frac{b^2}{(\omega_0^+)^2}(E_{0m}^+ + M)^2 \\ &\quad - \frac{3b^2}{2\omega_{0m}^+}(E_0^+ + M)\varepsilon s + 2\omega_0^+[\varepsilon Ms - (m + s)], \end{aligned} \quad (55)$$

$$\begin{aligned} (E_{0m}^-)^2 - M^2 &= 2\omega_0^-(|m + s| + 1) - \frac{b^2}{(\omega_0^-)^2}(E_{0m}^- - M)^2 \\ &\quad + \frac{3b^2}{2\omega_0^-}(E_{0m}^- - M)\varepsilon s \\ &\quad - 2\omega_0^-[s(1 - \varepsilon M) + (m + s)], \end{aligned} \quad (56)$$

$$\begin{aligned} (E_{1m}^+)^2 - M^2 &= 2\omega_1^+(|m| + 2) - \frac{b^2}{(\omega_1^+)^2}(E_{1m}^+ + M)^2 \\ &\quad - \frac{3b^2}{2\omega_1^+}(E_{1m}^+ + M)\varepsilon s + 2\omega_1^+[\varepsilon Ms - (m + s)], \end{aligned} \quad (57)$$

$$\begin{aligned} (E_{1m}^-)^2 - M^2 &= 2\omega_1^- (|m + s| + 2) - \frac{b^2}{(\omega_1^-)^2}(E_{1m}^- - M)^2 \\ &\quad + \frac{3b^2}{2\omega_1^-}(E_{1m}^- - M)\varepsilon s \\ &\quad - 2\omega_1^- [s(1 - \varepsilon M) + (m + s)], \end{aligned} \quad (58)$$

where ω_1^\pm is given by

$$\omega_1^\pm = \sqrt[3]{\frac{\Delta^\pm}{18} \frac{1}{\mathcal{A}^\pm}} - \sqrt[3]{\frac{2}{3\Delta^\pm} \mathcal{C}^\pm}, \quad (59)$$

with

$$\Delta^\pm = \sqrt{3} \sqrt{27[\mathcal{A}^\pm]^4 [\mathcal{D}^\pm]^2 + 4[\mathcal{A}^\pm]^3 [\mathcal{C}^\pm]^3 - 9[\mathcal{A}^\pm]^2 \mathcal{D}^\pm}, \quad (60)$$

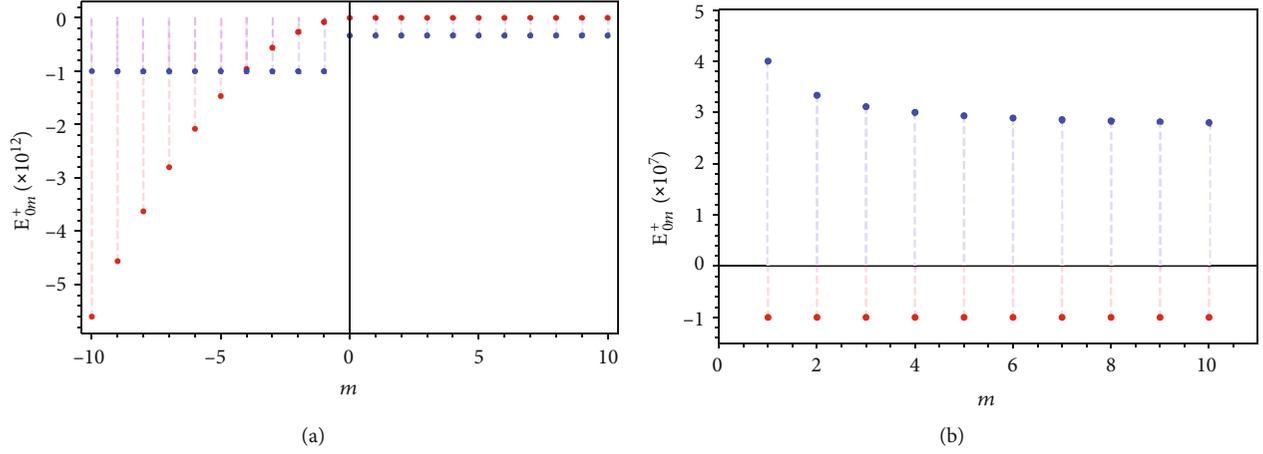


FIGURE 3: Illustration of the energy eigenvalues in the spin symmetry limit (equation (55)) as a function of the parameter m for (a) $s = 1$ and (b) $s = -1$. We use units such as $M = 1$, $n = 1$, $b = 1$, and $\varepsilon = 10^{-7}$.

