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Review Article

Time as a Quantum Observable, Canonically Conjugated to Energy, and Foundations of Self-Consistent Time Analysis of Quantum Processes

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Recent developments are reviewed and some new results are presented in the study of *time* in quantum mechanics and quantum electrodynamics as an observable, canonically conjugate to *energy*. This paper deals with the maximal Hermitian (but nonself-adjoint) operator for time which appears in nonrelativistic quantum mechanics and in quantum electrodynamics for systems with continuous energy spectra and also, briefly, with the four-momentum and four-position operators, for relativistic spin-zero particles. Two measures of averaging over time and connection between them are analyzed. The results of the study of time as a quantum observable in the cases of the discrete energy spectra are also presented, and in this case the quasi-self-adjoint time operator appears. Then, the general foundations of time analysis of quantum processes (collisions and decays) are developed on the base of time operator with the proper measures of averaging over time. Finally, some applications of time analysis of quantum processes (concretely, tunneling phenomena and nuclear processes) are reviewed.

1. General Introduction

During almost ninety years (e.g., [1, 2]), it is known that time cannot be represented by a self-adjoint operator, with the possible exception of special abstract systems (such as an electrically charged particle in an infinite uniform electric field) and a system with the limited from both below and above energy spectrum (to see later)). (Namely that fact that time cannot be represented by a self-adjoint operator is known to follow from the semiboundedness of the continuous energy spectra, which are bounded from below (usually by the value zero). Only for an electrically charged particle in an infinite uniform electric field, and for other very rare special systems, the continuous energy spectrum is not bounded and

extends over the whole energy axis from $-\infty$ to ∞.) This fact results to be in contrast with the known sircumstance that time, as well as space, in some cases plays the role just of a parameter, while in some other cases *is* a physical observable which *ought* to be represented by an operator. The list of papers devoted to the problem of time in quantum mechanics is extremely large (e.g., [3–51], and references therein). The same situation had to be faced also in quantum electrodynamics and, more in general, in relativistic quantum field theory (e.g., [12–14, 47, 50, 51]).

As to quantum mechanics, the first set of known and cited articles is [3–21]. The second set of papers on time as an observable in quantum physics [22–51] appeared from the end of the eighties and chiefly in the nineties and more recently, stimulated mainly by the need of a self-consistent definition for collision duration and tunneling time. It is noticeable that many of this second set of papers appeared however to ignore the Naimark theorem from [52], which had previously constituted an important basis for the results in [15–21]. This Naimark theorem states [52] that the nonorthogonal spectral decomposition $E(\lambda)$ of a Hermitian operator H is of the Carleman type (which is unique for the maximal Hermitian operator), that is, it *can be approximated* by a succession of the self-adjoint operators, the spectral functions of which do weakly converge to the spectral function $E(\lambda)$ of the operator H.

Namely, by exploiting that Naimark theorem, it has been shown by Olkhovsky and Recami [15–18, 21] (more details having been added in [22–27, 32–35, 47, 50, 51]) and, independently, by Holevo [19, 20] that, for systems with continuous energy spectra, time can be introduced as a quantum-mechanical observable, canonically conjugate to energy. More precisely, the time operator resulted to be maximal Hermitian, even if not self-adjoint. Then, in [23–25, 33–35, 50, 51] it was clarified that time can be introduced also for these systems as a quantum-mechanical observable, canonically conjugate to energy, and the time operator resulted to be quasi-self-adjoint (more precisely, it can be chosen as an almost self-adjoint operator with practically almost any degree of the accuracy).

We intend to justify the association of *time* with a quantum-physical observable, by exploiting the properties of the maximal Hermitian operators in the case of the continuous energy spectra, and the properties of quasi-self-adjoint operators in the case of the discrete energy spectra.

Then, we analyze the restricted sense of the positive operator value measure (POVM) approach, often used now (see, in particular [28–31, 36–46, 48, 49]). Finally, we do in a shorten way review our methods of time analysis and joint time-energy analysis which had already proved to be fruitful in tunnelling and nuclear processes.

2. Time as a Quantum Observable and General Definitions of Mean Times and Mean Durations of Quantum Processes

2.1. On Time as an Observable in Nonrelativistic Quantum Mechanics, for Systems with Continuous Energy Spectra

For systems with continuous energy spectra, the following simple operator, canonically conjugate to energy, can be introduced for time

$$\hat{t} = t, \quad \text{in the (time) } t\text{-representation}, \tag{2.1a}$$

$$\hat{t} = -i\hbar\frac{\partial}{\partial E}, \quad \text{in the (energy) } E\text{-representation}, \tag{2.1b}$$

which is not self-adjoint, but is *Hermitian*, and acts on square-integrable space-time wave packets in representation (2.1a), and on their Fourier transforms in representation (2.1b), once the point $E = 0$ is eliminated (i.e., once one deals only with moving packets, i.e., excludes any *nonmoving* back tails, as well as, of course, the zero flux cases). (Such a condition is enough for operator (2.1a) and (2.1b) to be a "*maximal Hermitian*" (or "*maximal symmetric*") operator [15–18, 21] (see also [26, 27, 33–35, 52, 53]), according to Akhiezer & Glazman's terminology. Let us explicitly notice that, anyway, the physically reasonable boundary condition $E \neq 0$ can be dispensed with, by having recourse to bilinear operators, as it is simply shown below in the form (2.26) and Appendix A.) It has been shown already in [15–18, 21]. The elimination of the point $E = 0$ is not restrictive since the "rest" states with the zero velocity, the wave packets with *nonmoving* rear tails, and the wave packets with zero flux are unobservable.

Operator (2.1b) is defined as acting on the space P of the continuous, differentiable, square-integrable functions $f(E)$ that satisfy the conditions

$$\int_0^\infty |f(E)|^2 dE < \infty, \qquad \int_0^\infty \left|\frac{\partial f(E)}{\partial E}\right|^2 dE < \infty, \qquad \int_0^\infty |f(E)|^2 E^2 dE < \infty, \tag{2.2}$$

and the condition

$$f(0) = 0, \tag{2.3}$$

which is a space P *dense* in the Hilbert space of L^2 functions defined (only) over the semiaxis $0 \leq E < \infty$. Obviously, the operator (2.1a) and (2.1b) is Hermitian, that is, the relation $(f_1, \hat{t} f_2) = ((\hat{t} f_1), f_2)$ holds, only if all square-integrable functions $f(E)$ in the space on which it is defined vanish for $E = 0$.

Also the operator $\hat{t}^2$ is Hermitian, that is, the relation $(f_1, \hat{t}^2 f_2) = ((\hat{t} f_1), (\hat{t} f_2)) = (\hat{t}^2 f_1, f_2)$ holds under the same conditions.

Operator $\hat{t}$ has no Hermitian extension because otherwise one could find at least one function $f_0(E)$ which satisfies the condition $f_0(0) \neq 0$ but that is inconsistent with the propriety of being Hermitian. So, according to [53], $\hat{t}$ is a maximal Hermitian operator and in accordance with the results of the mathematical theory of operators it is not a self-adjoint operator with equal deficiency indices but it has the deficiency indices (0,1). As a consequence, operator (2.1b) does not allow a unique orthogonal identity resolution.

Essentially because of these reasons, earlier Pauli (e.g., [1, 2]) rejected the use of a time operator; this had the result of practically stopping studies on this subject for about forty years. However, as far back as in [54] von Neumann had claimed that considering in quantum mechanics only self-adjoint operators could be too restrictive. To clarify this issue, let us quote an explanatory example set forth by von Neumann himself [54]: let us consider a particle, free to move in a spatial semiaxis ($0 \leq x < \infty$) bounded by a rigid wall located at $x = 0$. Consequently, the operator for the momentum x-component of the particle, which reads

$$\hat{p}_x = -i\hbar \frac{\partial}{\partial x} \tag{2.4}$$

is defined as acting on the space of the continuous, differentiable, square-integrable functions $f(x)$ that satisfy the conditions

$$\int_0^\infty |f(x)|^2 dx < \infty, \qquad \int_0^\infty \left|\frac{\partial f(x)}{\partial x}\right|^2 dx < \infty, \qquad \int_0^\infty |f(x)|^2 x^2\, dx < \infty, \tag{2.5}$$

and the condition

$$f(0) = 0, \tag{2.6}$$

which is a space *dense* Q in the Hilbert space of L^2 functions defined (only) over the spatial semiaxis $0 \leq x < \infty$. Therefore, operator $\widehat{p}_x = -i\hbar(\partial/\partial x)$ has the same mathematical properties as operator $\widehat{t}$ (2.1a) and (2.1b) and consequently it is not a self-adjoint operator but it is only a maximal Hermitian operator. Nevertheless, it is an observable with an obvious physical meaning. The same properties has also the radial momentum operator $\widehat{p}_r = -i\hbar(\partial/\partial r) + (1/r)$ $(0 < r < \infty)$.

By the way, one can easily demonstrate (e.g., [4, 5]) that in the case of (hypotetical) quantum-mechanical systems with the continuous energy spectra *bounded from below and from above* ($E_{\min} < E < E_{\max}$) the time operator (2.1a) and (2.1b) becomes a *really self-adjoint operator* and has a discrete time spectrum, with the "the time quantum" $\tau = \hbar/d$ where $d = |E_{\max} - E_{\min}|$.

In order to consider *time as an observable in quantum mechanics* and to define the observable mean times and durations, one needs to introduce not only the time operator, but also, in a self-consistent way, the measure (or weight) of averaging over time. In the simple one-dimensional (1D) and one-directional motion such *measure* (weight) can be obtained by the the simple quantity:

$$W(x,t)dt = \frac{j(x,t)dt}{\int_{-\infty}^{\infty} j(x,t)dt}, \tag{2.7}$$

where the probabilistic interpretation of $j(x,t)$ (namely in *time*) corresponds to the flux probability density of a particle passing through point x at time t (more precisely, passing through x during a unit time interval centered at t), when travelling in the positive x-direction. Such a measure had not been postulated, but is just a direct consequence of the well-known probabilistic (*spatial*) interpretation of $\rho(x,t)$ and of the continuity relation

$$\frac{\partial \rho(x,t)}{\partial t} + \operatorname{div} j(x,t) = 0 \tag{2.8}$$

for particle motion in the field of any hamiltonian in the desciption of the 1D Schroedinger equation. (The three-dimensional (3D) case is described in Appendix B.) Quantity $\rho(x,t)$ is the probability of finding a moving particle inside a unit space interval, centered at point x, at time t. The probability density $\rho(x,t)$ and the flux probability density $j(x,t)$ are related with the wave function $\Psi(x,t)$ by the usual definitions $\rho(x,t) = |\Psi(x,t)|^2$ and $j(x,t) = \operatorname{Re}[\Psi^*(x,t)(\hbar/i\mu)\partial\Psi(x,t)/\partial x]$. The measure (2.7) was firstly investigated in [21, 23–27, 32–35].

When the flux density $j(x,t)$ changes its sign, the quantity $W(x,t)dt$ is no longer positive definite and it acquires a physical meaning of a probability density *only* during those partial time intervals in which the flux density $j(x,t)$ does keep its sign. Therefore, let us introduce the two measures, by separating the positive and the negative flux-direction values (i.e., flux signs):

$$W_{\pm}(x,t)dt = \frac{j_{\pm}(x,t)dt}{\int_{-\infty}^{\infty} j_{\pm}(x,t)dt}, \tag{2.9}$$

with $j_{\pm}(x,t) = j(x,t)\Theta(\pm j)$ where $\Theta(z)$ is the Heaviside step function. It had been made firstly in [26, 27, 32–35]. Actually, one can rewrite the continuity relation (2.8) for those time intervals, for which $j = j_{+}$ or $j = j_{-}$ as follows:

$$\frac{\partial \rho_{>}(x,t)}{\partial t} = -\frac{\partial j_{+}(x,t)}{\partial x}, \qquad \frac{\partial \rho_{<}(x,t)}{\partial t} = -\frac{\partial j_{-}(x,t)}{\partial x}, \tag{2.10}$$

respectively. Relations in (2.10) can be considered as formal definitions of $\partial\rho_{>}/\partial t$ and $\partial\rho_{<}/\partial t$. Integrating them over time t from $-\infty$ to t, one obtains

$$\rho_{>}(x,t) = -\int_{-\infty}^{t} \frac{\partial j_{+}(x,t')}{\partial t'}dt', \qquad \rho_{<}(x,t) = -\int_{-\infty}^{t} \frac{\partial j_{-}(x,t')}{\partial t'}dt' \tag{2.11}$$

with the initial conditions $\rho_{>}(x,-\infty) = \rho_{<}(x,-\infty) = 0$. Then, it is possible to introduce the quantities

$$\begin{aligned} N_{>}(x,\infty;t) &\equiv \int_{x}^{\infty} \rho_{>}(x',t)dx' = \int_{-\infty}^{t} j_{+}(x,t')dt' > 0, \\ N_{<}(-\infty,x;t) &\equiv \int_{-\infty}^{x} \rho_{<}(x',t)dx' = -\int_{-\infty}^{t} j_{-}(x,t')dt' > 0, \end{aligned} \tag{2.12}$$

which have the meaning of probabilities for the particle wave packet $\Psi(x,t)$ to be located at time t on the semiaxis (x,∞) and $(-\infty,x)$, respectively, as functions of the flux densities $j_{+}(-\infty,t)$ and $j_{-}(x,t)$, provided that the normalization condition $\int_{-\infty}^{\infty}\rho(x,t)dx = 1$ is fulfilled. The right-hand parts of the last couple of equations have been obtained by integrating the rigt-hand parts of the expressions for $\rho_{>}(x,t)$ and $\rho_{<}(x,t)$, and by adopting the boundary conditions $j_{+}(-\infty,t) = j_{-}(-\infty,t) = 0$. Then, by differentiating $N_{>}(x,\infty;t)$ and $N_{<}(-\infty,x;t)$ with respect to t, one obtains

$$\frac{\partial N_{>}(x,\infty;t)}{\partial t} = j_{+}(x,t) > 0, \qquad \frac{\partial N_{<}(-\infty,x;t)}{\partial t} = -j_{-}(x,t) > 0. \tag{2.13}$$

Finally, from the last four equations one can easily infer that

$$W_+(x,t)dt = \frac{j_+(x,t)dt}{\int_{-\infty}^{\infty} j_+(x,t)dt} = \frac{\partial N_>(x,\infty;t)/\partial t}{N_>(x,\infty;\infty)},$$
$$W_-(x,t)dt = \frac{j_-(x,t)dt}{\int_{-\infty}^{\infty} j_-(x,t)dt} = \frac{\partial N_<(-\infty,x;t)/\partial t}{N_<(-\infty,x;\infty)}, \tag{2.14}$$

which justify the abovementined probabilistic interpretation of $W_\pm(x,t)$. Let us stress now that this approach does not assume any new physical postulate in the conventional (Copenhagen-interpretation) nonrelativistic quantum mechanics.

Then, one can eventually define the mean value $\langle t(x)\rangle$ of the time t at which a particle passes through position x (when travelling in only one positive x-direction), and $\langle t_\pm(x)\rangle$ of the time t at which a particle passes through position x, when travelling in the positive or negative direction, respectively,

$$\langle t(x)\rangle = \frac{\int_{-\infty}^{\infty} t j(x,t)dt}{\int_{-\infty}^{\infty} j(x,t)dt} = \frac{\int_0^\infty dE(1/2)\left[G^*(x,E)\hat{t}vG(x,E) + vG^*(x,E)\hat{t}G(x,E)\right]}{\int_0^\infty dE\, v\left|G(x,E)\right|^2}, \tag{2.15}$$

where $G(x,E)$ is the Fourier transform of the moving 1D wave packet

$$\Psi(x,t) = \int_0^\infty G(x,E)\exp\left(-\frac{iEt}{\hbar}\right)dE = \int_0^\infty g(E)\varphi(x,E)\exp\left(-\frac{iEt}{\hbar}\right)dE, \tag{2.16}$$

when going on from the time representation to the energy one,

$$\langle t_\pm(x)\rangle = \frac{\int_{-\infty}^{\infty} t j_\pm(x,t)dt}{\int_{-\infty}^{\infty} j_\pm(x,t)dt}, \tag{2.17}$$

and also the mean durations of particle 1D transmission from x_i to $x_f > x_i$ and 1D particle reflection from the region (x_i,∞) into $x_f \le x_i$:

$$\langle \tau_T(x_i,x_f)\rangle = \langle t_+(x_f)\rangle - \langle t_+(x_i)\rangle,$$
$$\langle \tau_R(x_i,x_f)\rangle = \langle t_-(x_f)\rangle - \langle t_+(x_i)\rangle, \tag{2.18}$$

respectively. (We recall that here we are confining ourselves to systems with continuous spectra only.) Of course, it is possible to pass in (2.17) also to integrals $\int_0^\infty dE\ldots$, similarly to (2.15) by using the unique Fourier (Laplace) transformations and the energy expansion of $j_\pm(x,t) = j(x,t)\theta(\pm j)$, but it is evident that they result to be rather bulky.

If one does now generalize the expressions (2.15) and (2.17) for $\langle t^n \rangle$ with a generic value $n = 2, 3, \ldots$, then we will be able to write down for $\langle f(t) \rangle$ with any *analytic* function of time $f(t)$, the *one-to-one* relation

$$\langle f(t) \rangle = \frac{\int_{-\infty}^{\infty} j(x,t) f(t) dt}{\int_{-\infty}^{\infty} j(x,t) dt} = \frac{\int_0^{\infty} dE (1/2) \left[G^*(x,E) f(\widehat{t}) v G(x,E) + v G^*(x,E) f(\widehat{t}) G(x,E) \right]}{\int_0^{\infty} dE\, v \left| G(x,E) \right|^2} \tag{2.19}$$

from the time to the energy representation. For free motion, one has $G(x,E) = g(E) \exp(ikx)$, $\varphi(x,E) = \exp(ikx)$, and $E = \hbar^2 k^2 / 2\mu = \mu v^2 / 2$, while the normalization condition is

$$\int_0^{\infty} \left| G(x,E) \right|^2 dE = \int_0^{\infty} \left| g(E) \right|^2 dE = 1, \tag{2.20}$$

and the boundary conditions are

$$\left[\frac{d^n g(E)}{dE^n} \right]_{E=0} = \left[\frac{d^n g(E)}{dE^n} \right]_{E=\infty} = 0, \quad \text{for } n = 0, 1, 2, \ldots. \tag{2.21}$$

Conditions (2.21) imply a very rapid decrease till zero of the flux densities near the boundaries $E = 0$ and $E = \infty$: this complies with the actual conditions of real experiments, and therefore they does not represent any restriction of generality.

In (2.19), $\widehat{t}$ is defined by relation (2.1b). One should explicitly notice that relation (2.19) does express the complete *equivalence* of the *time* and of the *energy representations* (with their own appropriate averaging weights). This equivalence is a consequence of the existence of the time operator. Actually, *for the time and energy operators it holds in quantum mechanics the same formalism as for all other pairs of canonically-conjugate observables.*

For quasimonochromatic particles, when $|g(E)|^2 \approx K\delta(E - \overline{E})$, K being a constant, quantity $j(x,t)$ goes into $\rho(x,t)$ and (2.19) goes into the more simple relation

$$\langle f(t) \rangle \equiv \frac{\int_{-\infty}^{\infty} j(x,t) f(t) dt}{\int_{-\infty}^{\infty} j(x,t) dt} \approx \frac{\int_{-\infty}^{\infty} \rho(x,t) f(t) dt}{\int_{-\infty}^{\infty} \rho(x,t) dt} \approx \frac{\int_0^{\infty} dE G^*(x,E) f(\widehat{t}) G(x,E)}{\int_0^{\infty} dE \left| G(x,E) \right|^2}, \tag{2.22}$$

because of the relations $j(x,t) v \rho(x,t) \approx \overline{v} \rho(x,t)$.

Now, one can see that two canonically conjugate operators, the time operator (2.1a), (2.1b), and (2.26) and the energy operator

$$\widehat{E} = \begin{cases} E & \text{in the energy } (E)\text{-representation,} \\ i\hbar \dfrac{\partial}{\partial t} & \text{in the time } (t)\text{-representation,} \end{cases} \tag{2.23}$$

satisfy the typical commutation relation

$$\left[\widehat{E}, \widehat{t} \right] = i\hbar. \tag{2.24}$$

Although up to now according to the Stone theorem [55] the relation (2.24) has been interpreted as holding *only* for the pair of the self-adjoint canonically conjugate operators, in both representations, and it was not directly generalized for *maximal Hermitian operators*, the difficulty of such direct generalization has in fact been by-passed by introducing $\widehat{t}$ with the help of the *single-valued* Fourier (Laplace) transformation from the t-axis ($-\infty < t < \infty$) to the E-semiaxis ($0 < E < \infty$) and by utilizing the peculiar mathematical properties of maximal symmetric operators (as in [19–21, 23–25, 33–35, 50, 51]), described in detail, for example, in [52, 53].

Actually, from (2.24) the uncertainty relation

$$\Delta E \, \Delta t \geq \frac{\hbar}{2} \tag{2.25}$$

(where the standard deviations are $\Delta a = \surd \mathrm{D}a$, quantity $\mathrm{D}a$ being the variance $\mathrm{D}a = \langle a^2 \rangle - \langle a \rangle^2$; and $a = E, t$, while $\langle \cdots \rangle$ denotes an average over t by the measures $W(x,t)dt$ or $W_{\pm}(x,t)dt$ in the t-representation or an average over E similar to the right-hand part of (2.19) in the E-representation) was derived by the simple generalizing of the similar procedures which are standard in the case of self-adjoint canonically conjugate quantities (see [17–21, 23–25, 33–35, 50, 51]). Moreover, relation (2.24) satisfies the Dirac "correspondence principle," since the classical Poisson brackets $\{q_0, p_0\}$, with $q_0 = t$ and $p_0 = -E$, are equal to unity [56]. In [21] (see also [23–25]) it was also shown that *the differences* between the mean times at which a wave packet passes through *a pair* of points obey the Ehrenfest correspondence principle; in other words, in [21, 23–25] the Ehrenfest theorem was suitably generalized.

After what precedes, one can state that, for systems with continuous energy spectra, the mathematical properties of the maximal Hermitian operators (described, in particular, in [49, 53]), like $\widehat{t}$ in (2.1a), (2.1b), and (2.26) are *sufficient* for considering them as quantum observables: namely, the *uniqueness* of the "spectral decomposition" (also called spectral function) for operators $\widehat{t}$, as well as for $\widehat{t}^n$ ($n > 1$) guarantees (although such an expansion is not orthogonal) the *equivalence* of the mean values of any analytic functions of time, evaluated either in the t- or in the E-representations. In other words, the existence of this expansion is equivalent to a *completeness relation* for the (formal) eigenfunctions of $\widehat{t}^n$ ($n > 1$), corresponding with any accuracy to *real eigenvalues* of the continuous spectrum; such eigenfunctions belonging to the space of the square-integrable functions of the energy E with the boundary conditions (2.2)-(2.3).

From this point of view, there is *no practical difference between self-adjoint and maximal Hermitian operators* for systems with continuous energy spectra. Let us underline that the *mathematica*, properties of $\widehat{t}^n$ ($n > 1$) are quite enough for considering *time* as*a quantum-mechanical observable* (like for energy, momentum, spatial coordinates, ...) *without having to introduce any new physical postulates.*

Now let us analyse the so-called *positive-operator-value-measure* (POVM) approach, often used in the second set of papers on time in quantum physics (e.g., in [28–31, 36–46, 48, 49]). This approach, in general, is well known in the various approaches to the quantum theory of measurements approximately from the sixties and had been applied in the simplest form for the time-operator problem in the case of the free motion already in [57]. Then, in [28–31, 36–46, 48, 49] (often with certain simplifications and abbreviations) it was affirmed that the generalized decomposition of unity (or POV measures) is reproduced from any self-adjoint extension of the time operator into the space of the extended Hilbert

space (usually, with negative values of energy E in the left semiaxis) citing the Naimark's dilation theorem from [58]. However, it was realized factually only for the simple cases like the particle free motion. As to our approach, it is based on *another* Naimark's theorem (from [52]), cited above, and *without* any extension of the physical Hilbert space of usual wave functions (wave packets) with the subsequent return projection to the previous space of wave functions, and, moreover, it had been published in [12–18, 21] (and independently in the papers of Holevo [19, 20], with the same principal idea) much earlier than [28–31, 36–46, 48, 49]. Being based on the earlier published remarkable Naimark theorem [52], it is *much more direct, simple and general, and at the same time mathematically not less rigorous* than POVM approach.

Let us note that it was introduced by Olkhovsky and Recami in [12–14] one more form of the time operator

$$\hat{t} = \left(-\frac{i\hbar}{2}\right)\frac{\overleftrightarrow{\partial}}{\partial E} \tag{2.26}$$

(the so-called *bilinear* form), where the meaning of the sign $\leftrightarrow$ is clear from the following definition: $(f, \hat{t}g) = (f, (-i\hbar/2)(\partial/\partial E)g) + ((-i\hbar/2)(\partial/\partial E)f, g)$. For this form the *direct* elimination of the point $E = 0$ is not necessary because it is eliminated automatically in $(f, \hat{t}f)$ and in $\int_{-\infty}^{\infty} tj(x,t)dt$ by such *bilinearity*. And such an elimination of the point $E = 0$ is not only more simple but also more physical than an elimination made in [28–31, 36–46, 48, 49], and it had been published (in [12–14]) much more earlier.

2.2. On the Momentum Representation of the Time Operator

In [19, 20], it had been demonstrated by Holevo that in the continuous spectrum case, instead of the energy (E-) representation, with $0 < E < \infty$, in (2.1a), (2.1b), (2.26), (2.2), (2.3), and (2.7) one can also use the momentum (k-) representation, with the advantage that $-\infty < k < \infty$:

$$\Psi(x,t) = \int_{-\infty}^{\infty} dk g(k)\varphi(x,k)\exp\left(-\frac{iEt}{\hbar}\right), \tag{2.27}$$

with $E = \hbar^2 k^2/2m,\ k \neq 0$. In such a case the time operator (2.1a), (2.1b), and (2.26) (acting on momentum eigenvector, defined on $-\infty < k < \infty$, is already *formally self-adjoint*, with the boundary conditions

$$\left[\frac{d^n g(k)}{dk^n}\right]_{k=-\infty} = \left[\frac{d^n g(k)}{dk^n}\right]_{k=\infty} = 0, \quad n = 0, 1, 2, \ldots, \tag{2.28}$$

except for the fact that we have excluded point $k = 0$; an exclusion which has now only the physical meaning of nonobserving the rest (motionless) state), being inessential mathematically (this had been considered in [19, 20, 50, 51]). In fact, it is one more argument in favor of that time is an observable in the same degree as any other quantity to which a self-adjoint operator corresponds.

Let us now compare choice (2.27) with choice (2.16); namely let us rewrite (2.27) as follows:

$$\begin{aligned}\Psi(x,t) = &\int_0^\infty dE(E)^{-1/2} g\left(\frac{(2mE)^{1/2}}{\hbar}\right)\varphi\left(x, \frac{(2mE)^{1/2}}{\hbar}\right)\exp\left(-\frac{iEt}{\hbar}\right)\\ &+\int_0^\infty dE(E)^{-1/2} g\left(-\frac{(2mE)^{1/2}}{\hbar}\right)\varphi\left(x, -\frac{(2mE)^{1/2}}{\hbar}\right)\exp\left(-\frac{iEt}{\hbar}\right).\end{aligned} \tag{2.29}$$

If we now introduce the weight

$$\tilde{g}(E) = \left(\frac{m}{2E\hbar^2}\right)^{1/4}\begin{bmatrix} g\left(\dfrac{(2mE)^{1/2}}{\hbar}\right)\\ g\left(-\dfrac{(2mE)^{1/2}}{\hbar}\right)\end{bmatrix} \tag{2.30}$$

as a "two-dimensional" vector, then

$$\int_{-\infty}^{\infty}\left|\Psi(x,t)\right|^2 dx = \int_0^\infty dE\left|\tilde{g}(E)\right|^2 < \infty, \tag{2.31}$$

the norm being

$$\tilde{g}(E) = g^*(E)\cdot g(E) > 0. \tag{2.32}$$

If the wave packet (2.27) is *one directional* and $g(k) \equiv g(k)\Theta(k)$, then the integral $\int_{-\infty}^{\infty} dk$ goes on to the integral $\int_0^\infty dk$ and the two-dimensional vector goes on to a scalar quantity. In such a case, the boundary conditions (2.2) and (2.3) can be replaced by relations of the same form, provided that the replacement $E \to k$ is performed.

2.3. The Second Measure of Time Averaging (in the Cases of Particle Dwelling in Spatial Regions)

One can easily see that the weight

$$dP(x,t) \equiv Z(x,t)dx = \frac{\left|\Psi(x,t)\right|^2 dx}{\int_{-\infty}^{\infty}\left|\Psi(x,t)\right|^2 dx} \tag{2.33}$$

can be considered as the meaning of the probability for a particle to be localized, or to *sojourn*, or to *dwell* in the spatial region $(x, x+dx)$ at the moment t, *independently from the motion processes*. As a consequence, the quantity

$$P(x_i, x_f, t) = \frac{\int_{x_1}^{x_f}\left|\Psi(x,t)\right|^2 dx}{\int_{-\infty}^{\infty}\left|\Psi(x,t)\right|^2 dx} \tag{2.34}$$

will have the meaning of the probability of particle dwelling in the spatial range (x_1, x_2) at the instant t. Taking into account the equality

$$\int_{-\infty}^{\infty} j(x,t)dt = \int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx, \tag{2.35}$$

which evidently follows from the 1D continuity relation (2.8), the mean dwell time can be presented in the following form:

$$\langle \tau_{\mathrm{dw}}(x_i, x_f) \rangle = \frac{\int_{-\infty}^{\infty} dt \int_{x_i}^{x_f} |\Psi(x,t)|^2 dx}{\int_{-\infty}^{\infty} j_{\mathrm{in}}(x_i,t)dt} \tag{2.36a}$$

with the flux density j_{in} for the initial "*dwelling*" *free* motion through point x_i. The expression (2.36a) can be rewritten in the following equivalent form

$$\langle \tau_{\mathrm{dw}}(x_i, x_f) \rangle = \frac{\left[\int_{-\infty}^{\infty} t j(x_f,t)dt - \int_{-\infty}^{\infty} t j(x_i,t)dt\right]}{\int_{-\infty}^{\infty} j_{\mathrm{in}}(x_i,t)dt}, \tag{2.36b}$$

taking into account the continuity relation (2.8) for the *total* flux density $j(x,t)$ in the interval (x_i, x_f) at time t (the details of the derivation one can see in [22, 47]).

Thus, in correspondence with two measures above, (2.7), (2.9), (2.36a), and (2.36b), when integrating on time, we get different two kinds of time distributions (mean values, variances, etc.) being with different physical meanings (referring to the particle moving, passing, transferring, traversing, transmitting, etc. in the case of the measures (2.7) and (2.9) and of particle staying, dwelling, living, sojourning, etc. in the case of the measure (2.36a) and (2.36b), resp.).

2.4. Extension of the Notion of Time as a Quantum-Physical Observable Quantity to Quantum Electrodynamics

The formal mathematical analogy between the stationary and time-dependent Schroedinger equation for nonrelativistic particles and the stationary and time-dependent Helmholtz equation for electromagnetic wave propagation was studied in [59–62]. In the time-dependent case, these equations are no longer mathematically equivalent, since the former is first-order in the time derivative whereas the latter is second order. However, here we will deal with the comparison of their solutions, considering not only the formal mathematical analogy between them but also such similarity of the probabilistic interpretation of the wave function for a particle and of an electromagnetic wave packet (being according to [63, 64] the "wave function for a single photon") which is sufficient for the identical definition of mean time instants and durations (and distribution variances, etc.) of propagation, collision, tunnelling, processes for particles and photons.

In the first quantization for the 1D case, the single-photon wave function can be probabilistically described by the wave packet (see, e.g., [63, 64])

$$\vec{A}(\vec{r},t) = \int_{k_0} \frac{d^3k}{k_0} \vec{\chi}(\vec{k}) \varphi(\vec{k},\vec{r}) \exp\left(-ik_0 t\right), \tag{2.37}$$

where, as usual, $\vec{A}(\vec{r},t)$ is the electromagnetic vector potential, and $\vec{r} = \{x,y,z\}$, $\vec{k} = \{k_x,k_y,k_z\}$, $k_0 \equiv \omega/c = \varepsilon/\hbar c$, $k \equiv |\vec{k}| = k_0$, where the gauge condition $\operatorname{div}\vec{A} = 0$ is assumed.

The axis x has been chosen as the propagation direction, $\vec{\chi}(\vec{k}) = \sum_{i=y}^{z} \chi_i(\vec{k})\vec{e}_i(\vec{k})$; with $\vec{e}_i\vec{e}_j = \delta_{ij}$, $x_i, x_j \equiv y, z$, $\chi i(\vec{k})$ is the probability amplitude for the photon to have momentum $\vec{k}$ and polarization $\vec{e}_j$ along x_j, and it is $\varphi(\vec{k},\vec{r}) = \exp(ik_x x)$ in the case of plane waves, while $\varphi(\vec{k},\vec{r})$ is a linear combination of evanescent (decreasing) and antievanescent (increasing) waves in the case of "photon barriers" (various band-gap filters, or even undersized segments of waveguides for microwaves, frustrated-total-internal-reflection regions for light, etc.). Although it is not possible to localize a photon in the direction of its polarization, nevertheless for 1D propagations, it is possible to use the space-time probabilistic interpretation of (2.37) and define the following *probability density*:

$$\rho_{\mathrm{em}}(x,t)dx = \frac{S_0\,dx}{\int S_0\,dx}, \quad S_0 = \iint s_0\,dy\,dz \tag{2.38}$$

($s_0 = [\vec{E}^*\vec{E} + \vec{H}^*\vec{H}]/4\pi$ being the energy density, the electromagnetic field being $\vec{H} = \operatorname{rot}\vec{A}$, $\vec{E} = -(1/c)\partial\vec{A}/\partial t$) *of a photon to be found (localized) in the spatial intervall* $(x, x+dx)$ *along axis* x *at the moment* t*, and the flux probability density*

$$j_{\mathrm{em}}(x,t)dt = \frac{S_x(x,t)dt}{\int S_x(x,t)dt}, \quad S_x(x,t) = \iint s_x\,dy\,dz \tag{2.39}$$

(with $s_x = c\,\mathrm{Re}[\vec{E}^*\vec{H}]_x/8\pi$ being the energy flux density, $\vec{H} = \operatorname{rot}\vec{A}$) *of a photon to pass through the point (plane)* x *in the time interval* $(t, t+dt)$, quite similarly to the probabilistic quantities for particles. The justification and convenience of such definitions is evident, every time that there is a coincidence of the wave packet group velocity and the velocity of the energy transport which was established for electromagnetic waves [65–67]. Hence, (1) in a certain sense, for the time analysis along the motion direction, the wave packet (2.24) is quite similar to a wave packet for nonrelativistic particles and (2) similarly to the conventional nonrelativistic quantum mechanics, one can define the mean time of photon (electromagnetic wave packet) passing through point x:

$$\langle t(x)\rangle = \int_{-\infty}^{\infty} t J_{\mathrm{em},x}\,dt = \frac{\int_{-\infty}^{\infty} t S_x(x,t)dt}{\int_{-\infty}^{\infty} S_x(x,t)dt}, \tag{2.40}$$

where the form (2.1b) of time operator is valid also for photons with natural boundary conditions $\chi_i(0) = \chi_i(\infty) = 0$ in the energy representation ($\varepsilon = \hbar c k_0$), quite similarly to (2.1b)–(2.3) for nonrelativistic particles in the energy representation.

The energy density s_0 and energy flux density s_x satisfy the relevant continuity equation

$$\frac{\partial s_0}{\partial t} + \frac{\partial s_x}{\partial x} = 0, \tag{2.41}$$

which is *Lorentz-invariant* for the spatially 1D propagation [32–35, 47, 50, 51]. As a consequence, it is self-evident that also in this case of photons we can use the same energy representation of the time operator as for particles in nonrelativistic quantum mechanics, and hence verify the equivalence of calculations of $\langle t(x)\rangle$, $\mathrm{D}t(x)$ and so on, in the both time and energy representations. Then, the same interpretation one can use for the propagation of electromagnetic wave packets (photons) in media and waveguides when collisions, reflections, and tunnelling can take place. Then, one can introduce the same form of the time operator as for particles in nonrelativistic quantum mechanics and hence verify the equivalence of calculations of mean values, variances, and so on, for time durations of photon motions, interactions and so on, with the measure (2.7)–(2.9), in the both time and energy representations [32–35, 47, 50, 51]. It is also possible to introduce the second measure in time averaging, quite similarly to (2.36a) and (2.36b). In other words, *in the cases of 1D photon propagations time is a quantum-physical observable also in quantum electrodynamics.*

In the case of fluxes which change their signs with time we introduce quantities $J_{\mathrm{em},x,\pm} = J_{\mathrm{em},x}\Theta(\pm J_{\mathrm{em},x})$ with the same physical meaning as for particles. Therefore, expressions for mean values and variances of distributions of propagation, tunnelling, transmission, penetration, and reflection durations can be obtained in the same way as in the case of nonrelativistic quantum mechanics for particles (with the substitution of J by J_{em}).

2.5. Time as an Observable and Time-Energy Uncertainty Relation for Quantum-Mechanical Systems with Discrete Energy Spectra

For systems with discrete energy spectra it is natural (following [23–25, 50, 51]) to introduce wave packets of the form

$$\psi(x,t) = \sum_{n=0}^{\cdots} g_n \varphi_n(x) \exp\left[-\frac{i(\varepsilon_n - \varepsilon_0)t}{\hbar}\right], \tag{2.42}$$

(where $\varphi_n(x)$ are orthogonal and normalized wave functions of system bound states which satisfy $\widehat{H}\varphi_n(x) = \varepsilon_n\varphi_n(x)$, $\widehat{H}$ being the system Hamiltonian, $\sum_{n=0}^{\cdots} |g_n|^2 = 1$, here we factually omitted a nonsignificant phase factor $\exp(-i\varepsilon_0 t/\hbar)$ as being general for all terms of the sum $\sum_{n=0}^{\cdots}$) for describing the evolution of systems in the regions of the purely discrete spectrum. Without limiting the generality, we choose moment $t = 0$ as an initial time instant.

Firstly, we will consider those systems, whose energy levels are spaced with distances for which the maximal common divisor is *factually existing*. Examples of such systems are *harmonic oscillator, particle in a rigid box,* and *spherical spinning top*. For these systems the wave packet (2.42) is a periodic function of time with the period (*Poincaré cycle time*) $T = 2\pi\hbar/D$, D being the maximal common divisor of distances between system energy level.

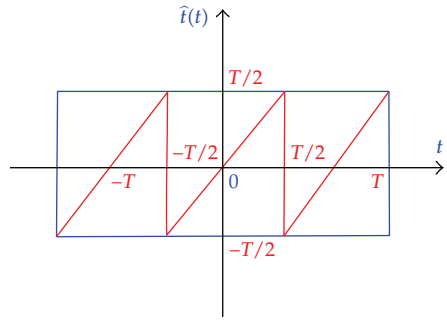

Figure 1: The periodical saw-tooth function for time operator for the case of (2.42).

In the t-representation the relevant energy operator $\widehat{H}$ is a self-adjoint operator acting in the space of *periodical* functions whereas the function $t\psi(t)$ does not belong to the same space. In the space of periodical functions the time operator $\hat{t}$, even in the eigen representation, has to be also a periodical function of time t. This situation is quite similar to the case of angular momentum (e.g., [68, 69]). Utilizing the example and result from [54], let us choose, instead of t, a *periodical* function

$$\hat{t} = t - T\sum_{n=o}^{\infty}\Theta\left(\frac{t-[2n+1]T}{2}\right) + T\sum_{n=0}^{\infty}\Theta\left(-\frac{t-[2n+1]T}{2}\right), \tag{2.43}$$

which is the so-called saw-function of t (see Figure 1).

This choice is convenient because the periodical function of time operator (2.43) is linear function (one-directional) within each Poincaré interval, that is, time conserves its flowing and its usual meaning of an *order* parameter for the system evolution.

The commutation relation of the *self-adjoint* energy and time operators acquires in this case (discrete energies and periodical functions) the form

$$[\hat{E},\hat{t}] = i\hbar\left\{1 - T\sum_{n=0}^{\infty}\delta\left(t-[2n+1]T\right)\right\}. \tag{2.44}$$

Let us recall (see, e.g., [70, 71]) that a generalized form of uncertainty relation holds

$$(\Delta A)^2 \cdot (\Delta B)^2 \geq \hbar^2\left[\langle N\rangle\right]^2 \tag{2.45}$$

for two self-adjoint operators $\hat{A}$ and $\hat{B}$, canonically conjugate each to other by the commutator

$$\left[\hat{A},\hat{B}\right] = i\hbar\widehat{N}, \tag{2.46}$$

$\widehat{N}$ being a third self-adjoint operator. One can easily obtain

$$(\Delta E)^2 \cdot (\Delta t)^2 \geq \hbar^2 \left[1 - \frac{T\left|\psi(T/2+\gamma)\right|^2}{\int_{-T/2}^{+T/2}\left|\psi(t)\right|^2 dt} \right], \tag{2.47}$$

where the parameter γ (with an arbitrary value between $-T/2$ and $+T/2$) is introduced for the univocality of calculating the integral on right part of (2.47) over dt in the limits from $-T/2$ to $+T/2$, just similarly to the procedure introduced in [68] (see also [70, 71]).

From (2.47) it follows that when $\Delta E \to 0$ (i.e., when $|g_n| \to \delta_{nn'}$) the right part of (2.47) tends to zero since $|\psi(t)|^2$ tends to a constant. In this case, the distribution of time instants of wave packet passing through point x in the limits of one Poincaré cycle becomes uniform. When $\Delta E \gg D$ and $|\psi(T+\gamma)|^2 \ll T\int_{-T/2}^{T/2}|\psi(t)|^2 dt$, the periodicity condition may be inessential for $\Delta t \ll T$, that is, (2.47) *passes to uncertainty relation* (2.11), which is just the same one as for systems with continuous spectra.

In principle, one can obtain the expression for the time operator (2.43) also in energy representation. If one will calculate the mean value $\langle t(x)\rangle$ of instants of particle passing through point x, then after a series of bulky transformations he will obtain the following expression:

$$\widehat{t} = \frac{i\hbar}{2}\sum_{n;>n}(-1)^{N_n - N_{n'}}\frac{\overleftrightarrow{\Delta}_{n'}}{\Delta_{n'}\varepsilon_n} \tag{2.48}$$

in the energy representation, where $N_n = (\varepsilon_n - \varepsilon_0)/D$; the bilinear operation, denominated by $\overleftrightarrow{\Delta}_n$, signifies

$$\begin{gathered} A_n^* \overleftrightarrow{\Delta}_{n'} A_n = A_n^* \Delta_{n'} A_n - A_n \Delta_{n'} A_n^*, \quad \Delta_{n'} A_n = A_{n'} - A_n, \\ \langle t(x)\rangle = \frac{\sum_{n=0}^{\infty} g_n^* \varphi_n^*(x)\widehat{t} g_n \varphi_n(x)}{\sum_{n=0}^{\infty}\left|g_n\varphi_n(x)\right|^2}. \end{gathered} \tag{2.49}$$

(Of course, one has to average over the flux density, but for the simplicity in this case it is possible to make averaging over $|\Psi(x,t)|^2$.) Operator (2.43) for two levels ($n = 0, 1$) acquires the more simple form

$$\widehat{t} = \frac{-i\hbar}{2}\frac{\overleftrightarrow{\Delta}}{\Delta\varepsilon}, \tag{2.50}$$

and when $D = \varepsilon_1 - \varepsilon_0 \to 0$, the expression (2.50) passes to the differential form

$$\widehat{t} = \frac{-i\hbar}{2}\frac{\overleftrightarrow{\partial}}{\partial\varepsilon}, \tag{2.51}$$

which coincides with (A.1) from Appendix A, that is, it is equivalemt to operator (2.1b) for the continuous energy spectra.

In general cases, for excited states of nuclei, atoms, and molecules, *level distances in discrete spectra have not strictly defined the maximal common divisor and hence, they have not the strictly defined time of the Poincaré cycle*. Also there is no strictly defined passage from the discrete part of the spectrum to the continuous part. Nevertheless, even for those systems one can introduce an approximate description (and with any desired degree of the accuracy within the chosen maximal limit of the level width, let us say, $\gamma_{\lim}$) by quasicycles with quasiperiodical evolution and for sufficiently long intervals of time the motion inside such systems (however, less than $\hbar/\gamma_{\lim}$) one can consider as *a periodical motion also with any desired accuracy*. For them one can choose (define) a time of the Poincare' cycle with any desired accuracy, including in one cycle as many quasicycles as it is necessary for demanded accuracy. Then, with the same accuracy the *quasi-self-adjoint time operator* (2.43) or (2.48) can be introduced and all time characteristics can be defined.

In the degenerate case when at-the-state (2.42) the sum $\sum_{n=0}^{\infty}$ contains only one term $(g_n \to \delta_{nn'})$, the evolution is absent and the time of the Poincare' cycle is equal formally to infinity.

If a system has both (continuous and discrete) regions of the energy spectrum, one can easily use the forms (2.1a), (2.1b), and (2.26) for the continuous energy spectrum and the forms (2.43) and (2.48) for the discrete energy spectrum.

3. Applications for Tunneling Phenomena

3.1. Introduction

The developments of the study of tunneling processes in nuclear physics (α-radioactivity, nuclear subbarrier fission, fusion, proton radioactivity and so on), then in various other fields of physics and especially the advent of high-speed electronic (and now microwave and optical) devices, based on tunnelling processes, generated an interest in the tunnelling time analysis and stimulated the publication of not only a lot of theoretical studies but already a lot of theoretical reviews on tunneling times (e.g., [72–80], apart from [26, 27, 32–35, 47]). And during many years, there had not been not only the consensus in the theoretical definition of the tunneling time for particles, but also there had been some declarations about the incompatibility of some approaches both quantitavely and in the physical interpretation. Among the reasons of such situation there had been the following ones:

(i) the problem of defining the tunneling time is closely connected with general fundamental problems of time as a *quantum-physical observable* and the general definition of quantum-collision durations, and the acquaintance with the principal solution of these problems had not got a wide prevalence yet till 2000–2004 (e.g., [47, 81]);

(ii) the motion of particles inside a potential barrier is a quantum phenomenon *without* any direct classical limit (namely for particles);

(iii) there are essential physical and mathematical *differences in initial, boundary, and external conditions* of various definition schemes.

Following [47, 80], we arrange the majority of approaches into several groups which are based on (1) the time-dependent wave packet description; (2) averaging over an introduced set of kinematic paths, distribution of which is supposed to describe the particle motion inside a barrier; (3) introducing a new degree of freedom, constituting a physical clock for measurements of tunnelling times. Separately, by one's self, the dwell time stands.

The last has ab initio the presumptive meaning of the time that the incident flux has to be turned on, to provide the accumulated particle storage in the barrier [22, 80].

The first group contains the so-called phase times, firstly mentioned in [82, 83] and applied to tunnelling in [84, 85], the times of the motion of wave packet spatial centroids, earlier considered for general quantum collisions in [12–14, 86, 87] and applied to tunnelling in [88, 89], and finally the Olkhovsky-Recami (O-R) method [26, 27, 32–35, 47, 90] of averaging over unidirectional fluxes, basing on the representation of time as a quantum-mechanical observable and on the generalization of the definitions, introduced in [21, 23–25, 91] for atomic and nuclear collisions. The second group contains methods, utilizing the Feynman path integrals [92–98], the Wigner distribution paths [99–102], and the Bohm approach [103]. The approaches with the Larmor clock [104–107] and the oscillatory barrier [108, 109] pertain to the third group.

Certainly, the basic self-consistent definition of tunnelling durations (mean values, variances of distributions, etc.) has to be elaborated quite *similarly* to the definitions of other physical quantities (distances, energies, momenta, etc.) on the base of utilizing all necessary properties of time as a quantum-physical observable (time operator, canonically conjugated to energy operator; the equivalency of the averaged quantities in time and energy representations with adequate measures, or weights, of averaging). For such definition, the description of solutions of the time-dependent Schroedinger equation by moving wave packets, which are typical in quantum collision theory (e.g., [110]), is quite natural for utilizing. Then one can expect that in the framework of the conventional quantum mechanics every known definition of tunnelling times can be shown, after appropriate analysis, to be (at least in the asymptotic region, used for typical boundary conditions in quantum collision theory) either a *particular* case of the general definition or an *equivalent* one or the definition which is valid *not* for tunnelling but for some accompanying process, different from tunnelling.

Here such a definition with the necessary formalism is presented (Section 3.2) and a brief comparison with various approaches is given (Sections 3.3–3.5), basing on the O-R formalism. In Section 3.6 the Hartman and Fletcher effect, with its generalization and its violations, is described. The tunneling through a double barrier is described in Section 3.7. The particle tunneling through three-dimensional barriers is presented in Section 3.8. The quaternion description of tunneling phenomena is mentioned in Section 3.9.

3.2. The O-R Formalism of Defining Tunnelling Durations, Based on Utilizing Properties of Time as a Quantum-Mechanical Observable

We confine ourselves to the simplest case of particles moving only along the x-direction, and consider a time-independent barrier in the interval $(0, a)$;—see Figure 1, in which a larger interval (x_i, x_f), containing the barrier region, is also indicated.

As it is well known, in the case of a rectangular potential barrier of the height V_0, the stationary wave function for a particle with mass m and energy $E < V_0$ has the usual form (e.g., [26, 27, 32, 47, 72–81, 90] and a lot of other papers):

$$\psi(k,x) = \begin{cases} \exp(ikx) + A_R \exp(-ikx), & x \le 0 \text{ (region I)}, \\ \alpha \exp(-\chi x) + \beta \exp(\chi x), & 0 \le x \le a \text{ (region II)}, \\ A_T \exp(ikx), & x \ge a \text{ (region II)}, \end{cases} \tag{3.1}$$

where $k = (2mE)^{1/2}/\hbar$, $\chi = [2m(V_0 - E)]^{1/2}/\hbar$, A_R, α, β and A_T are the amplitudes of the reflected, evanescent, antievanescent and transmitted waves, respectively.

Inside a barrier here we have not usual propagating waves but a superposition of an evanescent (decreasing) and antievanescent (growing) waves with an imaginary wave number $i\chi$. Just for this reason, for *particle* tunnelling (with subbarrier energies) through a barrier *any direct classical limit does not really exist*. However, one can see *the direct classical limit for waves* (more strictly, for time-dependent wave packet tunnelling, considered later). And we can remind real evanescent and antievanescent waves inside the layers with lesser refraction numbers between the layers with larger refraction numbers in the cases of the frustrated total internal reflection, well known in classical optics and in classical acoustics.

Following the definition of collision durations, put forth firstly in [21, 23–25, 91] and afterwards generalized in [26, 27, 32–35, 47] (see also [81]), we can eventually define the mean values of the time at which a particle passes through position x, travelling in the positive or negative direction of the x-axis, and the variances of the distributions of these times, respectively, as

$$\begin{aligned} \langle t_{\pm}(x)\rangle &= \frac{\int_{-\infty}^{\infty} t j_{\pm}(x,t)dt}{\int_{-\infty}^{\infty} j_{\pm}(x,t)dt}, \\ Dt_{\pm}(x) &= \int_{-\infty}^{\infty} t^2 j_{\pm}(x,t)dt, \end{aligned} \tag{3.2}$$

$j_{\pm}(x,t)$ being the positive or negative values, respectively, of the probability flux density $j(x,t) = \mathrm{Re}[(i\hbar/m)\Psi(x,t)\partial\Psi^*(x,t)/\partial x]$ for an evolving time-dependent normalized *wave packet* $\Psi(x,t)$. We recall here the *equivalence of canonically conjugated time and energy representations*, with appropriate measures of averaging, in the following sense: $\langle\cdots\rangle_t = \langle\cdots\rangle_E$ (index t is omitted in all expressions for $\langle\cdots\rangle_t$ for the sake of the simplicity). This equivalence is a consequence of the unique time-operator existence.

For transmissions from region I to region III we have

$$\langle\tau_T(x_i,x_f)\rangle = \langle t_+(x_f)\rangle - \langle t_+(x_i)\rangle, \tag{3.3}$$

$$D\tau_T(x_i,x_f) = Dt_+(x_f) + Dt_+(x_i), \tag{3.4}$$

with $-\infty < x_i \le 0$ and $a \le x_f < \infty$. For a pure tunnelling process one has

$$\begin{aligned} \langle\tau_{\mathrm{tun}}(0,a)\rangle &= \langle t_+(a)\rangle - \langle t_+(0)\rangle, \\ D\tau_{\mathrm{tun}}(0,a) &= Dt_+(a) + Dt_+(0). \end{aligned} \tag{3.5}$$

Similar expression we have for the penetration (into the barrier region II) temporal quantities $\langle\tau_{\mathrm{pen}}(x_i,x_f)\rangle$ and $D\tau_{\mathrm{pen}}(x_i,x_f)$ with $0 < x_f < a$. For reflections in any point $x_f < a$ one has

$$\begin{aligned} \langle\tau_R(x_i,x_f)\rangle &= \langle t_-(x_f)\rangle - \langle t_+(x_i)\rangle, \\ D\tau_R(x_i,x_f) &= Dt_-(x_f) + Dt_+(x_i). \end{aligned} \tag{3.6}$$

We stress that these definitions hold within the framework of conventional quantum mechanics, *without* introducing any new physical postulate.

In the asymptotic cases, when $|x_i| \gg a$,

$$\langle \tau_T^{\mathrm{as}}(x_i, x_f)\rangle = \langle t(x_f)\rangle_T - \langle t(x_i)\rangle_{\mathrm{in}}, \tag{3.7}$$

$$\langle \tau_T^{\mathrm{as}}(x_i, x_f)\rangle = \langle \tau_T(x_i, x_f)\rangle + \langle t_+(x_i)\rangle - \langle t(x_i)\rangle_{\mathrm{in}}, \tag{3.8}$$

where $\langle\cdots\rangle_T$ and $\langle\cdots\rangle_{\mathrm{in}}$ denote averagings over the fluxes corresponding to $\psi_T = A_T \exp(ikx)$ and $\psi_{\mathrm{in}} = \exp(ikx)$, respectively.

For initial wave packets

$$\Psi_{\mathrm{in}}(x,t) = \int_0^\infty G(k-\overline{k}) \exp\left(\frac{ikx - iEt}{\hbar}\right) dk, \tag{3.9}$$

(where $E = \hbar^2 k^2/2m$, $\int_0^\infty |G(k-\overline{k})|^2 dE = 1$, $G(0) = G(\infty) = 0$, $k > 0$) with sufficiently small energy (momentum) spreads when

$$\int_0^\infty v^n |GA_T|^2 dE \cong \int_0^\infty v^n |G|^2 dE, \quad n = 0, 1, \ v = \frac{\hbar k}{m}, \tag{3.10}$$

we get

$$\langle \tau_T^{\mathrm{as}}(x_i, x_f)\rangle \cong \langle \tau_T^{\mathrm{Ph}}(x_i, x_f)\rangle_E, \tag{3.11}$$

where

$$\begin{gathered} \langle\cdots\rangle_E = \frac{\int_0^\infty dE v |G(k-\overline{k})|^2 \cdots}{\int_0^\infty dE v |G(k-\overline{k})|^2}, \\ \tau_T^{\mathrm{Ph}}(x_i, x_f) = \left(\frac{1}{v}\right)(x_f - x_i) + \frac{\hbar d(\arg A_T)}{dE} \end{gathered} \tag{3.12}$$

are the *phase transmission time* obtained by the stationary-phase approximation. At the same approximation and with a small contribution of $Dt_+(x_i)$ into the variance $D\tau_T(x_i, x_f)$ (that can be realized for sufficiently large energy spreads, i.e., short wave packets) we get

$$D\tau_T(x_i, x_f) = \frac{\hbar^2 \langle (d|A_T|/dE)^2\rangle_E}{\langle |A_T|^2\rangle_E}. \tag{3.13}$$

For the opposite case of very small energy spreads (quasimonochromatic particles) it follows that, instead of the expression (3.13), the general expression (3.4) becomes just the item of $Dt_+(x_i)$ plus $D\tau_T(x_i, x_f)$ which is born by the barrier influence and formally is described by (3.13).

At the quasimonochromatric limit $|G|^2 \rightarrow \delta(E - E)$, E being $\hbar^2\overline{k}^2/2m$, we get for $\langle \tau_T^{\mathrm{as}}(x_i, x_f)\rangle \cong \langle \tau_T^{\mathrm{Ph}}(x_i, x_j)\rangle_E$ strictly the ordinary phase time, without averaging. For a rectangular barrier with height V_0 and $\chi a \gg 1$ (where $\chi = [2m(V_0 - E)]^{1/2}/\hbar$), the expressions (3.11) and (3.13), for $x_i = 0$, $x_f = a$ and $am/\hbar\overline{k} \gg Dt_+(x_i)$, pass into the known expressions

$$\tau_{\mathrm{tun}}^{\mathrm{Ph}} = \frac{2}{v\chi} \tag{3.14}$$

(coincident with the phase time [26, 27, 47, 83]), and

$$\left(D\tau_{\mathrm{tun}}^{\mathrm{Ph}}\right)^{1/2} = \frac{ak}{v\chi} \tag{3.15}$$

(coincident with one of the Larmor times [104–107] and the Buettiker-Landauer time [108] and also with the imaginary part of the complex time in the Feynman path-integration approach: see later Section 3.5), respectively.

For *real* weight amplitude $G(k - \overline{k})$, when $\langle t(0)\rangle_{\mathrm{in}} = 0$, from (3.8) we obtain

$$\left\langle \tau_{\mathrm{tun}}(0, a)\right\rangle = \left\langle \tau_{\mathrm{tun}}^{\mathrm{Ph}}\right\rangle - \left\langle t_+(0)\right\rangle. \tag{3.16}$$

By the way, if the measurement conditions are such that only the positive-momentum components of wave packets are registrated, that is, $\Lambda_+\Psi(x_i, t) = \Psi_{\mathrm{in}}(x_i, t)$, Λ_+ being the projector onto positive-momentum states, then for *any* x_i from $(-\infty, 0)$ and x_f from (a, ∞)

$$\left\langle \tau_T(x_i, x_f)\right\rangle_+ = \left\langle \tau_T^{\mathrm{Ph}}(x_i, x_f)\right\rangle_E, \tag{3.17}$$

$$\left\langle \tau_{\mathrm{tun}}(0, a)\right\rangle_+ = \left\langle \tau_{\mathrm{tun}}^{\mathrm{Ph}}\right\rangle_E, \tag{3.18}$$

because $\langle t(0)\rangle_+ = \langle t(0)\rangle_{\mathrm{in}}$.

In the particular case of quasimonochromatic electromagnetic wave packets, using the stationary-phase method under the same boundary and measurememt conditions as considered for particles, we obtain the identical expression for the *phase tunnelling time*

$$\tau_{\mathrm{tun,em}}^{\mathrm{Ph}} = \frac{2}{c}\chi_{\mathrm{em}} \quad \text{for } \chi_{\mathrm{em}}\, a \gg 1. \tag{3.19}$$

From (3.19), we can see that when $\chi_{\mathrm{em}} a > 2$ the effective tunnelling velocity

$$v_{\mathrm{tun}}^{\mathrm{eff}} = \frac{a}{\tau_{\mathrm{tun,em}}^{\mathrm{Ph}}} \tag{3.20}$$

is more than c, that is, superluminal. This result agrees with the results of the microwave-tunnelling measurements presented in [111–113] (see also [114] where moreover, the effective tunnelling velocity was identified with the group velocity of the final wave packet corresponding to a single photon).

3.3. Analysis of the Mean Dwell Time in the Light of the Olkhovsky-Recami Formalism

In Section 2, it was analyzed the meaning of two forms of the expression for the mean dwell time (2.36a) and (2.36b) from Section 2. Taking into account that the total flux $j(x_i,t) = j_{\text{in}}(x_i,t) + j_R(x_i,t) + j_{\text{int}}(x_i,t)$ and $j(x_f,t) = j_T(x_f,t)$ with j_{in}, j_R and j_T corresponding to the wave packets $\Psi_{\text{in}}(x_i,t)$, $\Psi_R(x_i,t)$ and $\Psi_T(x_f,t)$, constructed from the stationary wave functions ψ_{in}, $\psi_R = A_R \exp(-ikx)$ and ψ_T, respectively, and also

$$j_{\text{int}}(x,t) = \text{Re}\left\{\left(\frac{i\hbar}{m}\right)\left[\frac{\Psi_{\text{in}}(x,t)\partial\Psi_R^*(x,t)}{\partial x} + \frac{\Psi_R(x,t)\partial\Psi_{\text{in}}^*(x,t)}{\partial x}\right]\right\},$$
$$\int_{-\infty}^{\infty} j_{\text{int}}(x_i,t)dt = 0. \tag{3.21}$$

One obtains from (2.36a) and (2.36b) of Section 2,

$$\langle\tau^{\text{Dw}}(x_i,x_f)\rangle = \langle T\rangle_E\langle\tau_T(x_i,x_f)\rangle + \langle R(x_i)\rangle_E\langle\tau_R(x_i,x_f)\rangle, \tag{3.22}$$

with $\langle T\rangle_E = \langle|AT|^2 v\rangle_E/\langle v\rangle_E$, $\langle R(x_i)\rangle_E = \langle R\rangle_E + \langle r(x_i)\rangle$, $\langle R\rangle_E = \langle|A_R|^2 v\rangle_E/\langle v\rangle_E$, $\langle T\rangle_E + \langle R\rangle_E = 1$, and

$$\langle r(x)\rangle = \frac{\int_{-\infty}^{\infty}\left[J_+(x,t) - J_{\text{in}}(x,t)\right]dt}{\int_{-\infty}^{\infty} J_{\text{in}}(x,t)dt}. \tag{3.23}$$

One can see that $\langle r(x)\rangle$ is negative and tends to 0 when x tends to $-\infty$.