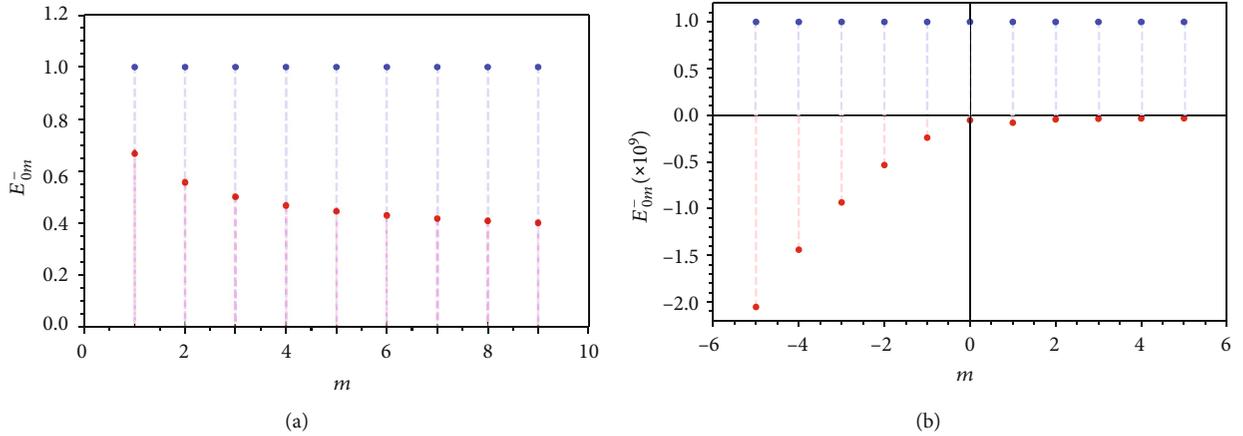


FIGURE 4: Illustration of the energy eigenvalues in the pseudospin symmetry limit (equation (56)) as a function of the parameter m for (a) $s = 1$ and (b) $s = -1$. We use units such as $M = 1$, $n = 1$, $b = 1$, and $\varepsilon = 10^{-7}$.

and the requirement that $27[\mathcal{A}^\pm]^4[\mathcal{D}^\pm]^2 + 4[\mathcal{A}^\pm]^3[\mathcal{E}^\pm]^3 > 0$. Such energies represent the first two energy levels of the system. For simplicity, we perform a numerical analysis only for the case $n=0$. In this way, by studying equation (55), we find that for $s=1$ and $m > 0$, the energy of the particle is infinitely degenerate, with respective eigenvalues $E_{0m}^+ = -1$ and $E_{0m}^+ = -0.33$, whereas for $m < 0$, only one of the roots is infinitely degenerate with energy $E_{0m}^+ = -1$ (Figure 3(a)). On the other hand, when we analyze equation (55) for $s=-1$, we verify that the energy spectrum is defined only for $m > 0$, and one of the roots is infinitely degenerate with eigenvalue $E_{0m}^+ = -1$ (Figure 3(b)). These characteristics are also present in the energies from equation (56). For $s=1$, the energies are defined only for $m > 0$ and one of the roots is infinitely degenerate, with eigenvalue $E_{0m}^- = 1$ (Figure 4(a)). For $s=-1$ and $m < 0$, there is an infinitely degenerate root with eigenvalue $E_{0m}^- = 1$ while for $m > 0$ both roots are infinitely degenerate with respective energies $E_{0m}^- = 1$ and $E_{0m}^- = -2.7 \times 10^7$ (Figure 4(b)).

To determine the energies corresponding to $n = 2, 3, 4, \dots$, we must make use of the above recipe. However, as we can

see from equation (53), the polynomials of degree $n \geq 3$ resulting from condition (48), in general, not all roots are physically acceptable. In the absence of deformation, the resulting equation describes the motion of an electron subject to interactions (15), (16), and (17). In the literature, models in this context can be found. With the presence of these interactions in the Dirac equation, the wave function depends on several parameters. Thus, if we consider a null deformation, a direct comparison with particular cases in the literature is not immediate. The reader can see some examples in Refs. [41, 71–73].

4. Conclusions

We have studied the relativistic quantum dynamics of a spin 1/2 charged particle with minimal and scalar couplings in the quantum deformed framework generated by the κ -Poincaré-Hopf algebra. The problem has been formulated using the κ -deformed Dirac equation in two dimensions. The κ -deformed Pauli equation was derived to study the dynamics of the system taking into account

the spin and pseudospin symmetry limits. For the κ -deformed Dirac-Pauli equation obtained (equation (26)), we have argued that only particular choices of radial function $V(r)$ lead to exactly solvable differential equations. We have considered the case where the particle interacts with a uniform magnetic field, a planar harmonic oscillator, and a linear potential. We have verified that the linear potential leads to a Coulomb-type term in the κ -deformed sector of the radial equation. The resulting equation obtained is a Heun-type differential equation. Analytical solutions for both spin and pseudospin symmetry limits enabled us to obtain expressions for the energy eigenvalues (through the use of equations (47) and (48)) and wave functions. Because of the limitations imposed by condition (48), we have derived expressions for the energies corresponding only to $n=0$ (equations (55) and (56)) and $n=1$ (equations (57) and (58)). We have shown that the energy eigenvalues and wave functions are modified by both the spin element s and the deformation parameter ε . We believe that future experiments may provide some estimate on the magnitude of the deformation parameter within the context of the model studied.

Data Availability

We did not use any data in this paper.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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