When $\Psi_{\text{in}}(x_i,t)$ and $\Psi_R(x_i,t)$ are sufficiently well separated in time, so that $\langle r(x_i)\rangle = 0$, it follows from (3.22) that the simple *weighted average rule*

$$\langle\tau^{\text{Dw}}(x_i,x_f)\rangle = \langle T\rangle_E\langle\tau_T(x_i,x_f)\rangle + \langle R\rangle_E\langle\tau_R(x_i,x_i)\rangle \tag{3.24}$$

is valid. For a rectangular barrier with $\chi a \gg 1$ and quasimonochromatic particles, the expressions (3.22) and (3.24) with $x_i = 0$ and $x_f = a$ pass to the known expressions

$$\langle\tau^{\text{Dw}}(x_i,x_f)\rangle = \left\langle\frac{\hbar k}{\chi V_0}\right\rangle_E, \tag{3.25}$$

(taking account of the interference term $\langle r(x_i)\rangle$)

$$\langle\tau^{\text{Dw}}(x_i,x_f)\rangle = \left\langle\frac{2}{\chi v}\right\rangle_E \tag{3.26}$$

(when the interference term $\langle r(x_i)\rangle$ is equal to 0).

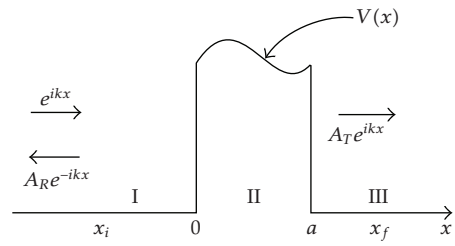

Figure 2: Schematic view of particle scattering and tunneling through a potential barrier.

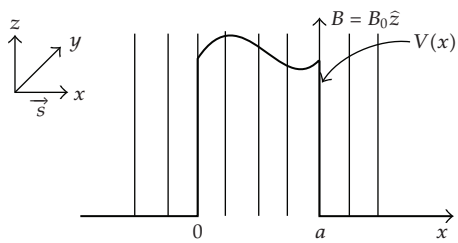

Figure 3: Orientations of spin amd magnetic field for the case of the "Larmor clock".

When $A_R = 0$, that is, a barrier is transparent, the mean dwell time (3.22) is automatically equal to

$$\langle \tau^{\mathrm{Dw}}(x_i, x_f) \rangle = \langle \tau_T(x_i, x_f) \rangle. \tag{3.27}$$

It is not clear how to define *directly* the variance of the dwell-time distribution. The approach, proposed in [115], is rather sophisticated, withan artificial abrupt switching on the initial wave packet. It is possible to define the variance of the dwell-time distribution *indirectly*, in particular, by means of relation (3.22), with the help of the variances of the transmission-time and reflection-time distributions, or by means of relation (2.36a) from Section 2, with the help of the variances of the positions x_1 and x_2.

3.4. Analysis of the Larmor and Buettiker-Landauer Clocks

One can often realize that the introducing of additional degrees of freedom as "clocks" does in a certain degree distort the true values of the tunnelling time. The Larmor clock uses the phenomenon of changing the spin orientation (the Larmor precession or spin-flip) in a weak homogeneous magnetic field covered the barrier region. If initially the particle spin is polarized in the x direction, after tunnelling the spin develops small y and z components (see Figure 3).

The Larmor times $\tau^{\mathrm{La}}_{y,T}$ and $\tau^{\mathrm{La}}_{z,T}$ are defined by the ratio of the spin-rotation angles around axes z and y (in turn defined by the developed y- and z-spin components, resp.) to the precession (rotation) frequency.

As to $\tau^{\mathrm{La}}_{z,T}$, in the reality it is not a precession but a jump to position "spin-up" or "spin-down" (spin-flip) accompanied by the Zeeman energy-level splitting [79, 104, 105]. Due to the Zeeman splitting, the component of the spin, that is parallel to the magnetic

field, corresponds to a higher tunnelling energy and hence tunnels preferentially, and namely therefore one can realize that this time is connected with the energy dependence of $|A_T|$ and coincides with the expression (3.15) (of course, at the same approximations when (3.15) is valid).

For an opaque rectangular barrier with $\chi a \gg 1$ the expressions

$$\langle \tau_{y,\mathrm{tun}}^{\mathrm{La}} \rangle = \langle \tau^{\mathrm{Dw}}(x_i, x_f) \rangle = \left\langle \frac{\hbar k}{\chi V_0} \right\rangle_E, \tag{3.28}$$

$$\langle \tau_{z,\mathrm{tun}}^{\mathrm{La}} \rangle = \left\langle \frac{ma}{\hbar \chi} \right\rangle_E \tag{3.29}$$

(for mean Larmor times) had been obtained in [79, 101–103].

In [26, 27, 32, 116], it was noted that, if the magnetic field region is infinite, the expression (3.28) passes into the expression (3.14) for the phase tunnelling time, after averaging over the small energy spread of the wave packet.

The work of the Buttiker-Landauer clock is connected withthe modulation cycle (absorption or emission of modulation quanta) caused by the oscillating part of a barrier, during tunneling. Also in this case one can realize that the coincidence of the Buttiker-Landauer time with (3.15) is connected withthe energy dependence of $|A_T|$ for the same reasons as for $\langle \tau_{z,\mathrm{tun}}^{\mathrm{La}} \rangle$.

3.5. Analysis of the Mean Tunnelling Times, Defined by Averaging over Kinematic Paths

The Feynman path-integral approach to quantum mechanics was applied in [92–98] to study and calculate the mean tunnelling time averaged over all possible paths, that have the same beginning and end, with the complex weight factor $\exp[iS(x(t))/\hbar]$, where S is the action associated with the path $x(t)$. Namely, such weighting of tunnelling times implies their distribution with a real and an imaginary component [79]. In [92], the real and imaginary parts of the obtained complex tunnelling time were found to be equal to $\langle \tau_{y,\mathrm{tun}}^{\mathrm{La}} \rangle$ and $\langle \tau_{z,\mathrm{tun}}^{\mathrm{La}} \rangle$, respectively.

An interesting development of this approach, the *instanton* version, is presented in [97, 98]. The instanton-bounce path is a stationary point of the Euclidean action. The latter is obtained by the analytic continuation to imaginary time in the Feynman-path integrals containing the factor $\exp(iS/\hbar)$. This path obeys a classical equation of motion in the potential barrier with the sign reversed. In [97, 98], the instanton bounces were considered as real physical processes. The bounce duration was calculated in real time and was found to be in good agreement with the one evaluated by the phase-time method. The temporal density of bounces was estimated in imaginary time and the obtained result coincided with (3.13) for the square root of the distribution variance at the limit of the phase-time approximation. Here one can see a manifestation of the virtual equivalence of the Schroedinger representation and the Feynman path-integral approach to quantum mechanics.

Another definition of the tunnelling time is connected with the Wigner distribution paths [99–102]. The basic idea of this approach, finally formulated by Muga, Brouard, and Sala, is that the distribution of the tunnelling times in the dynamical evolution of wave packets through barriers can be well approximated by a classical ensemble of particles with

a certain distribution function, namely the Wigner function $f(x,p)$, so that the flux at position x can be separated into positive and negative components:

$$J(x) = J^{+}(x) + J^{-}(x), \tag{3.30}$$

with $J^{+}(x) = \int_0^\infty (p/m) f(x,p)dp$ and $J^{-} = J - J^{+}$. Then *formally* the same expressions (3.3), (3.5), and (3.6) for the transmission, tunneling, and penetration durations and so on, as in the O-R formalism, were obtained with the substitution of $J^{\pm}$ instead of our $J_{\pm}$. The dwell time decomposition in this approach takes the form

$$\langle \tau^{\mathrm{Dw}}(x_i, x_f)\rangle = \langle T\rangle_E \langle \tau_T(x_i, x_f)\rangle + \langle R_M(x_i)\rangle_E \langle \tau_R(x_i, x_f)\rangle, \tag{3.31}$$

with $R_M(x) = \int_0^\infty |J^{-}(x,t)|dt$. Asymptotically, $R_M(x)$ tends to $\langle R\rangle_E$ and (3.31) takes *formally* the known form (3.24).

One more alternative is the stochastic method for wave packets [108]. It also leads to real times but its numerical implementation is not trivial [109].

In [110], the Bohm approach to quantum mechanics was used to choose a set of classical paths which do not cross. The Bohm formulation can provide, on the one hand, a strict equivalent to the Schroedinger equation, and on the other hand, a base for the nonstandard interpretation of quantum mechanics [79]. The obtained in [110] expression for the mean dwell time is not only positive definite but gives the unambiguous distinction between particles that are transmitted or reflected:

$$\tau_{\mathrm{Dw}}(x_i, x_f) = \int_0^\infty dt \int_{x_1}^{x_2} |\Psi(x,t)|^2 dx = T\tau_T(x_i, x_f) + R\tau_R(x_i, x_f), \tag{3.32}$$

with

$$\begin{aligned} \tau_T(x_i, x_f) &= \int_0^\infty dt \int_{x_1}^{x_2} |\Psi(x,t)|^2 \Theta(x - x_c)dx/T, \\ \tau_R(x_i, x_f) &= \int_0^\infty dt \int_{x_1}^{x_2} |\Psi(x,t)|^2 \Theta(x_c - x)dx/R, \end{aligned} \tag{3.33}$$

where T and R are here the mean transmission and reflection probability, respectively, the bifurcation line $x_c = x_c(t)$, separating transmitted and reflected trajectories, is defined by relation

$$T = \int_{-\infty}^{\infty} |\Psi(x,t)|^2 \Theta(x - x_c)dx. \tag{3.34}$$

Factually, in addition to the difference in the temporal integration in this and our formalisms ($\int_0^\infty dt$ and $\int_{-\infty}^{\infty} dt$, resp.), sometimes essential, this approach gives one more alternative—in separating the flux by the line x_c:

$$J(x,t) = [J(x,t)]_T + [J(x,t)]_R, \tag{3.35}$$

with

$$\begin{aligned}\left[J(x,t)\right]_T &= J(x,t)\Theta\left(x - x_c(t)\right),\\ \left[J(x,t)\right]_R &= J(x,t)\Theta\left(x_c(t) - x\right).\end{aligned} \tag{3.36}$$

3.6. On the Hartman and Fletcher Effect, Its Generalization and Its Violations

Firstly the Hartman and Fletcher effect (HFE) was revealed and studied in [84, 85] within the stationary-phase method for a 1D motion of quasimonochromatic nonrelativistic particles tunnelling through potential barriers. It consists in the absence of the dependence of the *phase tunnelling time*

$$\tau_{\text{tun}}^{\text{Ph}} = \frac{\hbar d\left(\arg A_T + ka\right)}{dE}, \tag{3.37}$$

(which is the *mean tunnelling time* $\langle \tau_{\text{tun}} \rangle$ within the stationary-phase method when it is possible to neglect the interference between incident and reflected waves out of a barrier [26, 27, 32], A_T and $E = \hbar^2 k^2/(2m)$ being the transmission amplitude and the particle kinetic energy, resp.) on the barrier width a for sufficiently large a. In particular, for a rectangular potential barrier $A_T = 4ik\chi[(k^2 - \chi^2)D_- + 2ik\chi D_+]^{-1}\exp[-(\kappa + i\chi)a]$, $D_\pm = 1 \pm \exp(-2\chi a)$, $\chi = [2m(V_0 - E)]^{1/2}/\hbar$, V_0 being the barrier height, and $\tau_{\text{tun}}^{\text{Ph}} \to 2/(\upsilon\chi)$ when $\chi a \gg 1$ ($\upsilon = \hbar k/m$ is the particle velocity before entering into a barrier).

Now we will test the validity of HFE for all other known theoretical expressions for mean tunnelling times. If we firstly take from the *mean dwell time* $\langle \tau_{\text{tun}}^{\text{Dw}} \rangle$, the *mean Larmor time* $\langle \tau_{y,\text{tun}}^{L} \rangle$ and the real part of the complex tunnelling time obtained by *averaging over the Feynman paths* $\operatorname{Re}\tau_{\text{tun}}^{F}$, which are equal $\hbar k/(\chi V_0)$ for quasimonochromatic particles and opaque rectangular barriers, we immediately easily see that also in these cases there is *no* dependence on the barrier width and consequently HFE is valid.

The validity of HFE for the *mean tunnelling time* within the O-R approach, is directly seen from the expression

$$\left\langle \tau_{\text{tun}} \right\rangle = \left\langle t_+(a) \right\rangle - \left\langle t_+(0) \right\rangle = \left\langle \tau_{\text{tun}}^{\text{Ph}} \right\rangle_E - \left\langle t_+(0) \right\rangle, \tag{3.38}$$

(where $\langle \cdots \rangle_E$ denotes averaging over the initial wave-packet energy spread and

$$\left\langle t_\pm(x) \right\rangle = \frac{\int_{-\infty}^{\infty} t j\pm(x,t)dt}{\int_{-\infty}^{\infty} j\pm(x,t)dt} \quad \text{with } j\pm(x,t) = \Theta(\pm j) j(x,t), \tag{3.39}$$

$j(x,t)$ being the probability flux density for a wave packet moving along axis x through a barrier located in the interval $(0,a)$ and it was confirmed in [26, 27, 32–35, 47] by numerous calculations for gaussian electron wave packets with narrow momentum spreads (see also Section 3.6).

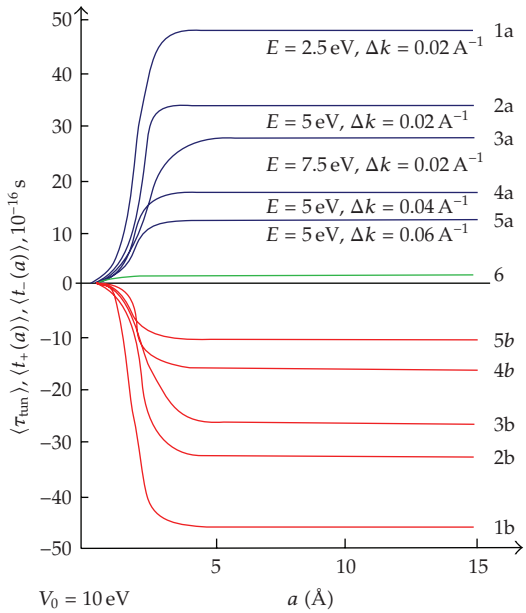

Figure 4: The dependences of $\langle\tau_{\rm tun}(0,a)\rangle$, $\langle\tau_{\rm tun}^{\rm Ph}\rangle$ and $\langle t_+(0)\rangle$ from a.

As to the other Larmor time $\tau_{z,\rm tun}^{L}$, from Sections 4.2 and 3.4 it follows that

$$\tau_{z,\rm tun}^{L} = \hbar\left[\frac{\langle (d|A_T|/dE)^2\rangle}{\langle |A_T|^2\rangle}\right]^{1/2}, \tag{3.40}$$

the *Bttiker-Landauer time* $\tau_{\rm tun}^{B-L}$ [111–113], and the *imaginary part of the complex tunnelling time* $\operatorname{Im}\tau_{\rm tun}^{F}$ [92], obtained within the Feynman approach, which are equal to (3.39), they become equal to $am/(\hbar\chi)$, that is, proportional to the barrier width a, in the opaque rectangular barrier limit (as one can see from (3.15) and (3.29)). These times are *not* mean times but *mean-square fluctuations* in the tunnelling-time distribution because they are equal to $[D_{\rm dyn}\tau_{\rm tun}]^{1/2}$ where $D_{\rm dyn}\tau_{\rm tun}$ is the dynamical tunneling-time variance caused by the barrier influence only and defined by the equation $D_{\rm dyn}\tau_{\rm tun} = D\tau_{\rm tun} - Dt_+(0)$ with $D\tau_{\rm tun} = \langle\tau_{\rm tun}^2\rangle - \langle\tau_{\rm tun}\rangle^2$, $\langle\tau_{\rm tun}^2\rangle = \langle[t_+(a) - \langle t_+(0)\rangle]^2\rangle + Dt_+(0)$ (it was shown in [26, 27, 33–35]). Hence, they are not connected with the peak (or group) velocities of tunnelling particles but with the relevant tunneling *velocity distribution over the barrier region.*

In Figure 4 the dependences of the values of $\langle\tau_{\rm tun}(0,a)\rangle$ from a are presented for electronic wave packets and rectangular barriers with the same parameters as in [47] ($V_0 = 10$ eV; mean electron energies $E = 2.5, 5, 7.5$ eV with $\Delta k = 0.02$ A^{-1} (curves 1a, 2a, and 3a, resp.); energy $E = 5$ eV with $k = 0.04$ A^{-1} and 0.06 A^{-1} (curves 4a and 5a, resp.)). The curves $\langle\tau_{\rm tun}^{\rm Ph}\rangle$ corresponding to different energies and Δk merge practically into one curve 6. Since the dependence of $\langle\tau_{\rm tun}^{\rm Ph}\rangle$ from a is very weak, the dependence of $\langle\tau_{\rm tun}(0,a)\rangle$ from a is defined mainly by the dependence of $\langle t_+(0)\rangle$ from a (curves 1b–5b, correspondent to 1a–5a, resp.).

All these calculations manifest the negative values of $\langle t_+(0)\rangle$. Such "acausal" advance can be interpreted as a result of the superposition and interference of incoming and reflected waves. The reflected-wave packet extinguishes the back edge of the incoming-wave packet, and the larger is the barrier width, the larger is the part of the back edge of the incoming-wave packet which is extinguished by the superimposing reflected-wave packet, -up to the saturation when the contribution of the reflected wave packet becomes almost constant and independent from a. Since all $\langle t_+(0)\rangle$ are negative, the values of $\langle \tau_{\text{tun}}(0,a)\rangle$ are always positive and, moreover, larger than $\langle \tau_{\text{tun}}^{\text{Ph}}\rangle$, in accordance with (3.16).

All presented here results are obtained for transparent media (without absorption and dissipation). As it was theoretically demonstrated in [114] in nonrelativistic quantum mechanics, *HFE vanishes for barriers with absorption.* As it follows from [114], if one describes the absorption by adding the imaginary term $-iV_1$ $(V_1 > 0)$ to V_0, then for small absorptions, when $V_1 \ll V_0$ and $V_1 m^{1/2} v\chi a/[2(V_0 - E)]^{3/2} \ll 2$, HFE does not vanish and remains practically valid. This was confirmed experimentally for electromagnetic (microwave) tunnelling in [115].

Now we will consider wave packets with large momentum spreads and with the initial condition of the wave-packet center motion from the distant point $x = x_0$ (with $|x_0| \gg a$) at the instant $t = 0$ in order to analyze the influence of rather strong wave-packet time spreading before the entering into the barrier [117].

First, we will formulate explicitly initial conditions which take into account the irreversibility of the wave packet spreading. Further, we will propose a particularly convenient form of the O-R tunnelling time which allows the control of the accuracy in numerical calculations. Finally, we will present and explain the strong decrease shown by the tunneling time as the momentum spread increases, in such condition that we can say that the the Hartman and Fletcher effect is violated. We analyze the case of the 1D tunnelling of particles along the x axis through a rectangular potential barrier with height V_0, localized in the interval $[0,a]$. The chosen here stationary wave function for a particle with mass m and energy $E < V_0$ has the usual form (3.1). The time dependent wave packet $\psi(x,t)$ is formed with wave functions (3.1):

$$\Psi(x,t) = \int_0^\infty g_0(k)\psi(k,x)\exp\left(-\frac{iEt}{\hbar}\right)dk, \tag{3.41}$$

with the weight amplitude

$$g(k) = g_0(k)\exp\left[-i(k-k_0)x_0\right] = C\exp\left\{-\left[\frac{(k-k_0)}{2\Delta k}\right]^2 - i(k-k_0)x_0\right\}, \tag{3.42}$$

$C = [(2\pi)^{3/2}\Delta k]^{-1/2}$ being the normalization coefficient. Here, since in calculations the only subbarrier part of the wave packet was considered, integrating over k in (3.42) had to be made at the limits from 0 to $(2mV_0)^{1/2}/\hbar$ [117].

The initial wave function turns out to be

$$\Psi_{\text{in}}(x,t) \cong C_0 s_{\text{in}}^{-1/2}\exp\left\{-\frac{\left[x-(x_0+vt)\right]^2}{2s_{\text{in}}} + \frac{ik_0x - iE_0t}{\hbar}\right\}, \tag{3.43}$$

whose center transits from point $x = x_0$ at the instant $t = 0$, moves along axis x from the left to the right with the velocity v and in the absence of barrier crosses the point $x = 0$ at the instant $t_0 = -x_0/v$.

The flux density $j(x,t) = \mathrm{Re}[(i\hbar/2m)\Psi(x,t)\partial\Psi^*(x,t)/\partial x]$, as usually in the O-R method, has been considered as a function of the arrival times at point x, so that the mean time instant for particles passing through point $x = a$ (the mean time of the particle exit from the barrier), has been chosen as

$$\langle t(a)\rangle = \frac{\int_0^\infty t j(a,t)dt}{\int_0^\infty j(a,t)dt}, \tag{3.44}$$

while the mean time instant for particle passing through point $x = 0$ (the mean time of the particle entrance into the barrier) is defined by relation

$$\langle t_+(0)\rangle = \frac{\int_0^\infty t j_+(0,t)dt}{\int_0^\infty j_+(0,t)dt}, \tag{3.45}$$

where $j_+(0,t)$ represents the positive values of the flux density $j(0,t)$, corresponding, therefore, to particles moving through point $x = 0$ from left to right (entering the barrier), and the integrals have been limited to positive times only, due to initial condition (3.43).

The tunnelling time τ is then defined as $\tau = \langle t(a)\rangle - \langle t_+(0)\rangle$.

The flux density $j(a,t)$ contains the wave function

$$\Psi(a,t) = \int_0^\infty g(k)\varphi_T(ka)\exp\left(-\frac{iEt}{\hbar}\right)dk, \tag{3.46}$$

where $\varphi_T(k,a)$ is the value of the stationary wave function (3.1) at point $x = a$, corresponding to the transmitted wave. Using the usual expression for the amplitude of the transmitted wave, we can $\varphi_T(k,a)$ represent in the form

$$\varphi_T(k,a) = \begin{cases} \dfrac{2ik\chi}{D_<}, & E < V_0, \\ \dfrac{2iikq}{D_>}, & E > V_0, \end{cases} \tag{3.47}$$

where $q = [2m(E - V_0)]^{1/2}/\hbar$,

$$\begin{aligned} D_< &= (k^2 - \chi^2)\sinh(\chi a) + 2ik\chi\cosh(\chi a), \\ D_> &= (k^2 + q^2)\sin(qa) + 2ikq\cos(qa). \end{aligned} \tag{3.48}$$

The normalization integral in (3.45),

$$N_a = \int_0^\infty j(a,t)dt \tag{3.49}$$

can be evaluated using (3.42), (3.46), and (3.47):

$$N_a = \frac{\hbar}{m}\mathrm{Re}\left\{\int_0^\infty dk g(k)\psi_T(k,a)\int_0^\infty dk' k' g(k')\psi_T(k',a)\int_0^\infty \exp\left[-i\frac{\hbar}{2m}(k^2-k'^2)t\right]dt\right\}. \tag{3.50}$$

If x_0 is chosen sufficiently far from the left of the barrier, the contribution of $j(a,t)$ in the integral (3.49) from $t \le 0$ is negligible small. In this case, the lower integration limit over t in (3.50) can be taken as $-\infty$. Then the integration over time gives $2\pi(m/\hbar k)\delta(k-k')$ and the integral (3.50) can be cast in the form

$$N_a = 2\pi\int_0^\infty g_0^2(k)\left|\psi_T(k,a)\right|^2 dk, \tag{3.51}$$

where

$$\left|\psi_T(k,a)\right|^2 = \begin{cases} \dfrac{4k^2\chi^2}{4k^2\chi^2 + V_0^2\sinh^2(\chi a)}, & E < V_0, \\ \dfrac{4k^2q^2}{4k^2q^2 + V_0^2\sin^2(qa)}, & E > V_0. \end{cases} \tag{3.52}$$

Now we evaluate the integral

$$I_a = \int_0^\infty t j(a,t)dt. \tag{3.53}$$

Using (3.50) and (3.51), we can write

$$I_a = \frac{\hbar}{m}\mathrm{Re}\left\{\int_0^\infty dk g(k)\psi_T(k,a)\int_0^\infty dk' g(k')\psi_T(k',a)\int_0^\infty k' t\exp\left[-i\frac{\hbar}{2m}(k^2-k'^2)t\right]dt\right\}, \tag{3.54}$$

which can be set in the form

$$I_a = 2\pi\frac{m}{\hbar}\int_0^\infty g_0^2(k)\frac{1}{k}\left\{\psi_1(k,a)\psi_2'(k,a) - \psi_2(k,a)\psi_2'(k,a) - x_0\left|\psi(k,a)\right|^2\right\}dk, \tag{3.55}$$

where $\psi_1(k,a) \equiv \mathrm{Re}\,\psi_T(k,a)$, $\psi_2(k,a) \equiv \mathrm{Im}\,\psi_T(k,a)$. After differentiating (3.47) over k, one can see the following:

$$\psi_1\psi_2' - \psi_1'\psi_2 = \begin{cases} -8k^2\chi\dfrac{\left[k^2(k^2-\chi^2)\chi a - V_0^2\cosh(\chi a)\sinh(\chi a)\right]}{\left[4k^2\chi^2 + V_0^2\sinh^2(\chi a)\right]^2}, & E < V_0, \\ 8k^2q\dfrac{\left[k^2(k^2+q^2)qa - V_0^2\cos(qa)\sin(qa)\right]}{\left[4k^2q^2 + V_0^2\sin^2(qa)\right]^2}, & E > V_0. \end{cases} \tag{3.56}$$

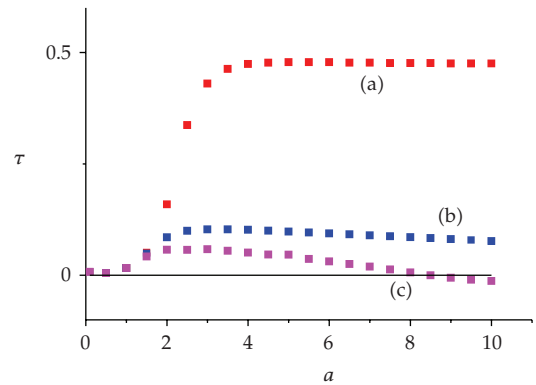

Figure 5: The tunneling time τ as a function of the barrier depth a, for different values of Δk: (a) $\Delta k = 0.01$ Å, (b) $\Delta k = 0.05$ Å, and (c) $\Delta k = 0.1$ Å.

From (3.51) and (3.56) it follows that, with an appropriate choice of x_0, the mean instant $\langle t(a)\rangle$ can be defined by the simple relation

$$\langle t(a)\rangle = \frac{m}{\hbar}\int_0^\infty g_0^2 \frac{1}{k}\left[\psi_1\psi_2' - \psi_2\psi_1' - x_0\left|\psi_T\right|^2\right]dk \Big/ \int_0^\infty g_0^2\left|\psi_T\right|^2 dk. \tag{3.57}$$

As regards the calculation of $\overline{\tau}_0$, the wave function entering the flux density $j(0,t)$ is

$$\psi(0,t) = \int_0^\infty g(k)\left[1 + A_R(k)\right]\exp\left(-\frac{iEt}{\hbar}\right)dk, \tag{3.58}$$

where

$$A_R(k) = \begin{cases} \dfrac{V_0\sinh(\chi a)}{D_<}, & E < V_0, \\ \dfrac{V_0\sin(qa)}{D_>}, & E > V_0. \end{cases} \tag{3.59}$$

Since for calculating $\langle t_+(0)\rangle$ we consider only the positive values of $j(0,t)$, the integrals $I_+ = \int_{-\infty}^{\infty} t j_+(0,t)dt$ and $N_+ = \int_{-\infty}^{\infty} j_+(0,t)dt$ in (3.45) can be obtained only numerically.

Figure 5 shows the results of the calculations of the tunnelling time τ as a function of the width a of the barrier, for electrons with energy $E_0 = 5$ eV through the rectangular barrier of potential with height 10 eV and $x_0 = -6/\Delta k$. We can see the manifestation of the HFE for $\Delta k = 0.01$ Å^{-1} (curve (a)), with the asymptotic behavior of the tunneling time approaching the constant value with increasing a. Curves (b) and (c) show, on the contrary, the strong decrease presented by the tunneling times when the wave packets are characterized by larger momentum spread $\Delta k = 0.05$ Å^{-1} and $\Delta k = 0.1$ Å^{-1}. For $\Delta k = 0.1$ Å^{-1} the tunneling time is even negative. The violation of the HFE is strongly evident.

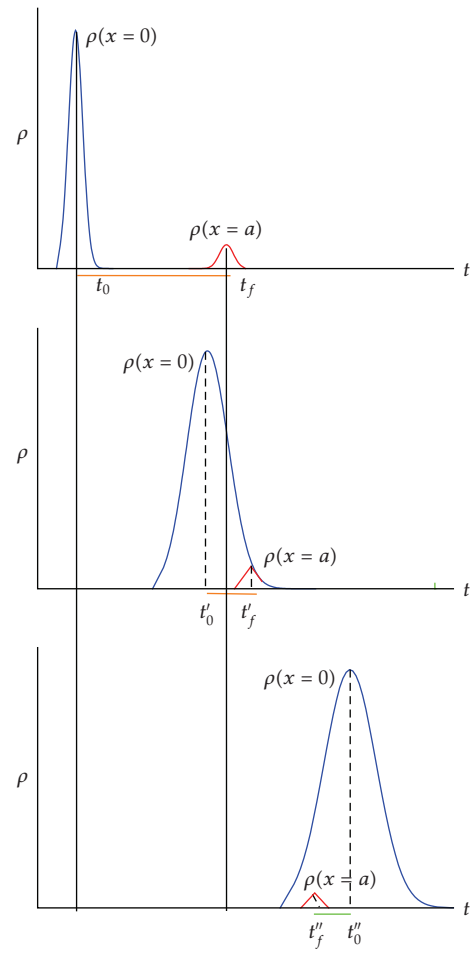

Figure 6: Qualitative description of the time shift between peaks t_f, t'_f, and t''_f, and t_0, t'_0, and t''_0 for the probability density $\rho(x=a,t)$ and $\rho(x=0,t)$ of a wave packet.

This effect can be explained with the following reasons [117]: the time spread of the Gaussian wave packet (3.43) is described by the relation

$$\Delta t \cong \Delta_0 t \left\{ 1 + \left[\frac{(\Delta v)^2 m}{2\hbar} \right]^2 t^2 \right\}^{1/2}, \tag{3.60}$$

where $\Delta_0 t \sim \hbar/\Delta E$, $\Delta v = \hbar \Delta k/m$. The value of Δt strongly increases in time t for large velocity spread Δv. So, the center of the initial wave packet during wave-packet approaching the barrier and increasing time t will appear even farther due to the essentially stronger increase of Δt with time t (proportional to Δk for large t) than the decrease of $|x_0|$, proportional to $1/\Delta k$. For the center of the transmitted wave packet in point $x = a$ such delay has to be considerably smaller because the value τ is smaller than the limited value $\overline{\tau^{\mathrm{ph}}}$ (τ^{ph} being the phase tunnelling time) due to the advance caused by the difference between $j_+(0,t)$ and $j_{\mathrm{in}}(0,t)$, j_{in} being $\mathrm{Re}[(i\hbar/2m)\Psi_{\mathrm{in}}(\partial\Psi^*_{\mathrm{in}}/\partial x)]$ (see, in particular, [26, 27]). So, one

can expect that the time interval between the transit of the center of the wave packet through the entrance and the exit of the barrier has to be smaller than in the case of the validity of the HFE, that is, for sufficiently large velocity spreads Δv becoming even negative.

Figure 2 shows qualitatively the behavior of the time shift of the outgoing wave packet with respect to the ingoing wave packet with different momentum spread Δk. t_0, t_0', t_0'' represent the time instants when the wave packet peak passes through the initial point $x = 0$ of the barrier. t_f, t_f', and t_f'' represent the transit time through the final point $x = a$ of the barrier.

In any case a point of the wave packet preparation has to be located at some finite distance from the barrier and therefore the wave packet center arrives to the barrier during a finite time interval. A wave packet is spreading during its motion and its width does always remain a finite one.

In conclusion, the O-R definition with strictly formulated initial conditions can be especially useful for investigations of the particle tunnelling accompanied by the quantum dissipation. A strong decrease of tunneling times, much more strong than in the case of the validity of the HFE, for wave packets with the large momentum spread can be easily explained by the sufficiently rapid spreading of such wave packets during the initial motion before the entrance into the barrier and also inside the barrier.

Some authors (see, e.g., [118–121]) have extended the study of particle tunneling phenomena to a completely relativistic case, using the Dirac equation. All these papers indicate the apparent superluminal tunneling through opaque barriers, showing a behavior similar to that of the HFE.

However, the complete review of the Dirac relativistic tunneling has to include the so-called Klein paradox when the reflection coefficient is greater than 1 and when the transmission coefficient has nonvanishing values (see, e.g., the papers [122–126]). Its origin is now usually described as the electropositron pair production for large potential step but it is not possible to develop a simple relationship between the *time-dependent* pair production process with a finite lifetime and the *time-independent* transmission coefficient in general. The problem of the Klein paradox and the Klein tunneling has to be studied in a self-consistent way and reviewed separately in another paper.

3.7. Tunneling through a Double Barrier

In this section, we confine ourselves by the approximation of not taking into account the multiple internal reflections between two separated barriers. Some words on such considering will be said at the end of Appendix F (before the subdivision, *the case of photon tunneling*).

Phase Time of Nonresonant Tunneling through Two Opaque Barriers

Now let us consider the stationary solution for 1D tunneling of a particle with mass m and kinetic energy $E = \hbar^2 k^2/2m = mv^2/2$, through two equal rectangular barriers with height V_0 ($V_0 > E$) and width a, the quantity l being the distance between them see Figure 7. The stationary Schroedinger equation is

$$-\frac{\hbar^2}{2}\frac{\partial^2}{\partial x^2}\psi(x) + V(x)\psi(x) = E\psi(x), \tag{3.61}$$

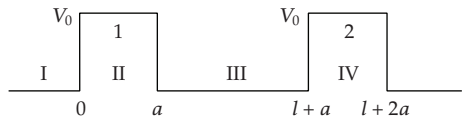

Figure 7: Tunneling through two successive potential berriers.

where $V(x) = 0$ outside the barriers and $V(x) = V_0$ inside the potential barriers. In various regions I $(x \le 0)$, II $(0 \le x \le a)$, III $(a \le x \le l+a)$, IV $(l+a \le x \le l+2a)$, and V $(x \ge l+2a)$, the solutions of (3.61) are the following:

$$\begin{aligned}
\psi_{\mathrm{I}} &= e^{ikx} + A_{1\mathrm{R}} e^{-ikx}, \\
\psi_{\mathrm{II}} &= \alpha_1 e^{-\chi x} + \beta_1 e^{\chi x}, \\
\psi_{\mathrm{III}} &= A_{1\mathrm{T}} \left[e^{ikx} + A_{2\mathrm{R}} e^{-ikx} \right], \\
\psi_{\mathrm{IV}} &= A_{1\mathrm{T}} \left[\alpha_2 e^{-\chi(x-l-a)} + \beta_2 e^{\chi(x-l-a)} \right], \\
\psi_{\mathrm{V}} &= A_{\mathrm{T}} e^{ikx}, \quad A_{\mathrm{T}} \equiv A_{1\mathrm{T}} A_{2\mathrm{T}},
\end{aligned} \tag{3.62}$$

where $\chi = [2m(V_0 - E)]^{1/2}/\hbar$, and quantities $A_{1\mathrm{R}}$, $A_{2\mathrm{R}}$, $A_{1\mathrm{T}}$, $A_{2\mathrm{T}}$, α_1, α_2, β_1, and β_2 are the reflection amplitudes, the transmission amplitudes, and the coefficients of the "evanescent" (decreasing) and "antievanescent" (increasing) waves for barriers 1 and 2, respectively. These 8 quantities can be easily obtained from 8 matching (continuity) conditions for the functions $\psi_{\mathrm{I,II,III,IV,V}}$ and their derivatives $d\psi_{\mathrm{I,II,III,IV,V}}/dx$ at points $x = 0, a, l+a, l+2a$. The obtained expressions for them for opaque barriers, when $\chi a \to \infty$, are

$$\begin{aligned}
A_{1\mathrm{R}} &\longrightarrow \frac{ik+\chi}{ik-\chi}, \qquad A_{1\mathrm{T}} \longrightarrow e^{-\chi a} e^{-ik(l+a)} A, \\
\alpha_1 &\longrightarrow \frac{2ik}{ik-\chi}, \qquad \beta_1 \longrightarrow e^{-2\chi a}(k - i\chi)\frac{\sin kl}{\chi} A, \\
A_{2\mathrm{R}} &\longrightarrow e^{2ik(l+a)} \frac{ik+\chi}{ik-\chi}, \qquad A_{2\mathrm{T}} \longrightarrow e^{2ik(l+a)} \frac{-4ik\chi}{(ik-\chi)^2}, \\
\alpha_2 &\longrightarrow e^{ik(l+a)} \frac{2ik}{ik-\chi}, \qquad \beta_2 \longrightarrow e^{ik(l+a)-2\chi a} \frac{-2ik(ik+\chi)}{(ik-\chi)^2},
\end{aligned} \tag{3.63}$$

where

$$A = \frac{2k\chi}{2k\chi \cos kl + \left(\chi^2 - k^2\right) \sin kl}. \tag{3.64}$$

From (3.63) one can derive that the phase tunneling time is

$$\tau_{\mathrm{tun}}^{\mathrm{Ph}} = \hbar \frac{\partial \arg \left[A_T e^{ik(l+2a)} \right]}{\partial E} \longrightarrow \frac{2}{v\chi}, \tag{3.65}$$

which is precisely the same as for one barrier and *does not depend* not only on the width a of the opaque barrier, but also on the distance l between two opaque barriers. This result is a striking generalization of the HFE, firstly obtained in [127]. It is important to stress that this result holds, however, for nonresonant tunneling, that is, for energies far from the resonances.

Esposito in [128] has generalized this result for the tunneling through an arbitrary number of finite rectangular opaque barriers and it has been shown that the total tunneling phase time depends neither on the barrier thickness nor on the interbarrier separation. It has been also shown the independence of the phase transit time (for nonresonant tunneling).

Now, we will consider the cases of the resonances (between two barriers) and the influence of not very far resonances.

The Resonant Tunneling

If one takes two arbitrary (not necessarily opaque) barriers (cf. Figure 7), the amplitude of the transmitted wave in this case is defined by the formula [88, 129]

$$A_T(k) = \frac{\exp(-2ika)}{D(k)}, \tag{3.66}$$

where

$$\begin{aligned} D(k) = {} & \cosh^2(\chi a) + \frac{1}{4}\sinh^2(\chi a)\left[\sigma^2\cos(2kl) - \delta^2\right] \\ & + i\sinh(\chi a)\left[\delta\cosh(\chi a) + \frac{1}{4}\sigma^2\sinh(\chi a)\sin(2kl)\right], \end{aligned} \tag{3.67}$$

$\delta = (\chi^2 - k^2)/k\chi$, and $\sigma = (k^2 + \chi^2)/k\chi$. The dimensionless constants δ and σ are connected by the relation

$$\sigma^2 = \delta^2 + 4. \tag{3.68}$$

A resonance is characterized by the condition [88, 120]

$$\left|A_T(k)\right|^2 = 1, \tag{3.69}$$

that is, the double barrier becomes totally transparent (without any reflections).

It is easy to see (cf. (3.67), and also [88, 129]) that the values of the wave number k for which condition (3.69) is satisfied can be found from the equation

$$\cot(kl) = -\frac{1}{2}\delta\tan(\chi a). \tag{3.70}$$

We can show that from (3.70) it is possible to find out also the values of parameters a, m, and V_0, at the resonance. Indeed, from (3.67) and (3.70) it follows that at a resonance it is

$$\left|D(k)\right|^2 = 1. \tag{3.71}$$

On introducing the functions

$$u = \cosh^2(\chi a) - \frac{1}{4}\delta^2 \sinh^2(\chi a),$$
$$v = \delta \cosh(\chi a)\sinh(\chi a), \tag{3.72}$$
$$w = \frac{1}{4}\sigma^2 \sinh^2(\chi a)$$

and using (3.68), one can infer that these functions are connected by the relation

$$u^2 + v^2 = (1 + w)^2. \tag{3.73}$$

Then, by using functions (3.72), the denominator of (3.67) can be written in the following form:

$$D = u + w\cos(2kl) + i[v + w\sin(2kl)]. \tag{3.74}$$

It follows from (3.74) and (3.73) that

$$|D|^2 = 1 + 2w\left[1 + w + u\cos(2kl) + v\sin(2kl)\right], \tag{3.75}$$

and condition (3.71) gets transformed into

$$1 + w + u\cos(2kl) + v\sin(2kl) = 0. \tag{3.76}$$

Taking (3.68) into account, we can write the term $1 + w$ of (3.76) in the following form:

$$1 + w = \cosh^2(\chi a) + \frac{1}{4}\delta^2\sinh^2(\chi a), \tag{3.77}$$

and this (last) equation can be easily transformed, afterwards, into (3.70). There are no other solutions to (3.76). Hence, (3.70), as well as (3.69) or (3.71) is a general resonance condition. In the region of a resonance, if we limit ourselves to the first two terms of the expansion of (3.67) into a series of powers of $E - E_r$ (quantity E_r being the resonance value of energy E), we obtain

$$D(k) = D_r + C_r\left(E - E_r\right), \tag{3.78}$$

where $D_r = D(k_r)$, $k_r = \sqrt{2mE_r}/\hbar$,

$$C_r = \frac{m}{\hbar^2 k}D_r', \tag{3.79}$$

and the index *prime* defines the derivative with respect to k. We can rewrite (3.78) in the following form:

$$D(k) = C_r\left(\frac{E - E_r + D_r C_r^*}{|C_r|^2}\right). \tag{3.80}$$

It follows from (3.70) that at the resonance it is

$$\begin{aligned} \cos(2k_r l) &= -\frac{u_r}{(1+w_r)}, \\ \sin(2k_r l) &= -\frac{v_r}{(1+w_r)}, \end{aligned} \tag{3.81}$$

where u_r, v_r and w_r are the values at $E = E_r$ of the functions u, v, and w, respectively. On inserting (3.81) into (3.74), we obtain

$$D(k_r) = \frac{(u_r + iv_r)}{(1+w_r)}. \tag{3.82}$$

Differentiating (3.74) with respect to k, and inserting the result into (3.81), we get

$$D'(k_r) = \left[u_r' - \frac{(u_r w_r' - 2l v_r w_r)}{(1+w_r)}\right] + i\left[v_r' - \frac{(v_r w_r' - 2l u_r w_r)}{(1+w_r)}\right]. \tag{3.83}$$

Then, on using (3.73), (3.79), (3.82), (3.83), and relation

$$uu' + vv' = (1+w)w', \tag{3.84}$$

which follows from (3.73), one finds that at the resonance

$$D_r C_r^* = i\frac{m}{\hbar^2 k_r}\left[\frac{u_r' v_r - u_r v_r'}{1+w_r} + 2l w_r\right], \tag{3.85}$$

where (by having recourse to (3.73), (3.79), (3.83), and (3.84))

$$|C_r|^2 = \left(\frac{m}{\hbar^2 k_r}\right)^2\left[(u_r')^2 + (v_r')^2 - (w_r')^2 + 4l\frac{u_r' v_r - u_r v_r'}{1+w_r}w_r + 4l^2 w_r^2\right]. \tag{3.86}$$

From (3.73) and (3.84) one gets

$$u'^2 + v'^2 - w'^2 = \frac{(u'^2 + v'^2)^2}{(1+w)^2}, \tag{3.87}$$

while, from (3.86) and (3.87), one obtains

$$|C_r|^2 = \left(\frac{m}{\hbar^2 k_r}\right)^2 \left(\frac{u_r' v_r - u_r v_r'}{1 + w_r} + 2l w_r\right). \tag{3.88}$$

By differentiating the functions (3.72) with respect to k, one can see that

$$u_r' v_r - u_r v_r' = \frac{1}{\chi_r} \frac{(1 + w_r)}{\left[\chi_r k_r a + 2\chi_r l w_r + \sigma_r^2 \cosh(\chi_r a)\sinh(\chi_r a)\right]}, \tag{3.89}$$

where χ_r, δ_r, and σ_r are the values of χ, δ, and σ at $E = E_r$. On inserting (3.89) into (3.85) and (3.88), we can then rewrite (3.80) in the noteworthy form

$$D(k) = C_r(E - E_r + i\beta), \tag{3.90}$$

where

$$\beta = \frac{\hbar^2 k_r \chi_r}{m} \left[\chi_r k_r a + 2\chi_r l w_r + \sigma_r^2 \cosh(\chi_r a)\sinh(\chi_r a)\right]^{-1}, \tag{3.91}$$

so that (3.80), (3.85), (3.88), and (3.90) yield

$$|C_r|^2 = \frac{1}{\beta^2}. \tag{3.92}$$

Finally, by inserting (3.90) into (3.66) and taking account of (3.92), we obtain that near a resonance it holds in general

$$|A_T(k)|^2 = \frac{\beta^2}{(E - E_r)^2 + \beta^2}. \tag{3.93}$$

which corresponds to nothing but a Breit and Wigner formula. In other words, one verifies that the Breit-Wigner's formula has a general validity for our (1D) tunneling near a resonance.

Let us start from the first two parts of relation (3.65), that is, concretely from the

$$\tau_{\text{tun}}^{\text{Ph}} = \frac{\hbar d \arg\left\{A_T \exp\left[ik(2a + l)\right]\right\}}{dE}. \tag{3.94}$$

By using (3.66) and (3.76), we can rewrite it in the following form:

$$\tau_{\text{tun}}^{\text{Ph}} = \frac{m}{\hbar k}\left[l - \frac{(D_1 D_2' - D_2 D_1')}{|D|^2}\right], \tag{3.95}$$

where

$$D_1 = u + w\cos(2kl), \qquad D_2 = v + w\sin(2kl) \tag{3.96}$$

(the index *prime* denoting again the derivative with respect to k). On inserting (3.96) into (3.95) and using (3.73), we get *in general* for the total tunneling phase time the remarkable formula:

$$\tau_{\text{tun}}^{\text{Ph}} = \frac{m}{\hbar k}\frac{P}{|D|^2}, \tag{3.97}$$

where

$$P = (1+2w)l + u'v - uv' + \left(u'w - uw'\right)\sin(2kl) + \left(vw' - v'w\right)\cos(2kl). \tag{3.98}$$

It should be noticed that (3.97) holds in general *for any (resonant and/or nonresonant) tunneling time* through two barriers. From (3.97), (3.98), (3.71), and (3.85), it follows that that the tunneling phase time at a resonance $\tau_{\text{tun},r}^{\text{Ph}}$ is given by the following expression:

$$\tau_{\text{tun},r}^{\text{Ph}} = \frac{m}{\hbar k\chi}\left[\sigma^2\cosh(\chi a)\sinh(\chi a) + \delta k a + (1+2w)\chi l\right]. \tag{3.99}$$

It follows from (3.65) and (3.66) that

$$\tau_{\text{tun}}^{\text{Ph}} = \hbar d \arg\left[\exp(ikl)/D\right]/dE. \tag{3.100}$$

Inserting (3.90) into (3.100), we get

$$\tau_{\text{tun}}^{\text{Ph}} = \hbar\frac{d}{dE}\arg\left[\frac{1}{C_r}\frac{\exp(ikl)}{E - E_r + i\beta}\right], \tag{3.101}$$

so that, near a resonance, the behavior of the tunneling phase time in terms of the energy is represented by the interesting following formula:

$$\tau_{\text{tun}}^{\text{Ph}} \cong \frac{m}{\hbar k}l + \hbar = \frac{\beta}{\left(E - E_r\right)^2 + \beta^2} \tag{3.102}$$

holding for any resonant tunneling through the two barriers. The first term represents the time associated with the particle free flight over the distance l between the two barriers; while the second term is the time delay caused by the quasibound state assumed by the particle in such an intermediate region.

Esposito in [128] has shown that for the arbitrary number of finite rectangular opaque barriers the resonant energy does not depend on the number of barriers.

The Dependence of the Tunneling Phase Time on the Width of the Setup, Far from the Resonances

When releasing the above condition, we have found in this paper a more complicate expression, given by formulae (3.65), (3.66), and (3.75) above. Anyway, from (3.75) and (3.72) it follows that, for opaque barriers, when $\chi a \gg 1$, it holds

$$|D|^2 \approx \frac{1}{32}\sigma^2 \exp(4\chi a)\left[\frac{1}{4}\sigma^2 + \left(1 - \frac{1}{4}\delta^2\right)\cos(2kl) + \delta\sin(2kl)\right]. \tag{3.103}$$

Differentiating the functions u, v, and w with respect to k, one can see that

$$\begin{aligned} u'v - uv' &= \frac{1}{\chi}(1+w)\left[\sigma^2\cosh(\chi a)\sinh(\chi a) + \delta k a\right], \\ u'w - uw' &= \frac{2}{\chi}\left[\delta(u+w) + \frac{1}{4}\sigma^2\cosh(\chi a)\sinh(\chi a)\right], \\ vw' - v'w &= \frac{1}{\chi}w\left[(4-\delta^2)\cosh(\chi a)\sinh(\chi a) - \delta k a\right]. \end{aligned} \tag{3.104}$$

Therefore, from (3.72), (3.98), (3.103), and (3.104), we get that, still for $\chi a \gg 1$,

$$P \approx \frac{1}{16}\sigma^2 \exp(4\chi a)\left[\frac{\sigma^2}{4} + \left(1 + \frac{\delta^2}{4}\right)\cos(2kl) + \delta\sin(2kl)\right] + \frac{1}{8}\sigma^2 l \exp(2\chi a) \tag{3.105}$$

and far from the resonances,

$$\tau_{\text{tun}}^{\text{Ph}} \approx \frac{2m}{\hbar k\chi} + 4\frac{m}{\hbar k} l \exp(-2\chi a)\left[\frac{\sigma^2}{4} + \left(1 + \frac{\delta^2}{4}\right)\cos(2kl) + \delta\sin(2kl)\right]^{-1}, \tag{3.106}$$

Namely, when χa increases, the second term in (3.106) decreases as $\exp(-2\chi a)$; while, at the limit when $\chi a \to \infty$, (3.106) goes into the right-hand side of (3.65).

Of course, our result (3.106) does not hold only for particles but—as well-known (see Section 2.4 and [59–62])—also for photons. This can explain the results explaining the experimental fact that $\tau_{\text{tun}}^{\text{Ph}}$ has been actually observed to increase (very slowly, almost linearly, and probably with very small oscillations) on l, which was recently found by Longhi et al. [130] and by Nimtz [131].

We would like now to underline that, in quantum experiments, the tunneling-time of a nonrelativistic particle is expected to be practically measurable for large χa values only (but not too large, of course, to avoid that the tunneled particles are too few). Therefore, in order to be able to reproduce theoretically any experimental results, it seems to be necessary studying the behavior of the transmission coefficient for large χa. Indeed, from (3.1) and (3.38) it follows that

$$|A_T|^2 \approx 32\frac{1}{\sigma^2}\exp(-4\chi a)\left[\frac{1}{4}\sigma^2 + \left(1 - \frac{1}{4}\delta^2\right)\cos(2kl) + \delta\sin(2kl)\right]^{-1}, \tag{3.107}$$

which shows that, with increasing χa, the transmission coefficient square $|A_T(k)|^2$ decreases as $\exp(-4\chi a)$, that is, even more quickly than the second term in the right-hand side of (3.106). Consequently, from the experimental point of view, *practically no tunneling can take place for those values of* χa which *make negligible the mentioned second term* in (3.106). Moreover, for the values of χa for which the tunneling probability is experimentally significant, the second term in the right-hand side of (3.106), which is slowly exponentially decreasing with χa, results to depend very weakly on the distance l between the barriers: more precisely, it will depend almost linearly (but very slightly) on l, with even slighter oscillations.

Some Words on Other Papers on Tunneling through a Double Barrier

In addition to [127, 129], in [132, 133] there are studies of the numerical and asymptotic analytical expressions of the wave packets travelled through a double-barrier potential (also not taking into account the multiple internal reflections between two separated barriers). In particular, in [132] the resonant and nonresonant dynamics of a Gaussian quantum wave packet tunneling through a double-barrier system has been analyzed as a function of the initial spectrum characteristics and of the potential parameters. The behavior of the tunneling time shows that there are situations where the Hartman effect occurs, while, when the resonances are dominant, the tunneling time can become very large and the HFE does not take place [133].

The authors of [134] have studied the relativistic quantum mechanical problem of a Dirac relativistic tunneling through two successive barriers, and have shown that in the limit of opaque barriers the generalized HFE also occurs. However, their results for the phase and dwell times show an almost linear increasing with the separation between the barriers and tend to saturate only when the barriers become extremely opaque.

3.8. Particle Tunneling through Three-Dimensional Barriers

Introduction

The particle tunneling through a three-dimensional (3D) barrier has been usually studied in a simplified way (in the framework of the WKB approximation and with using only the elementary time-dependent description) in applications for some concrete tasks such as α-decay (see, e.g., [135–140]). Here, we intend to study the nonrelativistic particle tunneling through a 3D potential barrier with a spherical symmetry without WKB approximation, following [141, 142]. We will consider, in the central part of the system, also the presence of a spherical well. We will refer to the various regions in this way: region (I), with $r > R_2$, represents the external region of null potential, region (II), delimited by R_1 and R_2, is the barrier region, and the internal region (III), with $r < R_1$, is the well. We will describe the impact of the particles with this potential as a sequence of two successive processes: in the first stage we think to an ingoing wave packet impinging from outside on the barrier, producing a reflected wave in the external region (I), tunneling through the barrier, and finally penetrating in the well where it is represented by an ingoing mode. In the second phase, we will consider the presence of an outgoing wave from the well (III), which, after a reflection against the internal side of the barrier, tunnels through the barrier and produces, finally, an outgoing mode in the external region (I). The scheme of these processes is sketched in Figure 8.

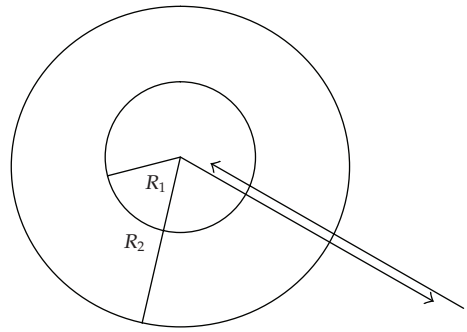

Figure 8: Schematic description of the impact process.

Multiple reflections inside the barrier can also develop, but, on account of the fact that this phenomenon is more sophisticated and usually present at a low level, we will neglect it in this first approach, considering them only afterwards. The present analysis is a first step for the self-consistent time-dependent study of the emission of protons or alpha-particles from a spherical compound nucleus, or of the photon emission from a glass–air spherical system.

3.8.1. Model Picture

(i) Impact from Outside

We will start by considering an initial wave packet, defined in the outer region (I) by means of a superposition of ingoing spherical waves, and moving from outside towards the barrier region (II) where the potential has value V_1:

$$r\Psi_0^{(1)}(r,t) = \int_0^{V_1} dE\, g(E) e^{-ikr-iEt/\hbar}, \quad (R_2 \le r < \infty), \tag{3.108}$$

where k and $E = \hbar^2 k^2/2m$ are the wave number and the kinetic energy, respectively.

A section of the potential along the r-axis is shown in Figure 9.

When the wave encounters the barrier, it is partially reflected with wave function:

$$r\Psi_R^{(1)}(r,t) = \int_0^{V_1} dE\, g(E) A_R^{(\mathrm{ex})} e^{-ikr-iEt/\hbar}, \quad (R_2 \le r < \infty). \tag{3.109}$$

At the same time, the wave packet tunnels through the barrier in region (II), where it is represented by the following function:

$$r\Psi_{\mathrm{II}}^{(1)}(r,t) = \int_0^{V_1} dE\, g(E) \left(\alpha_1 e^{-\chi r} + \beta_1 e^{\chi r}\right) e^{-iEt/\hbar}, \quad (R_1 \le r < R_2), \tag{3.110}$$

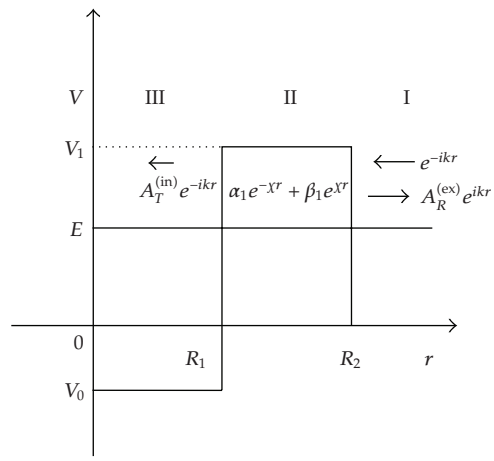

Figure 9: Schematic description of the impact of the wave with the barrier from outside.

where $\chi = \sqrt{2m(V_1 - E)/\hbar^2}$. After the tunnelling through the barrier, the wave penetrates inside the well and in region (III) we can write

$$r\Psi_T^{(1)}(r,t) = \int_0^{V_1} dE\, g(E) A_T^{(\text{in})} e^{-iKr - iEt/\hbar}, \quad (0 < r < R_1), \tag{3.111}$$

where $K = \sqrt{2m(E + V_0)/\hbar^2}$ is the wave number inside the well.

In general, we can say that the wave packet is described by the expression

$$r\Psi_J^{(1)}(r,t) = \int_0^{V_1} dE\, g(E) \Phi_J^{(1)}(k,r) e^{-iEt/\hbar}, \tag{3.112}$$

where the index J is I, II, III, 0, T or R depending on the particular mode considered, $\Phi_J^{(1)}$ being the stationary wave functions:

$$\begin{aligned} \Phi_{\text{I}}^{(1)} &= e^{-ikr} + A_R^{(\text{ex})} e^{ikr}, \quad R_2 \le r < \infty, \\ \Phi_{\text{II}}^{(1)} &= \alpha_1 e^{-\chi r} + \beta_1 e^{\chi r}, \quad R_1 \le r \le R_2, \\ \Phi_{\text{III}}^{(1)} &= A_T^{(\text{in})} e^{-iKr}, \quad 0 < r \le R_2. \end{aligned} \tag{3.113}$$

Furthermore, $g(E)$ is a normalized amplitude weight factor such as

$$\int_0^{V_1} dE \left| g(E) \right|^2 = N, \quad N < \infty. \tag{3.114}$$

We normalize (3.1) by the following condition:

$$4\pi \int_{R_2}^{\infty} r^2\, dr \left|\Psi_0^{(1)}(r,t)\right|^2 = 1, \tag{3.115}$$

which is strictly possible only in the limit V_1 going to infinity. For the sake of simplicity we neglect the contribution of the energy above the barrier, considering only cases where $E < V_1$.

All the previous formulas are written in the case of orbital momentum $l = 0$, and the extension of this theory to the general case with $l > 0$ can be obtained by replacing the terms $\exp(\pm ikr)$ and $\exp(\pm iKr)$ with the spherical Hanckel function of the first and second kinds.

In addition, we have that A_R^{ex}, α_1, β_1, and $A_T^{(\text{in})}$ are, respectively, the external reflection amplitude factor, the evanescent and antievanescent wave amplitude factors during the first tunnelling and the internal transmission amplitude factor. Their analytical expression can be found by imposing the continuity condition for both the stationary wave functions and their first derivatives at the points $r = R_2$ and $r = R_1$, finding for the external reflection coefficient the expression:

$$A_R^{\text{ex}} = -e^{-2ikR_2} a_R^{\text{ex}}, \quad a_R^{\text{ex}} = \frac{(\chi + ik)(iK + \chi) + e^{-2\chi(R_2-R_1)}(\chi - ik)(iK - \chi)}{(\chi - ik)(iK - \chi) + e^{-2\chi(R_2-R_1)}(\chi + ik)(iK + \chi)}, \tag{3.116}$$

which, in the limit $\chi(R_2 - R_1)$ tending to infinity, becomes $-e^{-2ikR_2}((\chi + ik)/(\chi - ik))$.

For the internal transmission coefficient we get

$$A_T^{\text{in}} = e^{-ikR_2 + i\kappa R_1} a_T^{\text{in}}, \quad a_T^{\text{in}} = \frac{4ik\chi e^{-\chi(R_2-R_1)}}{(\chi - ik)(iK - \chi) + e^{-2\chi(R_2-R_1)}(\chi + ik)(iK + \chi)} \tag{3.117}$$

that, in the same limit as before, tends to 0.

The calculation of the probability fluxes can be used for controlling if the quantities (3.116) and (3.117) satisfy the conservation law.

The fluxes $j_{\text{I}}^{(1)}$, $j_{\text{II}}^{(1)}$, and $j_{\text{III}}^{(1)}$ in the three regions I, II, and III are in general defined as

$$j_n^{(1)} = \frac{i\hbar}{2m}\left(r\Psi_n^{(1)} \frac{d}{dr}\left(r\Psi_n^{(1)*}\right) - r\Psi_n^{(1)*} \frac{d}{dr}\left(r\Psi_n^{(1)}\right)\right), \quad n = \text{I}, \text{II}, \text{III}. \tag{3.118}$$

One can use the approximation

$$\int_0^{\infty} j_n^{(1)}(r,t)dt \approx \int_{-\infty}^{\infty} j_n^{(1)}(r,t)dt. \tag{3.119}$$

For quasimonochromatic wave packet centred around the value $E = \langle E \rangle$, that is, when the factor $|g(E)|^2$ is a delta function $\delta(E - \langle E \rangle)$, with $\langle E \rangle$ in the open interval $(0, V_1)$, one obtains

$$\begin{gathered}
\frac{\int_{-\infty}^{\infty} j_R^{(1)}(r,t)dt}{\int_{-\infty}^{\infty} j_0^{(1)}(r,t)dt} \approx \left|A_R^{(\mathrm{ex})}(\langle E \rangle)\right|^2, \\
\frac{\int_{-\infty}^{\infty} j_{\mathrm{II}}^{(1)}(r,t)dt}{\int_{-\infty}^{\infty} j_0^{(1)}(r,t)dt} \approx \left|\alpha_1(\langle E \rangle)e^{-\chi(\langle E \rangle)r} + \beta_1(\langle E \rangle)e^{\chi(\langle E \rangle)r}\right|^2, \\
\frac{\int_{-\infty}^{\infty} j_{\mathrm{III}}^{(1)}(r,t)dt}{\int_{-\infty}^{\infty} j_0^{(1)}(r,t)dt} \approx \frac{K(\langle E \rangle)}{k(\langle E \rangle)}\left|A_T^{(\mathrm{in})}(\langle E \rangle)\right|^2.
\end{gathered} \tag{3.120}$$

Hence, from the conservation law for the probability fluxes

$$\int_{-\infty}^{\infty} j_0^{(1)}(r,t)dt = \int_{-\infty}^{\infty} j_R^{(1)}(r,t)dt + \int_{-\infty}^{\infty} j_T^{(1)}(r,t)dt \tag{3.121}$$

one obtains

$$\left|A_R^{(\mathrm{ex})}(\langle E \rangle)\right|^2 + \frac{K(\langle E \rangle)}{k(\langle E \rangle)}\left|A_T^{(\mathrm{in})}(\langle E \rangle)\right|^2 = 1, \tag{3.122}$$

that for $V_0 = 0$ becomes

$$\left|A_R^{(\mathrm{ex})}(\langle E \rangle)\right|^2 + \left|A_T^{(\mathrm{in})}(\langle E \rangle)\right|^2 = 1. \tag{3.123}$$

The transmission probability through the barrier from outwards is represented by the last of expressions (3.120).

The phase times $\tau_T^{\mathrm{ph(in)}}$ and $\tau_R^{\mathrm{ph(ex)}}$ can be defined as the evident generalization of the following 1D definitions:

$$\begin{gathered}
\tau_T^{\mathrm{ph(in)}} = \hbar\frac{\partial\left(\arg\left(A_T^{(\mathrm{in})}(E)e^{-ikR_1}\right)\right)}{\partial E} - \frac{-R_1}{v} = \hbar\frac{\partial\left(\arg\left(a_T^{(\mathrm{in})}(E)\right)\right)}{\partial E}, \\
\tau_R^{\mathrm{ph(ex)}} = \hbar\frac{\partial\left(\arg\left(A_R^{(\mathrm{ex})}(E)e^{-i\kappa R_2}\right)\right)}{\partial E} - \frac{-R_2}{v} = \hbar\frac{\partial\left(\arg\left(a_R^{(\mathrm{ex})}(E)\right)\right)}{\partial E},
\end{gathered} \tag{3.124}$$

where $v = \hbar k/m$. The quantities $\tau_R^{\mathrm{ph(ex)}}$ and $\tau_R^{\mathrm{ph(ex)}}$, in the limit $\chi(R_2 - R_1)$ approaching infinity, go to $2/v\chi$ and 0, respectively.

So, we can see from the previous expressions, for analogy with the same 1D quantities, the manifestation of the HFE effect and also the absence of dependence on the geometrical characteristics of the barriers R_1 and R_2.

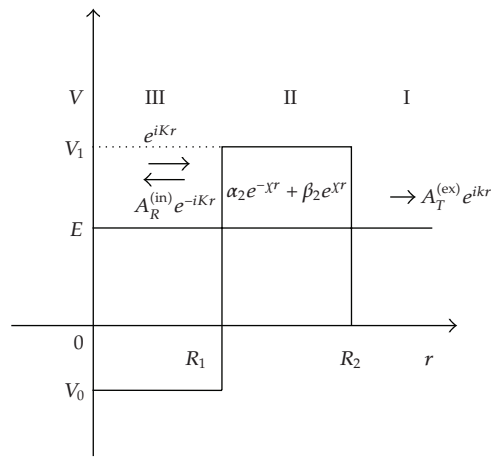

Figure 10: Schematic view of the second phase of the scattering process, the emission from inside.

(ii) Emission from the Barrier

Now we will study the evolution of a wave coming out from the central core of the system, with a wave function given by

$$r\Psi^{(\mathrm{in})}(r,t) = \int_0^{V_1} dE\, G(E) e^{iKr - iEt/\hbar}, \quad (0 \le r \le R_1) \tag{3.125}$$

constructed from the overlapping stationary solutions propagating in the positive r-direction from the well region. In Figure 10, the scheme of the various waves is presented.

When the wave impinges the barrier from inside, a reflected wave is formed, whose wave function is

$$r\Psi_R^{(2)}(r,t) = \int_0^{V_1} dE\, G(E) A_R^{(\mathrm{in})} e^{-iKr - iEt/\hbar}, \quad (0 \le r \le R_1). \tag{3.126}$$

Afterwards, in the region (II) of the barrier a system of evanescent and antievanescent waves develops

$$r\Psi_{\mathrm{II}}^{(2)}(r,t) = \int_0^{V_1} dE\, G(E) \left(\alpha_2 e^{-\chi r} + \beta_2 e^{\chi r}\right) e^{-iEt/\hbar}, \quad (R_1 \le r \le R_2), \tag{3.127}$$

while in the outer region a propagating wave will exist

$$r\Psi_{\mathrm{I}}^{(2)}(r,t) = \int_0^{V_1} dE\, G(E) A_T^{(\mathrm{ex})} e^{ikr - iEt/\hbar}, \quad (R_2 \le r < \infty). \tag{3.128}$$

As in the previous case we have that $G(E)$ is a normalized amplitude weight factor, and $A_R^{(\mathrm{in})}$, α_2, β_2, and $A_T^{(\mathrm{ex})}$ are the internal reflection, the evanescent and antievanescent and the

external transmission amplitude factors, respectively. With calculations similar to that of the first part we obtain the explicit expressions of the amplitude factors as follows:

$$A_R^{\text{in}} = -e^{-2ikR_1} a_R^{\text{in}}, \quad a_R^{\text{in}} = \frac{(\chi + iK)(ik - \chi) + e^{2\chi(R_1-R_2)}(\chi - iK)(ik + \chi)}{(\chi - iK)(ik - \chi) + e^{2\chi(R_1-R_2)}(\chi + iK)(ik + \chi)}, \tag{3.129}$$

$$A_T^{\text{ex}} = e^{-ikR_2+iKR_1} a_T^{\text{ex}}, \quad a_T^{\text{ex}} = \frac{4iK\chi e^{-\chi(R_2-R_1)}}{(\chi - iK)(ik - \chi) + e^{-2\chi(R_2-R_1)}(\chi + iK)(ik + \chi)}. \tag{3.130}$$

Also in this case we can demonstrate the conservation of the current fluxes

$$\left|A_R^{(\text{in})}(\langle E\rangle)\right|^2 + \frac{k(\langle E\rangle)}{K(\langle E\rangle)}\left|A_T^{(\text{ex})}(\langle E\rangle)\right|^2 = 1, \tag{3.131}$$

and we can introduce the phase times:

$$\begin{aligned} \tau_T^{\text{ph(ex)}} &= \hbar\frac{\partial\left(\arg\left(A_T^{(\text{ex})}(E)e^{-ikR_1}\right)\right)}{\partial E} - \frac{-R_1}{v} = \hbar\frac{\partial\left(\arg\left(a_T^{(\text{ex})}(E)\right)\right)}{\partial E}, \\ \tau_R^{\text{ph(in)}} &= \hbar\frac{\partial\left(\arg\left(A_R^{(\text{in})}(E)e^{-i\kappa R_2}\right)\right)}{\partial E} - \frac{-R_1}{v} = \hbar\frac{\partial\left(\arg\left(a_R^{(\text{in})}(E)\right)\right)}{\partial E}, \end{aligned} \tag{3.132}$$

with $v = \hbar k/m$. The quantities $\tau_T^{\text{ph(ex)}}$ and $\tau_R^{\text{ph(in)}}$, in the limit $\chi(R_2 - R_1)$ approaching infinity, go to $(1/v + 1/V)/\chi$ (with $V = \hbar K/m$) and 0, respectively.

(iii) Scattering Matrix

Finally, we will connect the two mechanisms of scattering described above in one single scattering event (see Figure 11), introducing the matrix of scattering S and considering the multiple reflections inside the potential well.

For this purpose we describe the stationary wave functions in the various regions as follows:

$$\begin{aligned} \Psi_{\text{I}}(k,r) &= e^{-ikr} - Se^{ikr}, \\ \Psi_{\text{II}}(k,r) &= \alpha e^{-\chi r} + \beta e^{\chi r}, \\ \Psi_{\text{III}}(k,r) &= A\left(e^{-iKr} - e^{iLr}\right). \end{aligned} \tag{3.133}$$

We can now find S, α, β, and A by connecting the various expression of the wave function and its derivative in $r = R_1$ and $r = R_2$.

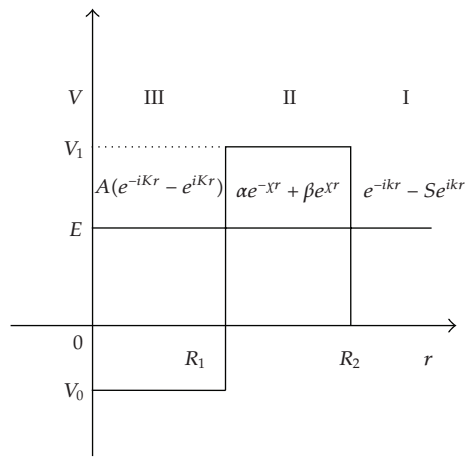

Figure 11: Schematic view of the whole scattering process.

Imposing the continuity conditions we have

$$S = e^{-2ikR_2} \frac{(K\cos KR_1 + \chi \sin KR_1)(\chi + ik) + e^{2\chi(R_1-R_2)}(K\cos KR_1 - \chi\sin KR_1)(\chi - ik)}{(K\cos KR_1 + \chi \sin KR_1)(\chi - ik) + e^{2\chi(R_1-R_2)}(K\cos KR_1 - \chi\sin KR_1)(\chi + ik)}, \tag{3.134}$$

$$A = e^{-2ikR_2} \frac{2ik\chi e^{2\chi(R_1-R_2)}}{(K\cos KR_1 + \chi \sin KR_1)(\chi + ik) + e^{2\chi(R_1-R_2)}(K\cos KR_1 - \chi\sin KR_1)(\chi + ik)}. \tag{3.135}$$

One can easily see that $|S| = 1$, and that

$$A = \frac{A_T^{(\mathrm{in})}}{1 + A_R^{\mathrm{in}}}, \tag{3.136}$$

$$S = -A_R^{(\mathrm{ex})} - iAA_T^{(\mathrm{ex})} = -A_R^{(\mathrm{ex})} + \frac{A_T^{(\mathrm{ex})} A_T^{(\mathrm{in})}}{1 + A_R^{(\mathrm{in})}}. \tag{3.137}$$

The physical meaning if the term $1/(1 + A_R^{\mathrm{in}})$ is directly connected to the presence of an infinite sequence of multiple internal reflections that can be described by the stationary wave functions

$$\begin{aligned} A_T^{(\mathrm{in})}\left(1 - A_R^{(\mathrm{in})} + \left(A_R^{(\mathrm{in})}\right)^2 - \left(A_R^{(\mathrm{in})}\right)^3 + \cdots\right)e^{-ikr} &= \frac{A_T^{(\mathrm{in})}}{1 + A_T^{(\mathrm{in})}} e^{-ikr}, \\ A_T^{(\mathrm{in})}\left(1 - A_R^{(\mathrm{in})} + \left(A_R^{(\mathrm{in})}\right)^2 - \left(A_R^{(\mathrm{in})}\right)^3 + \cdots\right)e^{ikr} &= \frac{A_T^{(\mathrm{in})}}{1 + A_T^{(\mathrm{in})}} e^{ikr} \end{aligned} \tag{3.138}$$

for the ingoing and the outgoing waves, respectively.

The scattering phase time $\tau_{\mathrm{sc}}^{\mathrm{ph}} = \hbar(\partial \arg Se^{ikR_2}/\partial E) - (-R_2/v)$ that in the limit $\chi(R_2 - R_1)$ approaching infinity goes to $2/v\chi$. So in this limit, the scattering phase time coincides with $\tau_R^{\mathrm{ph(ex)}}$.

Resonances

In all the previous analysis we have disregarded the possibility that resonances develop during tunneling and scattering. To take into account the insurgence of resonances we rewrite the scattering matrix S in the following form:

$$S = \alpha \frac{A_1\chi + ikA_2}{A_1\chi - ikA_2}, \tag{3.139}$$

where, from the comparison with (3.134), we set: $\alpha = e^{-2ikR_2}$, $A_1 = B_1 + e^{-2\chi(R_2-R_1)}B_2$, $A_2 = B_1 - e^{-2\chi(R_2-R_1)}B_2$, $B_1 = K\cos(KR_1) + \chi\sin(KR_1)$, $B_2 = K\cos(KR_1) - \chi\sin(KR_1)$.

In the region of a resonance, we can develop S into a series of powers of $(E - E_r)$, E_r being the eigenvalue of the energy for the resonance, considered as a solution of the transcendental equation $A_1(E_r) = 0$, and then we obtain the following:

$$S = \alpha \frac{(\partial(A_1\chi)/\partial E)_{E=E_r}(E - E_r) + i(kA_2)_{E=E_r}}{(\partial(A_1\chi)/\partial E)_{E=E_r}(E - E_r) - i(kA_2)_{E=E_r}}, \tag{3.140}$$

that shows better its resonant character if written in the form

$$S = \alpha \frac{E - E_r - i\Gamma/2}{E - E_r + i\Gamma/2}, \tag{3.141}$$

with

$$\frac{\Gamma}{2} = \frac{-k(E_r)A_2(E_r)}{\chi(E_r)(\partial A_1/\partial E)_{E=E_r}} = \frac{2k(E_r)e^{-2\chi(E_r)(R_2-R_1)}B_2(E_r)}{\chi(E_r)(\partial A_1/\partial E)_{E=E_r}}. \tag{3.142}$$

If $\chi(R_2 - R_1)$ is very large, we can neglect all the terms containing the negative exponential factor and write finally

$$\frac{\Gamma}{2} = 2k(E_r)e^{-2\chi(E_r)(R_2-R_1)}F(E_r), \tag{3.143}$$

with

$$F(E_r) = \frac{\hbar K(E_r)}{\chi(E_r)\sqrt{m/2}\left((1/\sqrt{V_0 + E_r})(1 + \chi(E_r)R_1) + (1/\sqrt{V_1 - E_r})(1 + K(E_r)R_1)\right)}. \tag{3.144}$$

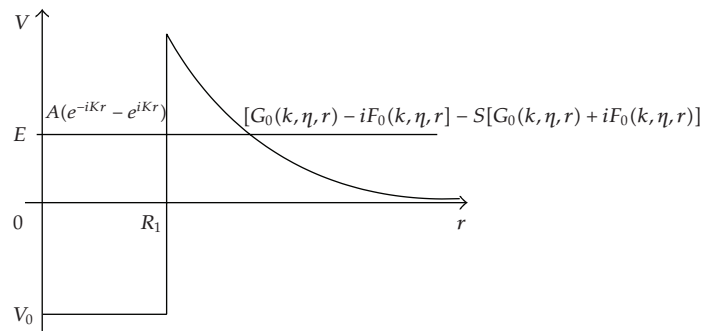

Figure 12: Schematic stationary description of a scattering in the presence of a coulomb barrier ($l = 0$).

The Case of a Rectangular Well with a Coulomb Barrier

In this case we can initiate from a simple Coulomb barrier

$$V = \frac{Z_1 Z_2 e^2}{r}, \tag{3.145}$$

($Z_1 e$ and $Z_2 e$ are the charges of the daughter nucleus and the emitted particle, resp.) instead of a rectangular potential barrier. Instead of the functions $h_l^{(1,2)}(kr)$ we have to use the coulomb functions $G_l(k,\eta,r) \pm iF_l(k,\eta,r)$ in the field of the coulomb barrier with

$$\begin{aligned} F_0(k,\eta,r) &\longrightarrow \sin(kr - \eta \ln 2kr + \sigma), \quad r \longrightarrow \infty \\ G_0(k,\eta,r) &\longrightarrow \cos(kr - \eta \ln 2kr + \sigma), \quad r \longrightarrow \infty \end{aligned} \tag{3.146}$$

(where $\eta = Z_1 Z_2 e^2 m/\hbar^2 k$ is the Sommerfeld parameter, $\sigma = \arg\Gamma(1 + i\eta)$).

In this case, taking also $l = 0$, we obtain

$$S = \frac{[G_0(k,\eta,R_1) - iF_0(k,\eta,R_1)]K\cos KR_1 - [G_0'(k,\eta,R_1) - iF_0'(k,\eta,R_1)]k\sin KR_1}{[G_0(k,\eta,R_1) + iF_0(k,\eta,R_1)]K\cos KR_1 - [G_0'(k,\eta,R_1) + iF_0'(k,\eta,R_1)]k\sin KR_1}, \tag{3.147}$$

(one can easily see that here $|S| = 1$ too)

$$A = \frac{[2ie^{ikR_2}k]}{[G_0'(k,\eta,R_1) + iF_0'(k,\eta,R_1)]k(1 - e^{2iKR_1}) + [G_0(k,\eta,R_1) + iF_0(k,\eta,R_1)]iK(1 + e^{2iKR_1})} \tag{3.148}$$

for the *scattering* (see Figure 12). Here, F_0' and G_0' signify the derivatives of F_0 and G_0 with respect to kR_1, respectively.

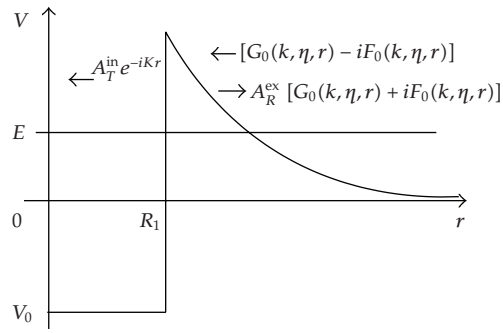

Figure 13: Schematic description of the impact of a wave packet with a Coulomb barrier from outside.

Similarly, in this case we also obtain

$$A_R^{\text{ex}} = \frac{-[G_0(k,\eta,R_1) + iF_0(k,\eta,R_1)]iK - [G_0'(k,\eta,R_1) + iF_0'(k,\eta,R_1)]k}{[G_0(k,\eta,R_1) + iF_0(k,\eta,R_1)]iK + [G_0'(k,\eta,R_1) + iF_0'(k,\eta,R_1)]k} \tag{3.149}$$

$$A_T^{\text{in}} = \frac{2ike^{iKR_1}}{[G_0(k,\eta,R_1) + iF_0(k,\eta,R_1)]iK + [G_0'(k,\eta,R_1) + iF_0'(k,\eta,R_1)]k}, \tag{3.150}$$

for the *impact from outside* (see Figure 13), and

$$A_R^{\text{in}} = e^{2iKR_1} \frac{[G_0(k,\eta,R_1) + iF_0(k,\eta,R_1)]iK - [G_0'(k,\eta,R_1) + iF_0'(k,\eta,R_1)]k}{[G_0(k,\eta,R_1) + iF_0(k,\eta,R_1)]iK + [G_0'(k,\eta,R_1) + iF_0'(k,\eta,R_1)]k}, \tag{3.151}$$

$$A_T^{\text{ex}} = e^{iKR_1} \frac{2iK}{[G_0(k,\eta,R_1) + iF_0(k,\eta,R_1)iK + [G_0'(k,\eta,R_1) + iF_0'(k,\eta,R_1)]k} \tag{3.152}$$

for the *emission from the barrier* (see Figure 14). In the derivation of (3.147), (3.148), (3.150), and (3.152) we had used the known relation $F_0G_0' - G_0F_0' = 1$ for the wronskian.

It is easy to be convinced that relations (3.122), (3.131), (3.136), and (3.137) are also valid for a coulomb barrier. We can also repeat the same reasonings on the physical meaning of the term $1/(1 + A_R^{\text{in}})$ which has been made in connection with the formula (3.137).

For very small k when $k \to 0$ (more precisely, when $2\eta \gg kR_1$),

$$\begin{aligned}
&G_0 \longrightarrow 2\left(\frac{\rho}{\pi}\right)^{1/2} I_0(2(2\pi\rho)^{1/2}) \exp(\pi\eta), \quad \text{with } I_0(2(2\pi\rho)^{1/2}) \longrightarrow 1,\\
&G_0' \longrightarrow -2\left(\frac{2\eta}{\pi}\right)^{1/2} K_0(2(2\pi\rho)^{1/2}) \exp(\pi\eta), \quad \text{with } K_0(2(2\pi\rho)^{1/2}) \longrightarrow \ln\left(\frac{1}{\gamma(2\pi\rho)^{1/2}}\right),\\
&F_0 \longrightarrow (\pi\rho)^{1/2} I_1(2(2\pi\rho)^{1/2}) \exp(-\pi\eta), \quad \text{with } I_1(2(2\pi\rho)^{1/2}) \longrightarrow (2\pi\rho)^{1/2},\\
&F_0' \longrightarrow (2\pi\eta)^{1/2} I_0(2(2\pi\rho)^{1/2}) \exp(-\pi\eta),
\end{aligned} \tag{3.153}$$

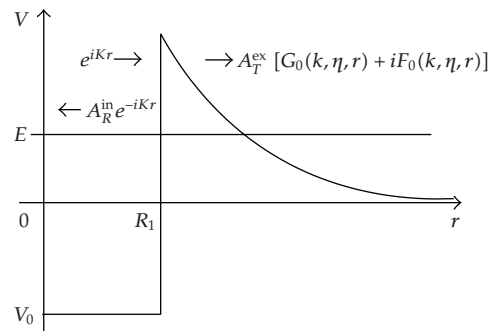

Figure 14: Schematic view of the emission from inside through a coulomb barrier.

$\gamma \approx 1.781\ldots$ being the Euler constant, and if $(2k^2/K^2\eta/\rho)[\ln\gamma^{-1}(2\pi\rho)^{-1/2}] \ll 1$, the *transmission* (*penetration*) *probability* from outside through the Coulomb barrier into the internal rectangular potential well $|A_T^{\mathrm{in}}|^2$ becomes

$$\left|A_T^{\mathrm{in}}\right|^2 \longrightarrow \left(\frac{\pi k}{K^2 R_1}\right)\exp(-2\pi\eta) \tag{3.154}$$

which contains the same exponential factor $\exp(-2\pi\eta)$ as the known quasiclassical (WKB) approximation, containing nevertheless the other pre-exponential factor. It is quite understandable because the quasiclassical approximation is not applicable near the point R_1 where there is an abrupt potential change and one cannot use the approximate notion of the turning point.

3.9. An Odd Description of Tunneling Phenomena

For the sake of the completeness, it is possible to mention the quaternion description of tunneling phenomena. In [143–145], the new quaternion description of tunneling phenomena has initiated (the quaternion description of quantum mechanics one can see, e.g., in [146]). The authors show a noteful difference between the complex and quaternionic formulations of tunneling phenomena which could be matter of further theoretical discussions and could represent the starting point for a possible experimental investigation.

4. Applications for Nuclear Reactions and Decays

4.1. Narrow Resonance $\rightarrow$ Exponential Decay

Let us firstly explain how a typical isolated Lorentzian (Breit-Wigner) resonance, in the cross-section of a quantum collision or nuclear reaction $\alpha \rightarrow \beta$, is connected with an exponential law of the decay function of the correspondent compound or radioactive nucleus (somewhat generalizing the similar derivation from [147] and mainly following [148]). We represent the

reaction amplitude $f_{\alpha\beta}(E)$ as

$$f_{\alpha\beta}(E) = \frac{C_{\alpha\beta}}{E - E_r + i\Gamma/2}, \tag{4.1}$$

where $C_{\alpha\beta}$ is a constant or a smooth function of the final-particle kinetic energy E in the region $(E_r - \Gamma/2, E_r + \Gamma/2)$, E_r and Γ being the resonance energy and width, respectively.

The final-particle wave packet in the 1D radial asymptotic limit is described by

$$\Psi_\beta(r_\beta, t) = r_\beta^{-1} \int_0^\infty dE\, g(E) f_{\alpha\beta}(E) \exp\left[ikr_\beta - \frac{iEt}{\hbar}\right], \tag{4.2}$$

where $g(E)$ is a smooth weight amplitude with an energy spread ΔE (usually $\Delta E \ll E_r$), m_β and r_β are the final-particle mass and radial coordinate, respectively, and $k = \sqrt{2m_\beta E}/\hbar$. For short-ranged interactions (including also screened Coulomb potentials), it can be rewritten as

$$\Psi_\beta(z_\beta, t) = \int_0^\infty dE\, g(E) T_{\alpha\beta}(E) \exp\left[ikz_\beta - \frac{iEt}{\hbar}\right], \tag{4.3}$$

with $T_{\alpha\beta}(E) = N_\beta(E) f_{\alpha\beta}(E)$, $T_{\alpha\beta}(E)$ is the T-matrix elements connected with the S-matrix elements by known relation $T_{\alpha\beta} = \delta_{\alpha\beta} - S_{\alpha\beta}$, $N_{\alpha\beta}(E)$ is an unessential smooth function of E, z_β is the axis along the direction of the final-particle emission imposed by the registration geometry, $z_\beta \geq R_\beta$, R_β is the interaction radius in the final channel. In the simplest case one can fix $z_\beta = R_\beta$ and

$$\Psi_\beta(z_\beta, t) = \int_0^\infty dE\, g(E) \widetilde{T}_{\alpha\beta}(E) \exp\left[-\frac{iEt}{\hbar}\right], \tag{4.4}$$

where $\widetilde{T}_{\alpha\beta}(E) = T_{\alpha\beta} \exp(ikR_\beta)$ is a smooth function of E: in accordance with the analytical S-matrix theory, $T_{\alpha\beta}(E)$ contains the factor $\exp(-ikR_\beta)$ and consequently this factor is being cancelled by $\exp(ikR_\beta)$ in $\widetilde{T}_{\alpha\beta}(E)$. For condition

$$\Gamma \ll \Delta E \ll E_r, \tag{4.5}$$

one can rewrite (4.4) in the following simplified form:

$$\Psi_\beta(R_\beta, t) = A \int_0^\infty dE \frac{\exp[-iEt/\hbar]}{E - E_r + i\Gamma/2}, \tag{4.6}$$

where A is a constant. In the approximation (4.5) and Γ = constant one obtain

$$\Psi_\beta(R_\beta,t) = \begin{cases} B\exp\left[-\dfrac{iE_r t}{\hbar-(\Gamma/2\hbar)t}\right], & \text{for } t>0, \\ 0, & \text{for } t<0, \end{cases} \tag{4.7}$$

(moving the lower integration limit in (4.15) from 0 to $-\infty$ and utilizing the residue theorem). Here B is a constant and more precisely there must be $t-t_{\text{in}}$ (with $t_{\text{in}} = \hbar(\partial \arg g/\partial E)$) instead of t. The form (4.7) is valid also with slight modifications in the cases when $\Gamma \sim E^{1/2}$ or Γ is a linear function of E (see, e.g., [149]).

The evolution of the particle β passing through position z_β during the unitary time interval, centered at t, is described by the probability flux density $j_\beta(z_\beta,t) = \text{Re}[\Psi_\beta(z_\beta,t)(i\hbar/2m)\partial\Psi^*_\beta(z_\beta,t)/\partial z_\beta]$ with the adequate normalization $\int_{-\infty}^{\infty} j_\beta(z_\beta,t)dt = 1$. The emission probability (per a time unit) in the vicinity of the compound nucleus (near $z_\beta = R_\beta$)

$$I(t) = \frac{j_\beta(R_\beta,t)}{\int_{-\infty}^{\infty} dt\, j_\beta(R_\beta,t)} \tag{4.8}$$

is equal to

$$I(t) = \left(\frac{\Gamma}{\hbar}\right)\exp\left(-\frac{\Gamma t}{\hbar}\right). \tag{4.9}$$

(On the presence of the violations of (4.9) for very small t and for very large t (see, in particular, *in* [57]).) In obtaining (4.9) we took into account that

$$\lim_{z_\beta\to R_\beta}\left(-\frac{i\hbar}{m_\beta}\right)T_{\alpha\beta}\partial\left[\exp(ikz_\beta)\right]\partial z_\beta = v\widetilde{T}_{\alpha\beta} \tag{4.10}$$

($v = \hbar k/m_\beta$). If $\Psi_\beta(R_\beta,t)$ has a form (4.7), the Fourier transform of Ψ_β is equal to

$$\int_0^\infty dt\,\Psi_\beta(R_\beta,t)\exp\left(-\frac{iEt}{\hbar}\right) = B\int_0^\infty dt\exp\left[-i(E-E_r)t/\hbar-\Gamma t/2\hbar\right] = \frac{iB}{E-E_r+i\Gamma/2}, \tag{4.11}$$

which is proportional to the amplitude from (4.1).

For $z_\beta > R_\beta$ one can rewrite (4.3) in a following way:

$$\Psi_\beta(z_\beta,t) = \int_0^\infty dk\frac{G(k)D(k)}{(k-k_0)(k+k_0)}\exp\left(ikz_\beta-\frac{iEt}{\hbar}\right), \tag{4.12}$$

with

$$G(k) = g\left(\frac{\hbar^2 k^2}{2m_\beta}\right)\left(\frac{dE}{dk}\right),$$
$$D(k) = \left(\frac{2m_\beta}{\hbar^2}\right)N_\beta(E)C_{\alpha\beta},$$
$$k_0 = \left(\frac{1}{\sqrt{2}}\right)\left(\sqrt{\sqrt{k_r^4+\gamma^2}+k_r^2} - i\sqrt{\sqrt{k_r^4+\gamma^2}-k_r^2}\right), \tag{4.13}$$
$$k_r = \frac{\sqrt{2m_\beta E_r}}{\hbar},$$
$$\gamma = \frac{\Gamma m_\beta}{\hbar^2}.$$

Since $G(k)$ and $D(k)$ are smooth functions of k

$$\Psi_\beta(z_\beta, t) \cong \int_{-\infty}^{\infty} dk \frac{G(k)D(k)}{(k-k_0)(k+k_0)} \exp\left(ikz_\beta - \frac{iEt}{\hbar}\right), \tag{4.14}$$

under the condition (4.5), then, introducing in (4.14) a new variable

$$y = \sqrt{\frac{i\hbar t}{2m_\beta}}\left(k - \frac{m_\beta z_\beta}{\hbar t}\right), \tag{4.15}$$

and performing the transformations quite similar to those which were used in [30–32], it is possible to obtain

$$\Psi_\beta(R_\beta, t) = \begin{cases} 0, & \text{for } z_\beta > v_r t, \\ \text{const} \cdot \exp\left[ik_r z_\beta - \dfrac{iE_r t}{\hbar} - \left(\dfrac{\Gamma}{2\hbar}\right)\left(\dfrac{t - z_\beta}{v_r}\right)\right], & \text{for } z_\beta \le v_r t, \end{cases} \tag{4.16}$$

with $v_r = \hbar k/m_\beta$. The wave function (4.16) can be applied for macroscopic distances z_β, near a detector which registers particles β.

Let us remember that an exponential law (4.9) and also the asymptotic (4.16) are valid only under conditions (4.5), that is, when all energies (or continuum states) around E_r are completely populated in the large region with the width $\Delta E \gg \Gamma$. If, on the contrary,

$$\Delta E \ll \Gamma. \tag{4.17}$$

The emission probability is nonexponential and does essentially depend on ΔE and the form of $g(E)$. If one will take the Lorentzian form also for $g(E)$, that is,

$$g(E') = \frac{g_0}{E' - E + i\Gamma/2}, \tag{4.18}$$

with g_0 = const or smooth inside ΔE under conditions (4.5) and (4.17), instead of (4.7) the expression

$$\Psi_\beta(R_\beta,t) = \begin{cases} 0, & \text{for } z_\beta > v_r(t-t^0_{\text{in}}); \\ \dfrac{\text{const}}{E-E_r+i\Gamma/2}\cdot\exp\left[ik_r z_\beta - \dfrac{iE_r t}{\hbar} - \left(\dfrac{\Delta E}{2\hbar}\right)\left(\dfrac{t-z_\beta}{v_r-t^0_{\text{in}}}\right)\right], & \text{for } z_\beta \le v_r(t-t^0_{\text{in}}) \end{cases} \tag{4.19}$$

(with $v = \hbar k/m_\beta$ and $t^0_{\text{in}} = \hbar(\partial \arg g_0/\partial E)$) will be obtained. The cross-section $\sigma_{\alpha\beta}$, which is proportional to

$$\sigma_{\alpha\beta} \sim \int_{t_{\min}}^{\infty} dt j_\beta(z_\beta,t), \tag{4.20}$$

where $t_{\min} = z_\beta/v + t^0_{\text{in}}$, $(t_{\min},\infty)$ being the operative registration time interval of detector, after integrating in (4.20) acquires the Breit-Wigner form

$$\sigma_{\alpha\beta} = \left|f_{\alpha\beta}\right|^2 = \frac{\text{const}}{(E-E_r)^2+\Gamma^2/4}. \tag{4.21}$$

4.2. The Real Possibility of the Phenomenon of the Delay-Advance in Proton Scattering by Nuclei

Near an isolated resonance the proton-nucleus (or nucleus-nucleus) scattering amplitude $F(E,\theta)$ is

$$F(E,\theta) = f(E,\theta) + f_{l,\text{res}}(E,\theta), \tag{4.22}$$

with

$$\begin{aligned} f(E,\theta) &= f_{\text{coul}}(E,\theta) + (2ik)^{-1}\sum_{\lambda\neq l}(2\lambda+1)P_\lambda(\cos\theta)\exp\left(2i\eta_\lambda\right)\left[\exp(2i\delta^b_\lambda)-1\right], \\ f_{l,\text{res}}(E,\theta) &= (2ik)^{-1}(2\lambda+1)P_\lambda(\cos\theta)\exp\left(2i\eta_\lambda\right)\left[\exp\left(2i\delta^b_l\right)\frac{E-E_{\text{res}}-i\Gamma/2}{E-E_{\text{res}}+i\Gamma/2}-1\right] \end{aligned} \tag{4.23}$$

is the Coulomb scattering amplitude, δ^b_l and η_l being the background nuclear l-scattering phase-shift and the Coulomb l-scattering phase shift, respectively, k is the wave number, θ is the scattering angle in the c.m.s. Rewriting (4.22) in the form

$$F(E,\theta) = \left[\frac{A(E-E_{\text{res}})+iB\Gamma}{2}\right]\left(\frac{E-E_{\text{res}}+i\Gamma}{2}\right)^{-1} \tag{4.24}$$

where

$$A = f(E,\theta) + (k)^{-1}(2l+1)P_l(\cos\theta)\exp\left(2i\eta_l + i\delta_l^b\right)\sin\delta_l^b,$$
$$B = f(E,\theta) + (ik)^{-1}(2l+1)P_l(\cos\theta)\exp\left(2i\eta_l + i\delta_l^b\right)\cos\delta_l^b, \tag{4.25}$$

we obtain (as it was firstly made in [150]) the following expression for the quasimonochromatic total scattering *duration* $\tau(E,\theta)$:

$$\tau(E,\theta) = \frac{2R}{v} + \frac{\hbar\partial\arg F}{\partial E} \equiv \frac{2R}{v} + \Delta\tau(E,\theta), \tag{4.26}$$

where $v = \hbar k/\mu$ is the projectile velocity and R is the interaction radius,

$$\Delta\tau(E,\theta) = \left(\frac{\hbar\Gamma}{2}\right)\left[\left(E - E_{\text{res}}\right)^2 + \frac{\Gamma^2}{4}\right]^{-1} - \left(\frac{\hbar\operatorname{Re}\alpha}{2}\right)\left[\left(\frac{E - E_{\text{res}} - \operatorname{Im}\alpha}{2}\right)^2 + \frac{(\operatorname{Re}\alpha)^2}{4}\right]^{-1} \tag{4.27}$$

is the *time delay*, $\alpha = \Gamma B/A$. When $0 < \operatorname{Re}(B/A) < 1$ and $|\operatorname{Im}\alpha/2| \ll E_{\text{res}}$, near $E = E_{\text{res}}$ the time delay $\Delta\tau$ can become negative, that is, an *advance* can appear instead of a delay.

Because of the existing Coulomb barrier, there is a certain indefiniteness in the choice of the exact value of the physical interaction radius R; it lies between the minimal value, determined by the equality of the energy E and the Coulomb barrier at the point of the exit of the final charged particle from the barrier, and the maximal value, determined by the practical vanishing of the external tail of the (screened) Coulomb barrier or by the evident causality condition ($\tau \geq 0$) in the case of negative values of $\Delta\tau$.

The time analysis of proton scattering by nuclei ^{12}C and ^{14}N at the range of isolated resonances distorted by the nonresonant background, with the help of the experimental study of the interference in the accompanied bremsstrahlung, had resulted in [151, 152] the revealing of the real possibility of such delay-advance phenomenon.

4.3. The Phenomenon of Time Resonances (Explosions)

Sometimes, in the cases of the dense and strongly overlapper resonances in high-energy nuclear reactions, when $\Delta E \gg \overline{\Gamma},\ D$ (ΔE, $\overline{\Gamma}$, and D being the experimental energy resolution, mean reasonance width and mean distance between resonances, resp., for any contributed resonant spin, parity and total moment quantum numbers), it is possible to approximate the reaction amplitude $f_{\alpha\beta}(E)$ by the form

$$f_{\alpha\beta}(E) = C_{\alpha\beta}^n \exp\left(-\frac{E\tau_n}{2\hbar} + \frac{iEt_n}{\hbar}\right), \tag{4.28}$$

where τ_n and t_n are constants (with the time dimension), τ_n and t_n determine the exponential dependence on energy for the correspondent cross-section and the linear dependence on

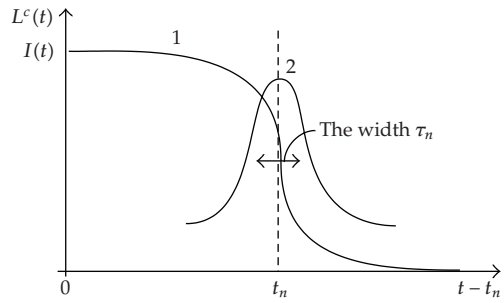

Figure 15: $L^c(t)$ (curve 1) and $I(t)$ (curve 2).

energy for the amplitude phase, respectively, $C^n_{\alpha\beta}$ is a constant or a very smooth function (inside ΔE) of the final-particle energy E, then the correspondent cross-section and the emission probability under the condition

$$\Delta E \ll \frac{2\hbar}{\tau_n} \tag{4.29}$$

will be

$$\sigma_{\alpha\beta} = \left|f_{\alpha\beta}\right|^2 = \text{const} \cdot \exp\left(-\frac{E\tau_n}{\hbar}\right), \tag{4.30}$$

$$I(t) = \left(\frac{\tau_n}{2\pi}\right)\frac{1}{\left(t - t_n\right)^2 + \tau_n^4/4}, \tag{4.31}$$

respectively, (the detailed physical and mathematical justification of the form like (4.28) see, e.g., in [148]). *The evolution of compound-nucleus surviving* (at instant t during the life and decay after the formation) can be described by the following function:

$$L^c(t) = 1 - \int_{(0)}^{t} dt\, I(t). \tag{4.32}$$

From (4.31)-(4.32), one can deduce the strongly nonexponential form of $L^c(t)$ and $I(t)$, like depicted in Figure 15.

When $f_{\alpha\beta}$ has a more general form like

$$f_{\alpha\beta}(E) = \sum_{n=1}^{v} C^n_{\alpha\beta} \exp\left(-\frac{E\tau_n}{2\hbar} + \frac{iEt_n}{\hbar}\right), \tag{4.33}$$

with several terms ($v = 2, 3, \ldots$), the cross-section $\sigma_{\alpha\beta} = |f_{\alpha\beta}|^2$ contains not only exponentially decreasing terms but also oscillating terms with factors $\cos[E(t_n - t_{n'})/\hbar]$ and $\sin[E(t_n - t_{n'})/\hbar]$.

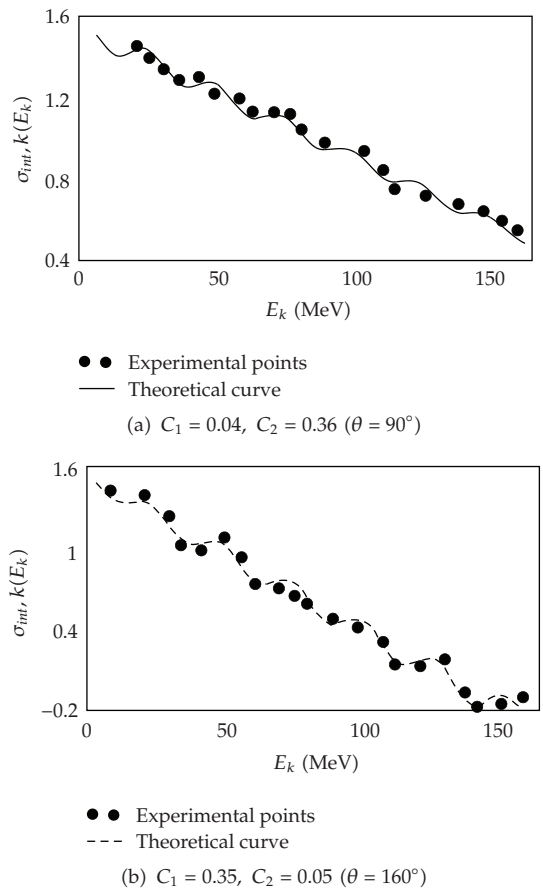

Figure 16: Inclusive process p + ^{12}C $\rightarrow$ ^{7}Be (2.1 GeV protons).

In the case of two terms ($v = 2$) in (4.33) formula (4.30) can be rewritten as

$$\begin{aligned}\sigma_{\alpha\beta} = {} & \left|f^1_{\alpha\beta}\right|^2 \exp\left(-\frac{E\tau_1}{\hbar}\right) + \left|f^2_{\alpha\beta}\right|^2 \exp\left(-\frac{E\tau_2}{\hbar}\right) \\ & + 2\mathrm{Re}\left\{f^1_{\alpha\beta} f^2_{\alpha\beta} * \exp\left[\frac{iE(t_1 - t_2)}{\hbar} - \frac{E(\tau_1 + \tau_2)}{2\hbar}\right]\right\},\end{aligned} \tag{4.34}$$

where the terms with ΔE are neglected if the conditions $\Delta E t_n \ll E\tau_n$ and $\Delta E \tau_n \ll E t_n$ are supposed.

In particular, for inclusive energy spectra of the kth final fragment we will use the following expression:

$$\begin{aligned}\sigma_{\mathrm{inc},k}(E_k) &= \sum_{n=1}^{2}\left|C_n \exp\left[\left(\frac{it_n - \tau_n}{2}\right)\frac{E_k}{\hbar}\right]\right|^2 \\ &= \sum_{n=1}^{2}\left|C_n\right|^2 \exp\left(-\frac{E_k\tau_n}{\hbar}\right) + 2\mathrm{Re}\, C_1^* C_2 \exp\left\{\left[\frac{i(t_2 - t_1) - (\tau_1 + \tau_2)}{2}\right]\frac{E_k}{\hbar}\right\}.\end{aligned} \tag{4.35}$$

In Figure 16, an example of the calculated inclusive energy spectra $\sigma_{\mathrm{inc},k}(E_k)$, in arbitrary units and in semilogarithmic scale, is presented in comparison with the experimental data (see from [148]).

Appendices

A. On the Bilinear Time Operator

One can rewrite the linear operator (2.1b) into the form of the *bilinear* operator

$$\hat{t} = \left(-\frac{i\hbar}{2}\right)\frac{\overleftrightarrow{\partial}}{\partial E} \tag{A.1}$$

with $(f, \hat{t}g) \equiv (f, (-i\hbar/2)(\partial/\partial E)g) + ((-i\hbar/2)(\partial/\partial E)f, g)$ and without changing the form (2.1a). Such bilinear form of the time operator had been firstly introduced in [12–14]. By adopting expression (A.1) for the time operator, the point $E = 0$ happens to be automatically emiminated both in $(f, \hat{t}f)$ and in the transformation of the integral $\int_{-\infty}^{\infty} tj(x,t)dt$ over time into the integral over energy.

B. The Measure of Averaging over Time in the 3D Case

In the 3D case of the Schroedinger equation, the continuity equation (2.8) in Section 2 has to be rewritten as

$$\frac{\partial \rho(\vec{r},t)}{\partial t} + \operatorname{div} j(\vec{r},t) = 0 \tag{B.1}$$

(see, e.g., [153, 154]), $\vec{r}$ being the radius vector of a particle moving in the extential potential or the radius vector for the relative motion of two interacting particles, $\vec{r} = \{x, y, z\}$ in Cartesian coordinates or $\vec{r} = \{r, \theta, \varphi\}$ in spherical coordinates. Within the framework of the quantum collision theory the motion direction is often described by any Cartesian axis (let us say, x, with fixed values of y, z) or by variable radial coordinate r, with fixed values of angular coordinates θ, φ. The term $\operatorname{div} j(\vec{r},t)$ acquires the form $\partial j_x(\vec{r},t)/\partial x$ or $(1/r^2)\partial[r^2 j_r(\vec{r},t)]/\partial r$, respectively.

Then, (2.7) from Section 2 has to be substituted by

$$W_x(\vec{r},t)dt = \frac{j_x(\vec{r},t)dt}{\int_{-\infty}^{\infty} j_x(\vec{r},t)dt} \tag{B.2}$$

or

$$W_r(\vec{r},t)dt = \frac{j_r(\vec{r},t)dt}{\int_{-\infty}^{\infty} j_r(\vec{r},t)dt}, \tag{B.3}$$

where the probabilistic interpretations of $j_x(\vec{r},t)$ and $j_r(\vec{r},t)$ in *time* are the same as the probabilistic interpretation, contained in Section 2 for the 1D flux $j(x,t)$, as (one can be convinced in this by the evident generalization).

When the flux density $j_x(\vec{r},t)$ (or $j_r(\vec{r},t)$) changes its sign, the quantity $W_x(\vec{r},t)$ (or $W_r(\vec{r},t)$) is no longer positive definite and it acquires a physical meaning of a probability density *only* during those partial time-intervals in which the flux density $j_x(\vec{r},t)$ (or $j_r(\vec{r},t)$) does keep its sign. Therefore, it is possible, quite similarly as $W_\pm(x,t)$ in Section 2, to introduce the two measures, by separating the positive and the negative flux-direction values (i.e., flux signs):

$$W_{x,\pm}(\vec{r},t)\,dt = \frac{j_{x,\pm}(\vec{r},t)\,dt}{\int_{-\infty}^{\infty} j_{x,\pm}(\vec{r},t)\,dt}, \tag{B.4}$$

$$W_{r,\pm}(\vec{r},t)\,dt = \frac{j_{r,\pm}(\vec{r},t)\,dt}{\int_{-\infty}^{\infty} j_{r,\pm}(\vec{r},t)\,dt}, \tag{B.5}$$

with $j_{x,\pm}(x,t) = j_x(x,t)\theta(\pm j_x)$ and $j_{r,\pm}(x,t) = j_r(x,t)\theta(\pm j_r)$, respectively.

C. Approximate Eigenvalues and Approximate Orthonormalized Eigenfunctions of the Time Operator

Following [4, 5] (see also [21, 23–25, 50, 51]), one can specify the following approximate eigenvalues and eigenfunctions of the operator $\widehat{t}$ in (2.1a), (2.1b), and (2.26) and simultaneously of the operator $\widehat{t}^2$:

$$\begin{aligned} \widehat{t}\varphi_t^{\delta,\eta}(E) &\approx t\varphi_t^{\delta,\eta}(E), \\ \widehat{t}^2\phi_t^{\delta,\eta}(E) &\approx t^2\phi_t^{\delta,\eta}(E), \end{aligned} \tag{C.1}$$

where

$$\varphi_t^{\delta,\eta}(E) = C\exp\left(\frac{iEt}{\hbar}\right) f_\delta(E) g_\eta(E), \tag{C.2}$$

C is an arbitrary constant, t is the continuous real eigenvalue of the operator $\widehat{t}$,

$$\begin{aligned} f_\delta(E) &= 2\frac{\sin \delta E/\hbar}{E/\hbar}, \\ \widehat{E} &= \begin{cases} 3\left(\dfrac{E}{\eta}\right)^2 - 2\left(\dfrac{E}{\eta}\right)^3 & \text{for } 0 \le E \le \eta, \\ 1, & \text{for } \eta \le E, \end{cases} \end{aligned} \tag{C.3}$$

δ is a positive parameter that describes the width of the wave packet formed from the functions $\exp(iEt/\hbar)$; the sequence of functions $g_\eta(E)$ has the limit as $\eta \to 0$ which is equal to the generalized function $\Theta(E)$; $\Theta(E)$ is equal to 1 for $E > 0$ and to 0 for $E \le 0$. As is readily seen by direct calculation of the left-hand side of (C.1), the functions $\varphi_t^{\delta,\eta}(E)$ approximate

the eigenfunctions of the operartors $\hat{t}$ and $\hat{t}^2$ the more accurately, and are the more nearly orthogonal for different t, the better the relations

$$\begin{gathered}\frac{\delta}{t} \ll \left(\frac{\delta\eta}{\hbar}\right)^{1/2} \ll 1, \\ \left(\frac{\delta}{t}\right)^2 \ll \left(\frac{\delta\eta}{\hbar}\right)^{3/2}\end{gathered} \tag{C.4}$$

hold as $\delta \to 0$ and $\eta \to 0$ and under the fulfilment of the condition (B.4), the variance

$$Dt = \langle \varphi_t^{\delta,\eta}(E) | \hat{t}^2 | \varphi_t^{\delta,\eta}(E) \rangle - \left| \langle \varphi_t^{\delta,\eta}(E) | \hat{t} | \varphi_t^{\delta,\eta}(E) \rangle \right|^2 \tag{C.5}$$

in the state $\varphi_t^{\delta,\eta}(E)$ tends to 0. The constant C can be chosen to make the norm $\|\varphi_t^{\delta,\eta}(E)\|$ equal to 1.

It is curious that the function (C.2) differs from simple wave packets of the form

$$\varphi_t^{\delta}(E) = C \exp\left(\frac{iEt}{\hbar}\right) f_\delta(E), \tag{C.6}$$

which are typical for "eigendifferentials" in the continuous spectrum of linear self-adjoint operators [54], only for the presence of the factor $g_\eta(E) \to \Theta(E)$.

D. The Duality of Time and Energy Operators

As it is known, in quantum theory there is a correspondence between energy E and *two* operators—operator, $i\hbar(\partial/\partial t)$ in the t-representation and Hamiltonian operator $\widehat{H}(\hat{p}_x, \hat{x}, \ldots)$. The duality of these operators is well seen from the Schroedinger equation $\widehat{H}\Psi = i\hbar(\partial\Psi/\partial t)$. The similar duality has to take place for time in quantum theory: besides the general form (2.1a), (2.1b), and (2.26) which is valid for any physical system (in the region of continuous energy spectrum) one can express the time operator in terms of the coordinate and momentum operators using the commutation relation between Hamiltonian operator and time one. So, if one will make the substitution

$$\begin{gathered}\hat{E} \longrightarrow \widehat{H}(\hat{p}_x, \hat{x}, \ldots), \\ \hat{t} \longrightarrow \hat{T}(\hat{p}_x, \hat{x}, \ldots),\end{gathered} \tag{D.1}$$

then he will obtain

$$\left[\widehat{H}, \hat{T}\right] = i\hbar, \tag{D.2}$$

which is similar to (2.24). The commutation relation (D.2) can be used for finding $\hat{T}(\hat{p}_x, \hat{x}, \ldots)$ for any concrete physical system with known $\widehat{H}(\hat{p}_x, \hat{x}, \ldots)$ (see, e.g., [11]). Choosing the coordinate representation or momentum one for $\widehat{H}(\hat{p}_x, \hat{x}, \ldots)$ and $\hat{T}(\hat{p}_x, \hat{x}, \ldots)$, one *does not*

change the formal expression for Hamiltonian $\widehat{H}(\widehat{p}_x, \widehat{x}, \ldots)$ but *does change the sign in the formal expression for time operator* $\widehat{T}(\widehat{p}_x, \widehat{x}, \ldots)$. It can be easily seen for a free particle:

$$\widehat{H} = \begin{cases} \dfrac{\widehat{p}_x^2}{2\mu}, \ \widehat{p}_x = -i\hbar\dfrac{\partial}{\partial x} & \text{in the coordinate representation,} \\ \dfrac{p_x^2}{2\mu} & \text{in the momentum representation,} \end{cases} \tag{D.3}$$

$$\widehat{T} = \frac{\mu}{2}\left[\widehat{p}_x^{-1}x + x\widehat{p}_x^{-1}\right], \quad \widehat{p}_x^{-1} = \frac{i}{\hbar}\int dx \cdots \text{ in the coordinate representation,} \tag{D.4a}$$

$$\widehat{T} = \frac{-\mu}{2}\left[p_x^{-1}\widehat{x} + \widehat{x}p_x^{-1}\right], \quad \widehat{x} = i\hbar\frac{\partial}{\partial p_x} \text{ in the momentum representation} \tag{D.4b}$$

(in the symmetrized form). By the way, formula (D.4b) is equivalent to $-i\hbar(\partial/\partial E)$ (since $E = p_x^2/2\mu$) and therefore is also a maximal Hermitian operator.

For plane-wave state $\exp(ikx)$ in both representations (D.4a) and (D.4b) we obtain the same result:

$$\widehat{T}\exp(ikx) = \frac{x}{v}\exp(ikx), \tag{D.5}$$

x/v being the time of a free motion with the velocity v over the distance x.

E. On Four-Position Operators in Quantum Field Theory

E.1. The Klein-Gordon Case: Three-Position Operators

The usual position operators, being Hermitian and, moreover, self-adjoint, are known to possess real eigen-values: that is, they yield a *point-like* localization. It is possible [49] to split operator $\widehat{x}$ into two bilinear parts as follows:

$$\widehat{x} = i\nabla_p \equiv \left(\frac{i}{2}\right)\overleftrightarrow{\nabla}_p + \left(\frac{i}{2}\right)\overleftrightarrow{\nabla}_p^{(+)}, \tag{E.1}$$

where $\Psi^*\overleftrightarrow{\nabla}_p\Phi \equiv \Psi^*\nabla p\Phi - \Phi\nabla p\Psi^*$ and $\Psi^*\overleftrightarrow{\nabla}_p^{(+)}\Phi \equiv \Psi^*\nabla_p\Phi + \Phi\nabla_p\Psi^*$, and where we always referred to a suitable space of *wave packets* (see, e.g., [12–14, 155–161]). Its Hermitian part [12–14, 155–161]

$$\widehat{x}_h \equiv \left(\frac{i}{2}\right)\overleftrightarrow{\nabla}_p, \tag{E.2}$$

which was expected to yield an (ordinary) point-like localization, was derived also by writing explicitly

$$(\Psi, \widehat{x}\Phi) = i\int\frac{d^3p}{p_0}\Psi^*(p)\nabla_p\Phi(p), \tag{E.3}$$

and imposing hermiticity, that is, the reality of the diagonal elements. The calculation yielded

$$\mathrm{Re}(\Phi, \hat{x}\Phi) = i\int \frac{d^3p}{p_0}\Phi^*(p)\overleftrightarrow{\nabla}_p\Phi(p), \tag{E.4}$$

just suggesting to adopt the Lorentz-invariant quantity (E.2) as a bilinear Hermitian position operator. Then, integrating by parts (and due to the vanishing of the surface integral), we verified that (E.2) is equivalent to the ordinary Newton-Wigner operator

$$\hat{x}_h \equiv \left(\frac{i}{2}\right)\overleftrightarrow{\nabla}_p \equiv i\nabla_p - \left(\frac{i}{2}\right)\frac{\vec{p}}{p^2 + m_0^2}. \tag{E.5}$$

We were left with the bilinear anti-Hermitian part

$$\hat{x}_a \equiv \left(\frac{i}{2}\right)\overleftrightarrow{\nabla}_p^{(+)}, \tag{E.6}$$

whose *average values* over the considered state (wave packet) were regarding as yielding [66] the sizes of an ellipsoidal localization region.

In general, the extended-type position operator $\hat{x}$ will give

$$\langle\Psi|\hat{x}|\Psi\rangle = (\vec{\alpha} + \Delta\vec{\alpha}) + i(\vec{\beta} + \Delta\vec{\beta}), \tag{E.7}$$

where $\Delta\vec{\alpha}$ and $\Delta\vec{\beta}$ are the mean-errors encountered when measuring the point-like position and the sizes of the localization region, respectively. It is to evaluate the commutators ($i, j = 1, 2, 3$):

$$\left[\left(\frac{i}{2}\right)\frac{\overleftrightarrow{\partial}}{\partial p_i}, \left(\frac{i}{2}\right)\frac{\overleftrightarrow{\partial}_{(+)}}{\partial p_j}\right] = \left(\frac{i}{2p_0^2}\right)\left(\frac{\delta_{ij} - 2p_ip_j}{p_0^2}\right), \tag{E.8}$$

where from the noticeable "*uncertainty correlations*" follow:

$$\Delta\alpha_i \cdot \Delta\beta_j \geq \left(\frac{1}{4}\right)\left\langle \frac{(\delta_{ij} - 2p_ip_j/p_0^2)}{p_0^2}\right\rangle_\Psi. \tag{E.9}$$

E.2. Four-Position Operators

It is tempting to propose as four-position operator the quantity $\hat{x}^\mu = \hat{x}_h^\mu + i\hat{x}_a^\mu$, whose Hermitian (Lorentz-covariant) part can be written as

$$\hat{x}_h^\mu \equiv -\left(\frac{i}{2}\right)\frac{\overleftrightarrow{\partial}}{\partial p_\mu}, \tag{E.10}$$

to be associated with its corresponding "operator" in four-momentum space

$$\widehat{p}_h^{\mu} \equiv +\left(\frac{i}{2}\right)\frac{\overleftrightarrow{\partial}}{\partial x_{\mu}}. \tag{E.11}$$

Let us firstly recall the proportionality between the 4-momentum operator and the 4-current density operator in the chronotopical space

$$\begin{aligned} m_0\widehat{\rho} \equiv \widehat{p}_0 &= \left(\frac{i}{2}\right)\frac{\overleftrightarrow{\partial}}{\partial t}, \\ m_0\widehat{\vec{j}} \equiv \widehat{\vec{p}} &= -\left(\frac{i}{2}\right)\frac{\overleftrightarrow{\partial}}{\partial \vec{r}}. \end{aligned} \tag{E.12}$$

Then, let us recall the canonical correspondence (in the 4-position and 4-momentum spaces, resp.) and introduce the 4-position operators in the 4-momentum space (cf. the previous section):

$$\widehat{t} = -\left(\frac{i}{2}\right)\frac{\overleftrightarrow{\partial}}{\partial p_0}, \tag{E.13a}$$

$$\widehat{\vec{x}} = \left(\frac{i}{2}\right)\frac{\overleftrightarrow{\partial}}{\partial \vec{p}}. \tag{E.13b}$$

Now, recalling the properties of the time operator as a maximal Hermitian operator in the nonrelativistic case, considered in Section 2.1, we can be easily convinced that the relativistic time operator (E.13a) (for the Klein-Gordon case) *has to be also self-adjoint bilinear operator for continuous energy spectra and a maximal Hermitian linear operator due to the boundedness by zero from below for the kinetic energy (or by m_0 from below for the total energy) for the free particle.*

Finally, comparing (E.12) and (E.13a) and (E.13b), we can conclude that the four-position "operator" (E.13a) and (E.13b) can be regarded as a 4-current density operator in the energy-momentum space [12–14].

Of course, similar considerations and conclusions can be carried on for the anti-Hermitian parts also [12–14].

F. On Multiple Internal Reflections during Tunnelling through a Barrier

Introduction

The analysis of multiple internal reflections inside potentials has been considered since a long (see, e.g., [162–164]). The problem is rather trivial for attractive potentials and for over-barrier energies in potential barriers, but things change drastically for under-barrier energies, namely for tunneling. Indeed, in this case decreasing (evanescent) and

increasing (antievanescent) waves *separately* correspond to zero—both stationary and (time-averaged) nonstationary—fluxes. Nonzero fluxes correspond *only* to linear combinations of *both* decreasing and increasing waves. As a consequence, evanescent and antievanescent waves cannot be regarded as physical propagating waves. This circumstance was overlooked anywhere up to now. It implies that a physical treatment of the problem requires a description of the tunneling in terms of wave packets.

Analyzing multiple internal reflections in particle and photon tunneling, we will follow [165, 166].

Evolution of Particle Tunneling through a 1D Rectangular Potential Barrier

We confine ourselves to the simplest case of particles moving along the x-direction and tunneling through a rectangular potential barrier of height V_0 in the interval $(0, a)$ (see Figure 1) where by I, II, and III we label the regions $x < 0$, $0 < x < a$ and $x > a$, respectively. As usually [26, 27, 32, 47], we will use the following expression for the stationary wave function $\psi(k, x)$

$$\psi \equiv \begin{cases} \psi_{\mathrm{I}} = \psi_{\mathrm{in}} + \psi_R, & x < 0, \\ \psi_{\mathrm{II}}, & 0 < x < a, \\ \psi_{\mathrm{III}} = \psi_T, & x > a, \end{cases} \tag{F.1}$$

with

$$\begin{aligned} \psi_{\mathrm{in}} &\equiv e^{ikx}, \\ \psi_R &\equiv A_R e^{-ikx}, \\ \psi_{\mathrm{II}} &\equiv \alpha e^{-\chi x} + \beta e^{\chi x}, \\ \psi_T &\equiv A_T e^{ikx}, \end{aligned} \tag{F.2$'$}$$

where $k = (2mE)^{1/2}/\hbar$, $\chi = (2m(V_0 - E))^{1/2}/\hbar$, and E and m are the particle kinetic energy and mass, respectively. The coefficients (amplitudes) A_R, A_T, α, and β can be analytically calculated and are well known.

The tunneling evolution has to be described by the *nonstationary description of actually moving wave packets.* These are built up in terms of the solutions $\psi(k, x)$ of the stationary Schroedinger equation by using the resolution of the evolution operator, namely, by integrating $\psi(k, x)\exp(-iEt/\hbar)$ over E from 0 to ∞ with a weight amplitude $g(E - \overline{\mathbf{E}})$,

$$\Psi(x, t) = \int dE\, g(E - \overline{\mathbf{E}})\psi(k, x)\exp\left(-\frac{iEt}{\hbar}\right), \tag{F.2}$$

where we assume the normalization condition $\int dE|g(E - \overline{\mathbf{E}})|^2 = 1$, quantity $\overline{\mathbf{E}}$ being the average kinetic energy.

By inserting in the integral (F.2) the initial (ψ_{in}), reflected (ψ_R) and transmitted (ψ_T) wave, instead of the total wave ψ, we obtain the *initial*, *final* reflected and transmitted wave packets, respectively, carrying a time delay during the motion or due to the interaction.

Of course, there is always a certain distortion in the wave packet form due to the energy dependence of A_R and A_T; but for a wide class of weight amplitudes such a distortion is negligible [6]. Moreover, we can get rid of the wave components with above-barrier energies by introducing the additional transformation

$$g(E-\overline{\mathbf{E}}) \longrightarrow G(E-\overline{\mathbf{E}}) \equiv g(E-\overline{\mathbf{E}})\Theta(E-V_0), \tag{F.3}$$

(where $\Theta(z)$ is the Heaviside step function) in order to avoid distortions of the under-barrier penetration (tunneling) due to the over-barrier transitions.

In the *conventional approach,* one requires that the stationary wave function $\varphi(k,x)$ and its first derivative be continuous across both potential discontinuities at $x = 0$ and $x = a$. This results in four equations in the four unknowns A_R, A_T, α, and β. If one does not take explicitly into account multiple internal reflections inside the barrier, the *general* tunneling evolution is quite simply described by passing from the stationary solution $\varphi(k,x)$ to the nonstationary wave packet $\Psi(x,t)$, defined by (F.2)-(F.3), but in a more detailed description including multiple internal reflections one has to take a different course.

Our model of the tunneling process of a nonrelativistic particle is as follows. We are going to solve the problem of a wave packet incident on the first (initial) potential wall;

(1) without taking into account the second (final) potential barrier wall, because the wave packet has not yet reached it in consequence of its finite propagation speed;

(2) being careful of not breaking the requirement of the finiteness of the wave function (packet) for infinitely wide barriers (when increasing waves have to be absent);

(3) constructing the waves in successive steps of multiple internal reflections from both barrier walls, *in such a way that they are analytical continuations of the related expressions, corresponding to current waves in the simpler case of above-barrier energies*.

We can therefore distinguish three subsequent steps in the evolution of the tunneling process.

Step 1 (the particle starts tunneling the barrier by crossing the first wall at $x = 0$). At this initial step, in region I we have the *initial* time-dependent wave packet

$$\Psi_{\text{in}}(x,t) = \int dE\, G(E-\overline{\mathbf{E}})\varphi_{\text{in}}(k,x)\exp\left(-\frac{iEt}{\hbar}\right), \quad x<0, \tag{F.4}$$

plus the externally reflected (from the initial potential wall) time-dependent wave packet

$$\Psi_R^{\text{ex}}(x,t) = \int dE\, G(E-\overline{\mathbf{E}})\varphi_R^{\text{ex}}(k,x)\exp\left(-\frac{iEt}{\hbar}\right), \quad x<0. \tag{F.5}$$

The sum of the wave packets (F.4) and (F.5) transforms continuously, in the passage across the initial potential wall, into the internal time-dependent wave packet inside the barrier (in region II). In the hypothesis that the tunneling packet does not feel the final potential wall, the

corresponding flux is directed initially only towards the second wall, that is, the penetrated wave packet does contain only decreasing waves:

$$\Psi^1_{\text{pen}}(x,t) = \int dE\, G(E-\overline{\mathbf{E}})\alpha_0 \exp(-\chi x)\exp\left(-\frac{iEt}{\hbar}\right), \quad 0 < x < a, \tag{F.6}$$

by α_0 we denote the coefficient of initial penetration. Then, passing from the time-dependent wave packets to the corresponding stationary wave functions, and requiring continuity at $x = 0$, we obtain for Step 1 (entrance of the initial wave packet inside the barrier) the following two equations in the two unknowns A^0_R, α_0:

$$\begin{gathered}\exp(ikx)\big|_{x=0} + A^0_R \exp(-ikx)\big|_{x=0} = \alpha_0 \exp(\chi x)\big|_{x=0}, \\ ik\left[\exp(ikx) - A^0_R \exp(-ikx)\right]\big|_{x=0} = -\chi\alpha_0,\end{gathered} \tag{F.7}$$

where A^0_R is the coefficient of the initial (external) reflection. Let us stress that *the stationary flux for* $\alpha_0 \exp(-\chi x)$ *is equal to zero and the total flux for* $\Psi^1_{\text{pen}}(x,t)$, *integrated over time* t, *is zero, too*.

Step 2 (the particle crosses the second barrier wall at $x = a$). The wave packet, penetrated inside region II, reaches the second wall of the barrier. It therefore transforms into a wave packet, transmitted through the final wall and propagated into region III, plus a wave packet, reflected from the same wall and penetrated back into region II. Quite similarly to (F.7) we obtain for this step, by the continuity requirement at $x = a$, the following two equations:

$$\begin{gathered}\alpha_0 \exp(-\chi x)\big|_{x=a} + \beta_0 \exp(\chi x)\big|_{x=a} = A^0_T \exp(ikx)\big|_{x=a}, \\ \chi\left[-\alpha_0 \exp(-\chi x) + \beta_0 \exp(\chi x)\right]\big|_{x=a} = -ikA^0_T \exp(ikx)\big|_{x=a},\end{gathered} \tag{F.8}$$

which can be solved to yield the unknown coefficients β_0 (amplitude of the evanescent wave reflected from the second wall in region II) and A^0_T (amplitude of the wave transmitted through the second wall in region III).

Step 3 (the particle, bounced back from the second wall, crosses again the first wall moving in the negative x-direction). The wave packet reflected from the second wall is incident inside the barrier upon the first wall. Then, it transforms into a wave packet transmitted through this wall (as an addition to the packet reflected back in region I), and in a wave packet reflected from the same wall forward inside region II. Again, quite similarly to (F.7) and (F.8), we get the following two equations in the unknown coefficients α_1 (amplitude of the evanescent wave reflected into region II) and A^1_R (amplitude of the wave transmitted through the first wall in region I):

$$\begin{gathered}\alpha_1 \exp(-\chi x)\big|_{x=0} + \beta_0 \exp(\chi x)\big|_{x=0} = A^1_R \exp(ikx)\big|_{x=0}, \\ \chi\left[-\alpha_1 \exp(-\chi x) + \beta_0 \exp(\chi x)\right]\big|_{x=0} = -ikA^1_R.\end{gathered} \tag{F.9}$$

Step 3 corresponds, of course, to a first internal reflection. The process can be iterated by considering successive processes of internal reflections of the gradually decreasing (in consequence of preceding internal incidences with the walls) wave packet from the barrier walls (with partial transmissions through the walls outside). Such a description of the tunneling process is just the *multiple internal reflection approach*. It is easily seen that any of the subsequent steps can be reduced to one of the first three steps considered above. Moreover, we obtain, from the continuity requirement, the following recurrence relations:

$$\begin{gathered}\alpha_0 = \frac{2k}{k+i\chi}, \qquad \beta_n = \alpha_n \frac{i\chi - k}{i\chi + k}\exp(-2\chi a), \qquad \alpha_{n+1} = \beta_n \frac{i\chi - k}{i\chi + k}, \\ A_R^0 = \frac{k - i\chi}{k + i\kappa\chi}, \qquad A_T^n = \alpha_n \frac{2i\chi}{i\chi + k}\exp(-\chi a - ika), \qquad A_R^{n+1} = \beta_n \frac{2i\chi}{i\chi + k}\end{gathered} \tag{F.10}$$

for the unknown coefficients α_n, β_n, A_T^n, and A_R^n ($n = 0, 1, \ldots$) at the successive steps of evolution of the tunneling wave packet. The number n labels the successive steps of evolution of the wave packet inside the barrier, starting from $n = 0$ (beginning of the wave-packet penetration inside the barrier). For $n \neq 0$, the corresponding evolution step is the internal reflection from any barrier wall until the arriving to the other wall. Odd values $n = 2\mu + 1$ correspond to reflections from the first wall, with amplitude α_μ, whereas even values $n = 2(v + 1)$ correspond to reflections from the second wall, with amplitude β_v (e.g., step $n = 2$ discussed above describes reflection back inside from the second wall with amplitude β_0, while step $n = 3$ describes reflection forward inside from the first wall with amplitude α_1).

The *general* evolution of the initial wave packet, tunneling through the barrier, is obtained by summing on all possible steps. It is easy to see that

$$\begin{gathered}A_T = \sum_{n=0}^{\infty} A_T^n = \frac{4ik\chi\exp(-\chi a - ika)}{F}, \\ A_R = \sum_{n=0}^{\infty} A_R^n = \frac{k_0^2 D_-}{F}, \\ \alpha = \sum_{n=0}^{\infty} \alpha_n = \frac{2k(k + i\chi)}{F}, \\ \beta = \sum_{n=0}^{\infty} \beta_n = \frac{2k(i\chi - k)\exp(-2\chi a)}{F},\end{gathered} \tag{F.11}$$

where $F = (k^2 - \chi^2)D_- + 2ik\chi D_+$, $D_\pm = 1 \pm \exp(-2\chi a)$, $k_0^2 = k^2 + \chi^2 = 2mV_0/\hbar^2$.

All these results for the coefficients α, β, A_T, and A_R coincide with those derived by a standard treatment of the tunneling process based on the general expressions for the stationary wave function $\psi(k, x)$, (F.1), and for the nonstationary wave packet $\Psi(x, t)$, (F.2)-(F.3). Moreover, by replacing

$$i\chi \longrightarrow k_1, \tag{F.12}$$

where $k_1 = [2m(E - V_0)]^{1/2}/\hbar$ is the wave vector for the case of *above-barrier* energies ($E > V_0$), all the expressions (F.10)-(F.11) for α_n, β_n, A_T^n, A_R^n, α, β, A_T, and A_R transform into

the corresponding ones obtained in the analysis of the ordinary particle propagation above barrier in terms of multiple internal reflections.

In the limit $\chi a \to \infty$ (i.e., for infinitely wide and/or high barriers), the infinite series of multiple internal reflections reduces to only one step (the first) and we get (F.4)–(F.7) as total solution, with

$$A_T = 0, \qquad A_R = A_R^0, \qquad \alpha = \alpha_0, \qquad \beta = 0, \tag{F.13}$$

instead of (F.11). This case—together with the similar case of a subbarrier energy particle tunneling into a semiclosed barrier region with a dead stopper at the second wall—has been analyzed in [167].

Intermediate Reflection and Traverse Times, Total Tunneling and Reflection Times

In order to discuss the relevant times involved in the multiple reflection description of the tunneling process, we will exploit the general definition of phase times as times of propagation of the wave packet maximum (peak) for the quasimonochromatic wave packets (F.1)–(F.3). We have, respectively:

(1)

$$t_{\text{inc}} = \frac{\hbar\partial(\arg g)}{\partial E} \tag{F.14}$$

for the incident phase time at the barrier beginning ($x = 0$); we take this as the zero (origin) time;

(2)

$$\tau_{\text{refl}}^1 = t_{\text{refl}}^1 - t_{\text{inc}} = \frac{\hbar\partial \arg A_R^0}{\partial E} = \frac{2}{v\chi} \tag{F.15}$$

(with $v = \hbar k/m$ being the mean—or group—incident velocity) for the 1-step (external) reflection phase time;

(3)

$$\tau_{\text{tr}}^1 = t_{\text{tr}}^1 - t_{\text{inc}} = \frac{a}{v} + \frac{\hbar\partial\left(\arg A_T^0\right)}{\partial E} = \frac{2}{v\chi} \tag{F.16}$$

for the 1-step traverse phase time at the barrier end ($x = a$).

Similarly, we obtain the following expressions for the n-step reflection and traverse phase times:

$$\begin{aligned} \tau_{\text{refl}}^n &= \frac{4(v+1)}{v\chi}, \quad n = v+1,\ v = 0, 1, \ldots, \\ \tau_{\text{tr}}^n &= \frac{2(2\mu+1)}{v\chi}, \quad n = 2\mu+1,\ \mu = 0, 1, \ldots. \end{aligned} \tag{F.17}$$

Then, the total tunneling and reflection phase times are defined by:

$$\tau_{\text{tun}} = \frac{a}{v} + \frac{\hbar\partial\sum_{n=0}^{\infty}\arg A_T^n}{\partial E} = \frac{(v\chi)^{-1}\left[k_0^2\,\text{sh}(2\chi a) + 2\chi a k^2(\chi^2 - k^2)\right]}{\left[4k^2\chi^2 + k_0^2\,\text{sh}^2(\chi a)\right]},$$
$$\tau_{\text{refl}} = \frac{\hbar\partial\sum_{n=0}^{\infty}\arg A_R^n}{\partial E} = \tau_{\text{tun}}. \tag{F.18}$$

In the limit $\chi a \to \infty$ (for infinitely wide and/or high barriers) one gets

$$\tau_{\text{tun}} = \tau_{\text{refl}} = \tau_{\text{refl}}^1 = \tau_{\text{tr}}^1 = \frac{2}{v\chi}. \tag{F.19}$$

We see that for $\mu > 0$ and $v \geq 0$ all α_μ, β_v, A_T^v, and A_R^μ are exponentially decreasing for $\chi a \to \infty$. Let us stress that *not only* τ_{tun}, *but also all* τ_{tr}^n $(n = 1, 2, \ldots)$, *exhibit the HFE*, that is, the independence of τ_{tun} from a for large a (which implies an infinite growing of the tunneling velocity a/τ_{tun} for $a \to \infty$).

Study of the Physical Meaning of the Evanescent-Wave Packets with the Help of the Virtual Momentum Fourier Expansion and the Instanton Approaches

We want now to discuss the physical meaning of the wave packet (F.6) constructed from evanescent waves by exploiting the virtual momentum Fourier expansion and the instanton approach.

The Fourier Expansion

We will expand the wave packet (F.6) constructed from evanescent waves as a virtual momentum $(-\infty < q < \infty)$ Fourier integral:

$$\begin{aligned}\Psi_{\text{pen}}^1(x,t) &= \int dE\, G(E - \overline{\mathbf{E}})\alpha_0 \exp(-\chi x)\exp\left(-\frac{iEt}{\hbar}\right)\\ &= (2\pi)^{-1}\int_{-\infty}^{\infty} dq\, g_\eta(q)\exp(-qx)\\ &\quad\times\int dE\, G(E - \overline{\mathbf{E}})\alpha_0 \exp\left(-\frac{iEt}{\hbar}\right)\left[-\chi - iq\right]^{-1}\left[\exp(-\chi a - iqa) - 1\right],\end{aligned} \tag{F.20}$$

where the infinitesimally narrow-step function $g_\eta(q)$, defined as

$$g_\eta(q) = \begin{cases}(n+1)\left(\dfrac{q}{\eta}\right)^n - n\left(\dfrac{|q|}{\eta}\right)^{n+1}, & n \geq 2,\ 0 \leq |q| \leq \eta,\\ 1, & \eta \leq |q|,\end{cases} \tag{F.21}$$

within the infinitesimal interval $(-\eta, \eta)$ $(\eta > 0, \eta \to 0^+)$, is inserted in order to eliminate the point $q = 0$. By passing to the variable $\varepsilon = E - E_q$ (with $E_q = \hbar^2 q^2/2m$), (F.20) can be rewritten as

$$\Psi^1_{\text{pen}}(x,t) = (2\pi)^{-1}\int_{-\infty}^{\infty} dq\, g_\eta(q) \exp\left(iqx - \frac{iE_q t}{\hbar}\right) \\ \times \int dE\, G(E - \overline{\mathbf{E}})\alpha_0 \exp\left(-\frac{i\varepsilon t}{\hbar}\right)[-\chi - iq]^{-1}[\exp(-\chi a - iqa) - 1]. \tag{F.22}$$

In (F.22) one can see *current waves* $\exp(iqx - iE_q t/\hbar)$ and the oscillating factor $\exp(-i\varepsilon t/\hbar)$, damped with time. The main contribution to (F.22) originates from the current waves which satisfy the condition

$$\varepsilon \longrightarrow 0, \quad \text{or} \quad E_q \approx E, \tag{F.23}$$

when the dampingis practically absent even for large t,—more precisely, when

$$\frac{\varepsilon t}{\hbar} \leq 1, \tag{F.24}$$

and also for q within the range

$$-\chi \leq q \leq \chi. \tag{F.25}$$

(By the way, it is striking that generally in this approach both symmetrically forward $(q > 0)$ and backward $(q < 0)$ motions are possible.) Either condition (F.23) or (F.25) is satisfied when both k and χ are within the range Δk, covered by the weight amplitude $g(E - \overline{\mathbf{E}})$.

The Instanton Approach

We will now analyze the propagation of the wave packet (F.6) using the well-known method of transforming the space-time metric inside the barrier region, namely the formal inspection of the wave-packet motion along an imaginary time axis $t \to i\tau$, which is typical of the instanton approach (see, e.g., [116, 168, 169] and references therein). In this case, χ becomes imaginary: $\chi \to -i\kappa$ $(\kappa > 0)$, and therefore one gets *nonzero fluxes* along the axis τ; moreover, $E \to E'$. So, we get for $\Psi^1_{\text{pen}}(x,t)$

$$\Psi^1_{\text{pen}}(x,t) = \int dE\, \alpha_0 G(E - \overline{\mathbf{E}}) \exp\left(i\kappa x - \frac{E\tau}{\hbar}\right). \tag{F.26}$$

By introducing the virtual-energy $(-\infty < \acute{\mathbf{E}} < \infty)$ Fourier transform

$$\exp\left(-\frac{E'\tau}{\hbar}\right) = \hbar^{-1}(2\pi)^{-1/2}\int_{-\infty}^{\infty} D(E', \acute{\mathbf{E}}) \exp\left(\frac{i\acute{\mathbf{E}}\tau}{\hbar}\right) d\acute{\mathbf{E}}, \tag{F.27}$$

with

$$D(E',\acute{\mathbf{E}}) = \hbar(2\pi)^{-1/2}\int_0^\infty \exp\left(-\frac{E'\tau}{\hbar}\right)\exp\left(-\frac{i\acute{\mathbf{E}}\tau}{\hbar}\right)d\tau = \hbar(2\pi)^{-1/2}\left(E'+i\acute{\mathbf{E}}\right)^{-1}, \tag{F.28}$$

equation (F.26) can be rewritten as

$$\Psi^1_{\text{pen}}(x,t) = (2\pi)^{-1}\int dE\, G(E-\overline{\mathbf{E}})\alpha_0 \exp\left(i\kappa - \frac{iE_\kappa\tau}{\hbar}\right)\int_{-\infty}^{\infty} d\acute{\mathbf{E}}\, \exp\left[\frac{i(E_\kappa+\acute{\mathbf{E}})\tau}{\hbar}\right]\left[E'+i\acute{\mathbf{E}}\right]^{-1}, \tag{F.29}$$

where $E_\kappa = \hbar^2\kappa^2/2m$. So, it is easily seen that along the axis τ inside the barrier, the functions (F.29) correspond to wave-packet motions with group velocities $\pm dE_\kappa/d|\kappa| = \hbar|\kappa|/m$, accompanied by oscillations described by the factor $\exp[i(E_\kappa+\acute{\mathbf{E}})\tau/\hbar]$, damped for $\tau \to \infty$.

In (F.29) one can see *current waves* $\exp\left(i\kappa x - iE_\kappa\tau/\hbar\right)$ and the oscillating factor $\exp[i(E_\kappa+\acute{\mathbf{E}})\tau/\hbar]$ damped with time. The main contribution to (F.29) originates from the current waves which satisfy the condition

$$\frac{\left|E_\kappa+\acute{\mathbf{E}}\right|\tau}{\hbar} \le 1, \tag{F.30}$$

when the dampingis practically absent, and also for $\acute{\mathbf{E}}$ within the range

$$-E \le \acute{\mathbf{E}} \le E. \tag{F.31}$$

Either condition (F.30)-(F.31) is satisfied when both E and E_κ are within the range ΔE (or k and κ are within the range $\Delta k = ((2m\Delta E)^{1/2}/\hbar)$, covered by the weight amplitude $g(E-\overline{\mathbf{E}})$, precisely as in the previous case. The only difference between these approaches is the absence of backward motion in the instanton representation.

Some Words on Multiple Internal Reflections for Particle Energies above a Single Barrier and for Two Separated Barriers with Even Particle Subbarrier Energies

For the case of the above-barrier energies, there is an evident influence of multiple internal reflections of propagating waves. In [170–174] it was presented the detailed study of such influence and it was shown the presence of a multitude of various secondary reflected and transmitted peaks in addition to two main treflected and transmitted wave packet peaks. In [175], the influence of multiple internal reflections was extended for transmissions of relativistic Dirac wave packets with energies above a single barrier.

In [176], the influence of multiple internal reflections was further extended, taking into account multiple internal reflections in the motion of free (nonrelativisctic) propagating wave packets between two separated barriers. It was shown the existence of multiple peaks, due to multiple reflections in the free region between two barriers, in addition to the assumption of a single outgoing peak in the case of neglecting such multiple reflections as it was made in Section 3.7 on the base of [127–129].

The Case of Photon Tunneling

It follows from [165, 166] (see also [47] and Section 2.4) that all the results and conclusions of Appendix F for subbarier energies, concerning both multiple internal reflections and total phase tunneling and reflection times, are also valid for the tunneling of photons in a 1D rectangular wave guide (with the only substitution $v \to c$ in (2.27)–(2.31)).

In the particular case of quasimonochromatic wave packets, using the stationary-phase method under the same boundary and measurement conditions as considered for particles in Section 3.2, we obtained there the identical expression (3.19) which manifests the superluminal photon effective tunneling velocity. This result agrees with the experimental findings of the microwave-tunneling measurements presented in [170–174]. Analyzing now the multiple internal reflections for photons, one can also see that (as in the particle case) not only τ_{tun} but also all τ_{tr}^{n} $(n = 1, 2, \ldots)$ manifest the HFE.

Summary: Conclusions and Perspectives

(1) The time operator (2.1a), (2.1b), and (2.26) is general for any quantum collision and process in the continuum energy spectrum within nonrelativistic quantum mechanics and quantum electrodynamics (with self-consistent definition of averaging measures over time for the 1D particle and photon motion). Of course, it cannot be defined in the cases with zero fluxes and unmoving particles, but for the same cases *apriori* there is no evolution processes at all. The uniqueness of the maximal symmetric time operator (2.1a), (2.1b), and (2.26) does directly follow from the uniqueness of the Fourier transformations from time representation to energy one.

Two measures of averaging over time and connection between them are analyzed. The foundations of the self-consistent time analysis for quantum (in particular, tunneling and nuclear) processes are in fact developed on the base of the time operator with the proper measure of averaging over time (2.7) and/or (2.9). They include the mathematically rigorous and one-to-one time and energy representations of the Olkhovsky-Recami definition of mean durations and variances in distributions of durations for all really known Hamiltonians in various quantum processes and collisions (including all kinds of multiple internal reflections between barriers and inside barriers among them).

An actual perspective for the nearest future is opened for generalizing the time analysis of quantum processes for more complicated particle and photon motions (e.g., such as along helixes and motions through 2D and 3D (nonspherical) potentials and barriers).

As to systems with the discrete energy spectra, the form (2.43) for the time operator corresponds to the class of bound-state wave functions just similarly to the situation with the azimuth-angle operator. It is general for processes in the discrete energy spectrum (or a system has a purely discrete energy spectrum either it has a discrete spectrum as a part of its total energy spectrum). The time operator cannot be defined in the case of *one* bound state with *zero* flux and *unmoving* particle, and there is no evolution here too.

Once more we underline that at the limit of infinitesimally close levels formula (2.50) passes to formula (2.1b) for systems with continuous spectra.

All commutation relations, analyzed here, (2.24) and (2.44) and also uncertainty relations (2.25) and (2.47) are set side by side with the similar relations for other pairs of canonically conjugate observables (such as, for coordinate $\hat{x}$ and momentum $\hat{p}_x$ in the case of (2.25), and for azimuthal angle φ and angular momentum $\hat{L}_z$ in the case of (2.47)). Our relations do not replace, but essentially extend the meaning of the time and energy

uncertainties given in [177, 178]. Also they are consistent with the conclusions of [179, 180] where, in particular, it was directly said about absence of objections against those limitations on the time and energy measurements which can be derived from the mathematic formalism with introducing the adequate energy and time operators and the corresponding statistical fluctuations. We hope that these relations which are deduced with applying the properties of the time operator can help to attenuate endless debates about the status of the time-energy uncertainty relation.

(2) Finally, we state that *not only time but any other quantities, to which maximal symmetrical (Hermitian) operators correspond* (e.g., a momentum in a semispace with a rigid wall and a radial momentum, both are defined over the semibounded axis from 0 to ∞) can be considered as *quantum-physical observables* in the same degree as the quantities to which self-adjoint operators correspond, *without introducing any new physical postulates.* The same conclusion is valid for quasi-self-adjoint operators like (2.37) and (2.43).

(3) Similar derivations and conclusions with quite evident generalization can be carried out for time operator in relativistic quantum mechanics (the Klein-Gordon case and the Dirac case). It is rather perspective (but, of course, not always simple) to develop the analysis of three- and four-position operators for other relativistic cases, especially to analyze the localization problems (the Dirac particles, 2D and 3D particles and photon motions, etc.)

(4) Actually, the time operator (2.1a), (2.1b), and (2.26) has been rather fruitfully used in the case of the tunneling times (see [26, 27, 32–35, 47]). We have established that practically all earlier known particular tunneling times appear to be the special cases of the mean tunneling time or of the square root of the variance in the tunneling-time distribution (or pass into them under some boundary conditions), defined within the general O-R approach. It had been carried out in some reviews (in particular, in [26, 27, 32, 47, 81]) also the connection of other earlier known approaches or simultaneously elaborated approaches with the O-R approach which had been recognized as the most self-consistent definition of the tunnelling time within the conventional quantum mechanics (see, e.g., [81]).

It is meaningful to stress also that, although any direct classical limit for particle tunnelling through potential barrier with subbarrier energies is really absent, *there is the direct classical limit for wave packet tunneling*. Let us recall *real evanescent and antievanescent waves*, well known in *classical* optics and in *classical*acoustics (see, e.g., [176, 181–185]).

(5) The HFE is now extended for all expressions of mean tunneling times, however, with sufficiently narrow momentum spreads of initial particle wave packets (and, of course, for quasimonochromatical particles). The violations of the HFE are revealed and explained for the presence of the absorption and also for the cases of the rather large momentum spreads of initial particle wave packets.

(6) It is elaborated the rigorous combined resonant and nonresonant description of the 1D particle tunneling through two potential rectangular barriers.

(7) There are derived and analyzed new general expressions for the elastic scattering S-matrix, for internal and external transmissions and reflection probability amplitudes and also the connection between them in a 3D tunnelling process through spherically symmetric potential (rectangular and Coulomb) barriers, taking into account the multiple internal reflections from internal barrier wall into the potential well. In the case of a rectangular barrier, there are also derived the expressions for the tunnelling and reflection phase time, and it is shown the occurrence of the HFE. Of course, in the realistic 3D situations in nuclear physics one has to use typical phenomenological potentials (like the Saxon-Woods well) and consider also the charge distribution inside and at the surface of the daughter nucleus.

We have also taken into account the resonances and written in this case the explicit analytical expression for the resonant S-matrix, with the resonance width proportional to the very small exponential factor $e^{-2\chi(R_2-R_1)}$ (in the case of a rectangular barrier with $l = 0$). Similarly one can write the resonant S-matrix for the Coulomb barrier, with the resonance width proportional to the very small exponential factor $e^{-2\pi\eta}$ and for cases with $l > 0$.

The presented method can be used as the initial phase for the joint time-dependent study of nuclear reactions (beginning from the scattering) and decays for any value of l, and not only for spherically symmetric interactions but also for nonspherical cases (beginning from the coupled-channel method). Also the results for the case of the Coulomb barrier can be used as an initial phase for analysis of the subbarrier low-energy astrophysical nuclear-fusion reactions (near the Gamow energy). In distinction from the 1D WKB approximation which is used till now for such analysis, in the 3D case it is important even for very small k to take into account not only the exponential factor $e^{-2\pi\eta}$ but also the multiple internal reflections with

$$A_R^{\text{in}} \longrightarrow e^{2iKR_1}, \qquad \frac{1}{1 + A_R^{\text{in}}} \longrightarrow \left[1 + \exp\left(2ikR_1\right)\right]^{-1} \tag{F.32}$$

for both resonance and nonresonant collisions.

Then, for this same system, we have derived the elastic scattering S-matrix (and its connection with the transmission and reflection probability amplitudes) and all quantities related taking into account the multiple internal reflection.

The results we have obtained can be used for the study of the alpha and proton radioactivity decay, if one will use the Coulomb barrier instead of the rectangular one and may be introduce the hard core inside the internal well and then modify all the potentials into more realistic ("smoothed" like Woods-Saxon well, etc.) potentials. Also they can be used in another field, with the appropriate modifications, for the study of the photon emission from a glass sphere surrounded by a spherical air layer and externally by another glass sphere.

(8) The analysis of the possibility and the study of the multiple internal reflections, of wave packets with subbarrier energies, not only between barriers but also inside barriers (and not only for particles but also for photons) by several approaches is carried out in Appendix F. Inside barriers we have used (i) the analytic continuation from axis of real momentum values to axis of imaginary momentum values, (ii) the Fourier expansion, and (iii) the instanton approach.

One can see an interesting perspective to research the multiple internal reflections also experimentally (in particular, by frustrated total internal reflections by 2D and 3D barriers in photon (see [181–183] where namely such optical experiments are firstly described) and acoustical tunneling) with presented here theoretic foundations of time analysis of quantum processes.

Another interesting perspective, firstly in theoretical reseach, is revealed in [186, 187] by establishing of not only *incoherent* multiple internal reflections for particles and also photons but also *coherent* multiple internal reflections from barriers inside wells between barriers—-in the cases of the tunneling of small bound systems through certain Coulomb barriers during long decays of more complex systems (e.g., long-living alpha-radioactive nuclei and heavy nuclei, undergoing the spontaneous fission, etc.)

(9) Some results of the time analysis of nuclear collisions and decays are briefly reviewed. In the regions of isolated resonances, distorted by the nonresonant background, the principal possibility of the negative values of time delays (i.e., advances) is shown and

concretely the time analysis of proton scattering by nuclei ^{12}C and ^{14}N at the range of isolated resonances distorted by the nonresonant background, accompanied by the bremsstrahlung, resulted in the discovery of the real possibility of the delay-advance phenomenon in the proton emission during scattering [151, 152].

The time analysis of high-energy nuclear reactions, with highly excited (at the range of very dense overlapping energy resonances) final compound-fragment formations, resulted in the revealing of the new phenomenon of time resonances (or explosions) of such formations [148].

Results of the time analysis for other nuclear processes are presented in [21, 23–25, 188–191].

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References

[1] W. Pauli, in *Handbuch der Physik*, S. Fluegge, Ed., vol. 5/1, p. 60, Springer, Berlin, Germany, 1926.
[2] W. Pauli, *General Principles of Quantum Mechanics*, Springer, Berlin, Germany, 1980.
[3] H. Paul, "Über quantenmechanische Zeitoperatoren," *Annalen der Physik*, vol. 9, no. 7, pp. 252–261, 1962.
[4] E. Fick and F. Engelmann, "Quantentheorie der Zeitmessung," *Zeitschrift für Physik A*, vol. 175, no. 3, pp. 271–282, 1963.
[5] F. Engelmann and E. Fick, "Quantentheorie der Zeitmessung—II," *Zeitschrift für Physik A*, vol. 178, no. 5, pp. 551–562, 1964.
[6] B. A. Lippmann, "Operator for time delay induced by scattering," *Physical Review*, vol. 151, no. 4, pp. 1023–1024, 1966.
[7] M. Razavy, "Quantum-mechanical conjugate of the hamiltonian operator," *Il Nuovo Cimento B*, vol. 63, no. 1, pp. 271–308, 1969.
[8] T. T. Gien, "On the operators for time of motion and delay time induced by scattering," *Canadian Journal of Physics*, vol. 47, no. 3, pp. 279–289, 1969.
[9] T. T. Gien, "Delay time and phase of the initial state," *Canadian Journal of Physics*, vol. 48, no. 6, pp. 639–652, 1970.
[10] G. R. Allcock, "The time of arrival in quantum mechanics—I. Formal considerations," *Annals of Physics*, vol. 53, no. 2, pp. 253–285, 1969.
[11] D. M. Rosenbaum, "Super Hilbert space and the quantum-mechanical time operators," *Journal of Mathematical Physics*, vol. 10, pp. 1127–1144, 1969.
[12] V. S. Olkhovsky and E. Recami, "Space-time shifts and cross-sections in collisions between relativistic wave packets," *Il Nuovo Cimento A*, vol. 53, no. 3, pp. 610–624, 1968.
[13] V. S. Olkhovsky and E. Recami, "About collision-process lifetimes and causality," *Il Nuovo Cimento A*, vol. 63, no. 3, pp. 814–826, 1969.
[14] V. S. Olkhovsky and E. Recami, "About a space-time operator in collision descriptions," *Lettere Al Nuovo Cimento*, vol. 4, no. 24, pp. 1165–1173, 1970.
[15] V. S. Olkhovsky, "On the problem of time operator and collision duration," *Ukrainskiy Fizicheskaya Zhurnal*, vol. 18, p. 1910, 1973.
[16] V. S. Olkhovsky, E. Recami, and A. J. Gerasimchuk, "Time operator in quantum mechanics—I: nonrelativistic case," *Il Nuovo Cimento A*, vol. 22, no. 2, pp. 263–278, 1974.

[17] E. Recami, "A time operator and the time-energy uncertainty relation," in *The Uncertainty Principle and Foundation of Quantum Mechanics*, pp. 21–28, John Wiley & Sons, London, UK, 1977.
[18] E. Recami, "An operator for observable time," in *Proceedings of the 13th Winter School in Theoretical Physics*, vol. 2, pp. 251–256, Wroclaw, Poland, 1976.
[19] A. S. Holevo, "Estimation of shift parameters of a quantum state," *Reports on Mathematical Physics*, vol. 13, no. 3, pp. 379–399, 1978.
[20] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory*, vol. 1 of *North-Holland Series in Statistics and Probability*, North-Holland, Amsterdam, The Netherlands, 1982.
[21] V. S. Olkhovsky, "To the investigation of nuclear reactions and decays by analysis of their durations," *Soviet Journal of Particles and Nuclei*, vol. 15, no. 2, pp. 130–148, 1984.
[22] W. Jaworski and D. M. Wardlaw, "Time delay in tunneling: transmission and reflection time delays," *Physical Review A*, vol. 37, no. 8, pp. 2843–2854, 1988.
[23] V. S. Olkhovsky, "Nonstationary characteristics in study of nuclear reaction mechanism and kinetics and compound-nucleus properties," *Nukleonika*, vol. 35, no. 1–3, pp. 99–144, 1990.
[24] V. S. Olkhovsky, "Time analysis of nuclear collisions and decays," *Atti della Accademia Peloritana dei Pericolanti. Classe di Scienze Fisiche, Matemàtiche e Naturali*, vol. 70, p. 21, 1992.
[25] V. S. Olkhovsky, "On time as a quantum-physical observable quantity," in *Mysteries, Puzzles, and Paradoxes in Quantum Mechanics*, R. Bonifacio, Ed., vol. 461 of *AIP Conference Proceedings*, pp. 272–276, American Institute of Physics, Woodbury, NY, USA, 1999.
[26] V. S. Olkhovsky and E. Recami, "Recent developments in the time analysis of tunneling processes," *Physics Reports*, vol. 214, no. 6, pp. 339–356, 1992.
[27] V. S. Olkhovsky, E. Recami, F. Raciti, and A. K. Zaichenko, "More about tunnelling times, the dwell time and the "Hartman effect"," *Journal de Physique I*, vol. 5, no. 10, pp. 1351–1365, 1995.
[28] P. Busch, M. Grabowski, and P. J. Lahti, "Time observables in quantum theory," *Physics Letters A*, vol. 191, no. 5-6, pp. 357–361, 1994.
[29] D. H. Kobe and V. C. Aguilera-Navarro, "Derivation of the energy-time uncertainty relation," *Physical Review A*, vol. 50, no. 2, pp. 933–938, 1994.
[30] Ph. Blanchard and A. Jadczyk, "Time of events in quantum theory," *Helvetica Physica Acta*, vol. 69, no. 5-6, pp. 613–635, 1996.
[31] N. Grot, C. Rovelli, and R. S. Tate, "Time of arrival in quantum mechanics," *Physical Review A*, vol. 54, no. 6, pp. 4676–4690, 1996.
[32] V. S. Olkhovsky and A. Agresti, "Developments in time analysis of particles and photon tunnelling," in *Proceedings of the Adriatico Research Conference on Tunnelling and Its Implications (ICTP '96)*, pp. 327–355, World Scientific, Trieste, Italy, July-August 1996.
[33] V. S. Olkhovsky, "Time analysis of tunneling of particles and photons," *Physics of the Alive*, vol. 5, pp. 19–37, 1997 (Russian).
[34] V. S. Olkhovsky, "Developments in examining time as a quantum-physical observable," *Physics of the Alive*, vol. 6, no. 2, pp. 17–29, 1998.
[35] V. S. Olkhovsky, "Recent developments on time as a quantum-physical observable quantity," *Atti della Accademia Peloritana dei Pericolanti. Classe di Scienze Fisiche, Matemàtiche e Naturali*, pp. 193–209, 1998.
[36] J. León, "Time-of-arrival formalism for the relativistic particle," *Journal of Physics A*, vol. 30, no. 13, pp. 4791–4801, 1997.
[37] R. Giannitrapani, "Positive-operator-valued time observable in quantum mechanics," *International Journal of Theoretical Physics*, vol. 36, no. 7, pp. 1575–1584, 1997.
[38] Y. Aharonov, J. Oppenheim, S. Popescu, B. Reznik, and W. G. Unruh, "Measurement of time of arrival in quantum mechanics," *Physical Review A*, vol. 57, no. 6, pp. 4130–4139, 1998.
[39] H. Atmanspacher and A. Amann, "Positive-operator-valued measures and projection-valued measures of noncommutative time operators," *International Journal of Theoretical Physics*, vol. 37, no. 2, pp. 629–650, 1998.
[40] M. Toller, "Localization of events in space-time," *Physical Review A*, vol. 59, no. 2, pp. 960–970, 1999.
[41] J. Kijowski, "Comment on "Arrival time in quantum mechanics" and "Time of arrival in quantum mechanics"," *Physical Review A*, vol. 59, no. 1, pp. 897–899, 1999.
[42] V. Delgado, "Quantum probability distribution of arrival times and probability current density," *Physical Review A*, vol. 59, no. 2-3, pp. 1010–1020, 1999.
[43] J. G. Muga, J. P. Palao, and C. R. Leavens, "Arrival time distributions and perfect absorption in classical and quantum mechanics," *Physics Letters A*, vol. 253, no. 1-2, pp. 21–27, 1999.

[44] P. Kochański and K. Wódkiewicz, "Operational time of arrival in quantum phase space," *Physical Review A*, vol. 60, no. 4, pp. 2689–2699, 1999.
[45] D. H. Kobe, H. Iwamoto, M. Goto, and V. C. Aguilera-Navarro, "Tunneling time through a barrier using the local value of a "time" operator," *Physical Review A*, vol. 64, no. 2, Article ID 022104, 8 pages, 2001.
[46] J. G. Muga, I. L. Egusquiza, J. A. Damborenea, and F. Delgado, "Bounds and enhancements for negative scattering time delays," *Physical Review A*, vol. 66, no. 4, Article ID 042115, 8 pages, 2002.
[47] V. A. Olkhovsky, E. Recami, and J. Jakiel, "Unified time analysis of photon and particle tunnelling," *Physics Reports*, vol. 398, no. 3, pp. 133–178, 2004.
[48] A. Góźdź and M. Dębicki, "Time operator and quantum projection evolution," *Physics of Atomic Nuclei*, vol. 70, no. 3, pp. 529–536, 2007.
[49] Z.-Y. Wang and C.-D. Xiong, "How to introduce time operator," *Annals of Physics*, vol. 322, no. 10, pp. 2304–2314, 2007.
[50] V. S. Olkhovsky and E. Recami, "Time as a quantum observable," *International Journal of Modern Physics A*, vol. 22, no. 28, pp. 5063–5067, 2007.
[51] V. S. Olkhovsky and E. Recami, "New developments in the study of time as a quantum observable," *International Journal of Modern Physics B*, vol. 22, no. 12, pp. 1877–1897, 2008.
[52] M. Neumark, "Spectral functions of a symmetric operator," *Izvestiya Academii Nauk SSSR. Seriya Matematicheskaya*, vol. 4, no. 3, pp. 277–318, 1940 (Russian).
[53] N. I. Akhiezer and I. M. Glazman, *Theory of Linear Operators in Hilbert Space*, Pitman, Boston, Mass, USA, 1981.
[54] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik*, Springer, Berlin, Germany, 1932.
[55] M. H. Stone, "Linear transformations in Hilbert space—III. Operational methods and group theory," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 16, no. 2, pp. 172–175, 1930.
[56] D. ter Haar, *Elements of Hamiltonian Mechanics*, Pergamon Press, Oxford, NY, USA, 1971.
[57] P. M. Gopych and I. I. Zalyubovsky, "The exponentiality of main law of radioactive decay," *Soviet Journal of Particles and Nuclei*, vol. 19, p. 785, 1988.
[58] M. Neumark, "Positive definite operator functions on a commutative group," *Izvestiya Academii Nauk SSSR. Seriya Matematicheskaya*, vol. 7, no. 5, pp. 237–244, 1943.
[59] R. Y. Chiao, P. G. Kwiat, and A. M. Steinberg, "Analogies between electron and photon tunneling: a proposed experiment to measure photon tunneling times," *Physica B*, vol. 175, no. 1–3, pp. 257–262, 1991.
[60] A. Ranfagni, D. Mugnai, P. Fabeni, and G. P. Pazzi, "Delay-time measurements in narrowed waveguides as a test of tunneling," *Applied Physics Letters*, vol. 58, no. 7, pp. 774–776, 1991.
[61] Th. Martin and R. Landauer, "Time delay of evanescent electromagnetic waves and the analogy to particle tunneling," *Physical Review A*, vol. 45, no. 4, pp. 2611–2617, 1992.
[62] A. M. Steinberg, "Conditional probabilities in quantum theory and the tunneling-time controversy," *Physical Review A*, vol. 52, no. 1, pp. 32–42, 1995.
[63] S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory*, chapter 5.3, Row, Peterson, Evanston, Ill, USA, 1961.
[64] A. I. Akhiezer and V. B. Berestezky, *Quantum Electrodynamics*, Fizmatgiz, Moscow, Russia, 1959.
[65] J. D. Jackson, *Classical Electrodynamics*, John Wiley & Sons, New York, NY, USA, 2nd edition, 1975.
[66] L. Brillouin, *Wave Propagation and Group Velocity*, vol. 8 of *Pure and Applied Physics*, Academic Press, New York, NY, USA, 1960.
[67] L. B. Felsen and N. Marcuvitz, *Radiation and Scattering of Waves*, chapter 1.5, Prentice-Hall, Englewood Cliffs, NJ, USA, 1973.
[68] D. Judge and J. T. Lewis, "On the commutator $[L_{Z,\varphi}]$," *Physics Letters*, vol. 5, no. 3, p. 190, 1963.
[69] P. Carruthers and M. M. Nieto, "Phase and angle variables in quantum mechanics," *Reviews of Modern Physics*, vol. 40, no. 2, pp. 411–440, 1968.
[70] A. S. Davydov, *Quantum Mechanics*, Pergamon Press, Oxford, UK, 1976.
[71] A. S. Davydov, *Quantum Mechanics*, Pergamon Press, Oxford, UK, 1982.
[72] E. H. Hauge and J. A. Støevneng, "Tunneling times: a critical review," *Reviews of Modern Physics*, vol. 61, no. 4, pp. 917–936, 1989.
[73] Z. H. Huang, P. H. Cutler, T. E. Feuchtwang, E. Kazes, H. Q. Nguyen, and T. E. Sullivan, "Model studies of tunneling time," *Journal of Vacuum Science and Technology A*, vol. 8, no. 1, pp. 186–191, 1990.

[74] M. Büttiker, "Traversal, reflection and dwell time for quantum tunneling," in *Electronic Properties of Multilayers and Low Dimensional Semiconductors*, J. Chamberlain, L. Eaves, and J. Portal, Eds., pp. 297–315, Plenum Press, New York, NY, USA, 1990.

[75] C. R. Leavens and G. Aers, "Tunneling times for one-dimensional barriers," in *Scanning Tunneling Microscopy and Related Methods*, R. Behm, N. Garcia, and H. Rohrer, Eds., pp. 59–76, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1990.

[76] A.-P. Jauho, "Tunneling times in semiconductor heterostructures," in *Quantum Transport in Semiconductors*, D. Ferry and C. Jacoboni, Eds., p. 179, Plenum Press, New York, NY, USA, 1991.

[77] A.-P. Jauho, "Tunneling times in semiconductor heterostructures: a critical review," in *Hot Carriers in Semiconductor Nanostructures: Physics and Applications*, J. Shah, Ed., pp. 121–150, Academic Press, London, UK, 1992.

[78] M. Jonson, "Tunneling times quantum mechanical tunneling," in *Quantum Transport in Semiconductors*, D. Ferry and C. Jacoboni, Eds., pp. 193–237, Plenum Press, New York, NY, USA, 1991.

[79] R. Landauer and Th. Martin, "Barrier interaction time in tunneling," *Reviews of Modern Physics*, vol. 66, no. 1, pp. 217–228, 1994.

[80] G. Privitera, G. Salesi, V. S. Olkhovsky, and E. Recami, "Tunnelling times: an elementary introduction," *Rivista del Nuovo Cimento della Societa Italiana di Fisica*, vol. 26, no. 4, pp. 1–55, 2003.

[81] M. Abolhasani and M. Golshani, "Tunneling times in the copenhagen interpretation of quantum mechanics," *Physical Review A*, vol. 62, no. 1, Article ID 012106, 7 pages, 2000.

[82] D. Bohm, *Quantum Theory*, Prentice-Hall, New York, NY, USA, 1952.

[83] E. P. Wigner, "Lower limit for the energy derivative of the scattering phase shift," *Physical Review*, vol. 98, no. 1, pp. 145–147, 1955.

[84] T. E. Hartmann, "Tunneling of a wave packet," *Journal of Applied Physics*, vol. 33, no. 12, pp. 3427–3433, 1962.

[85] J. R. Fletcher, "Time delay in tunnelling through a potential barrier," *Journal of Physics C*, vol. 18, no. 2, pp. L55–L59, 1985.

[86] V. S. Olkhovsky, "About the time duration of scattering processes," *Il Nuovo Cimento B*, vol. 48, no. 1, pp. 178–182, 1967.

[87] R. Vedrinski, "Stationary approach to the definition of time delay in the scattering process," *Il Nuovo Cimento B*, vol. 51, no. 2, pp. 487–491, 1967.

[88] E. H. Hauge, J. P. Falck, and T. A. Fjeldly, "Transmission and reflection times for scattering of wave packets off tunneling barriers," *Physical Review B*, vol. 36, no. 8, pp. 4203–4214, 1987.

[89] R. Landauer and Th. Martin, "Time delay in wave packet tunneling," *Solid State Communications*, vol. 84, no. 1-2, pp. 115–117, 1992.

[90] V. S. Olkhovsky, E. Recami, and A. K. Zaichenko, "Reply to the comments by leavens on Olkhovsky-Recami's approach to the "tunnelling time problem": more about the time analysis of tunnelling processes," *Solid State Communications*, vol. 89, no. 1, pp. 31–35, 1994.

[91] T. Ohmura, "Wave packet theory of scattering. Time delay and spread," *Progress of Theoretical Physics. Supplement*, no. 29, pp. 108–134, 1964.

[92] D. Sokolovski and L. M. Baskin, "Traversal time in quantum scattering," *Physical Review A*, vol. 36, no. 10, pp. 4604–4611, 1987.

[93] E. Pollak and W. H. Miller, "New physical interpretation for time in scattering theory," *Physical Review Letters*, vol. 53, no. 2, pp. 115–118, 1984.

[94] E. Pollak, "The significance of imaginary time in quantal reactive scattering," *The Journal of Physical Chemistry* , vol. 83, no. 3, pp. 1111–1120, 1985.

[95] D. Sokolovski and J. N. L. Connor, "Negative probability and the distributions of dwell, transmission, and reflection times for quantum tunneling," *Physical Review A*, vol. 44, no. 3, pp. 1500–1504, 1991.

[96] D. Sokolovski and J. N. L. Connor, "Quantum interference and determination of the traversal time," *Physical Review A*, vol. 47, no. 6, pp. 4677–4680, 1993.

[97] D. Mugnai and A. Ranfagni, "Instanton bounces as real physical processes," *Europhysics Letters*, vol. 6, no. 1, pp. 1–5, 1988.

[98] D. Mugnai and A. Ranfagni, "Complex classical trajectories in tunnelling: how instanton bounces can become real processes," *Il Nuovo Cimento D*, vol. 14, no. 5, pp. 541–551, 1992.

[99] H. W. Lee and M. O. Scully, "Wigner phase-space description of a Morse oscillator," *The Journal of Chemical Physics*, vol. 77, no. 9, pp. 4604–4610, 1982.

[100] H.-W. Lee and M. O. Scully, "The Wigner phase-space description of collision processes," *Foundations of Physics*, vol. 13, no. 1, pp. 61–72, 1983.
[101] J. G. Muga, S. Brouard, and R. Sala, "Transmission and reflection tunneling times," *Physics Letters A*, vol. 167, no. 1, pp. 24–28, 1992.
[102] S. Brouard, R. Sala, and J. G. Muga, "Systematic approach to define and classify quantum transmission and reflection times," *Physical Review A*, vol. 49, no. 6, pp. 4312–4325, 1994.
[103] C. R. Leavens and G. Aers, "Bohm trajectories and the tunnelling time problem," in *Scanning Tunnelling Microscopy III*, R. Wiesendager and H.-J. Güntherodt, Eds., p. 105, Springer, New York, NY, USA, 1993.
[104] A. I. Baz', "Lifetime of Intermediate States," *Soviet Journal of Nuclear Physics*, vol. 4, pp. 182–188, 1967.
[105] A. I. Baz', "A quantum mechanical calculation of the collision time," *Soviet Journal of Nuclear Physics*, vol. 5, p. 161, 1967.
[106] V. F. Rybachenko, "On particle penetration through a barrier," *Soviet Journal of Nuclear Physics*, vol. 5, p. 484, 1967.
[107] M. Büttiker, "Larmor precession and the traversal time for tunneling," *Physical Review B*, vol. 27, no. 10, pp. 6178–6188, 1983.
[108] M.-Q. Chen and M. S. Wang, "Traversal time for quantum tunneling," *Physics Letters A*, vol. 149, no. 9, pp. 441–444, 1990.
[109] V. Delgado, S. Brouard, and J. G. Muga, "Does positive flux provide a valid definition of tunnelling times?" *Solid State Communications*, vol. 94, no. 12, pp. 979–982, 1995.
[110] C. R. Leavens, "The "tunneling time problem": fundamental incompatibility of the Bohm trajectory approach with the projector and conventional probability current approaches," *Physics Letters A*, vol. 197, no. 2, pp. 88–94, 1995.
[111] M. Büttiker and R. Landauer, "Traversal time for tunneling," *Physical Review Letters*, vol. 49, no. 23, pp. 1739–1742, 1982.
[112] M. Büttiker and R. Landauer, "Traversal time for tunneling," *Physica Scripta*, vol. 32, no. 4, pp. 429–434, 1985.
[113] M. Büttiker and R. Landauer, "Traversal time for tunneling," *IBM Journal of Research and Development*, vol. 30, no. 5, pp. 451–454, 1986.
[114] F. Raciti and G. Salesi, "Complex-barrier tunnelling times," *Journal de Physique I*, vol. 4, no. 12, pp. 1783–1789, 1994.
[115] G. Nimtz, H. Spieker, and M. Brodowsky, "Tunneling with dissipation," *Journal de Physique I*, vol. 4, no. 10, pp. 1379–1382, 1994.
[116] L. S. Schulman, "Ray optics for diffraction: a useful paradox in a path-integral context," in *The Wave-Particle Dualism*, S. Diner, D. Fargue, G. Lochak, and F. Selleri, Eds., D. Reidel, Dordrecht, The Netherlands, 1984.
[117] V. S. Olkhovsky, V. Petrillo, and A. K. Zaichenko, "Decrease of the tunneling time and violation of the Hartman effect for large barriers," *Physical Review A*, vol. 70, no. 3, Article ID 034103, 4 pages, 2004.
[118] P. Krekora, Q. Su, and R. Grobe, "Effects of relativity on the time-resolved tunneling of electron wave packets," *Physical Review A*, vol. 63, no. 3, Article ID 032107, 8 pages, 2001.
[119] P. Krekora, Q. Su, and R. Grobe, "Critique of the Wigner tunneling speed and a proposed alternative," *Physical Review A*, vol. 64, no. 2, Article ID 022105, 8 pages, 2001.
[120] V. Petrillo and D. Janner, "Relativistic analysis of a wave packet interacting with a quantum-mechanical barrier," *Physical Review A*, vol. 67, no. 1, Article ID 012110, 7 pages, 2003.
[121] S. De Leo and P. P. Rotelli, "Dirac equation studies in the tunneling energy zone," *The European Physical Journal C*, vol. 51, no. 1, pp. 241–247, 2007.
[122] A. Hansen and F. Ravndal, "Klein's paradox and its resolution," *Physica Scripta*, vol. 23, no. 6, pp. 1036–1042, 1981.
[123] V. L. Telegdi, "The Oskar Klein Centenary," in *Klein's Paradox Revisited*, U. Lindström, Ed., pp. 112–121, World Scientific, Singapore, 1995.
[124] H. Nitta, T. Kudo, and H. Minowa, "Motion of a wave packet in the Klein paradox," *American Journal of Physics*, vol. 67, no. 11, pp. 966–971, 1999.
[125] N. Dombey and A. Calogeracos, "Seventy years of the Klein paradox," *Physics Report*, vol. 315, no. 1–3, pp. 41–58, 1999.
[126] S. De Leo and P. P. Rotelli, "Barrier paradox in the Klein zone," *Physical Review A*, vol. 73, no. 4, Article ID 042107, 7 pages, 2006.

[127] V. S. Olkhovsky, E. Recami, and G. Salesi, "Superluminal tunneling through two successive barriers," *Europhysics Letters*, vol. 57, no. 6, pp. 879–884, 2002.
[128] S. Esposito, "Multibarrier tunneling," *Physical Review E*, vol. 67, no. 1, Article ID 016609, 7 pages, 2003.
[129] V. S. Olkhovsky, E. Recami, and A. K. Zaichenko, "Resonant and non-resonant tunneling through a double barrier," *Europhysics Letters*, vol. 70, no. 6, pp. 712–718, 2005.
[130] S. Longhi, P. Laporta, M. Belmonte, and E. Recami, "Measurement of superluminal optical tunneling times in double-barrier photonic band gaps," *Physical Review E*, vol. 65, no. 4, Article ID 046610, 6 pages, 2002.
[131] G. Nimtz, private communications.
[132] V. Petrillo and V. S. Olkhovsky, "Tunnelling time of a Gaussian wave packet through two potential barriers," *Central European Journal of Physics*, vol. 3, no. 3, pp. 339–350, 2005.
[133] V. Petrillo and V. S. Olkhovsky, "Time asymptotic expansion of the tunneled wave function for a double-barrier potential," *Europhysics Letters*, vol. 74, no. 2, pp. 327–333, 2006.
[134] J. T. Lunardi and L. A. Manzoni, "Relativistic tunneling through two successive barriers," *Physical Review A*, vol. 76, no. 4, Article ID 042111, 6 pages, 2007.
[135] G. Gamow, "Zur Quantentheorie des Atomkernes," *Zeitschrift für Physik*, vol. 51, no. 3-4, pp. 204–212, 1928.
[136] R. W. Gurney and E. U. Condon, "Wave mechanics and radioactive disintegration," *Nature*, vol. 122, no. 3073, p. 439, 1928.
[137] R. W. Gurney and E. U. Condon, "Quantum mechanics and radioactive disintegration," *Physical Review*, vol. 33, no. 2, pp. 127–140, 1929.
[138] B. Buck, A. C. Merchant, and S. M. Perez, "New look at α decay of heavy nuclei," *Physical Review Letters*, vol. 65, no. 24, pp. 2975–2977, 1990.
[139] B. Buck, A. C. Merchant, and S. M. Perez, "decay calculations with a realistic potential," *Physical Review C*, vol. 45, no. 5, pp. 2247–2253, 1992.
[140] A. Sobieczewski, "Present view of stability of heavy and superheavy nuclei," *Physics of Particles and Nuclei*, vol. 25, no. 2, p. 295, 1994.
[141] V. S. Olkhovsky, V. Petrillo, J. Jakiel, and W. Kantor, "Particle tunneling and scattering in a spherical three-dimensional potential with a barrier," in *Proceedings of the International Conference on Current Problems in Nuclear Physics and Atomic Energy*, pp. 496–506, Kyiv, Ukraine, May-June 2006.
[142] V. S. Olkhovsky, V. Petrillo, J. Jakiel, and W. Kantor, "Particle tunneling and scattering in a three-dimensional potential with a barrier," *Central European Journal of Physics*, vol. 6, no. 1, pp. 122–127, 2008.
[143] S. De Leo and G. C. Ducati, "Quaternionic diffusion by a potential step," *Journal of Mathematical Physics*, vol. 47, no. 10, Article ID 102104, 9 pages, 2006.
[144] S. De Leo and G. C. Ducati, "Quaternionic wave packets," *Journal of Mathematical Physics*, vol. 48, no. 5, Article ID 052111, 10 pages, 2007.
[145] S. De Leo, G. C. Ducati, and T. M. Madureira, "Analytic plane wave solutions for the quaternionic potential step," *Journal of Mathematical Physics*, vol. 47, no. 8, Article ID 082106, 15 pages, 2006.
[146] C. Doran and A. Lasenby, *Physical Applications of Geomatric Algenra: Part 1: Non-Relativistic Physics*, Cambridge University Press, Cambridge, UK, 2003.
[147] A. I. Baz', A. M. Perelomov, and Ya. B. Zel'dovich, *Scattering, Reactions and Decays in Non-Relativistic Quantum Mechanics*, Israel Program for Scientific Translations, Jerusalem, Israel, 1969.
[148] V. S. Olkhovsky, M. E. Dolinska, and S. A. Omelchenko, "The possibility of time resonance (explosion) phenomena in high-energy nuclear reactions," *Central European Journal of Physics*, vol. 4, no. 2, pp. 223–240, 2006.
[149] V. V. Davydovsky and V. S. Olkhovsky, "The influence of the T-matrix energy structure on the decay law of a resonance state," *Ukrainian Journal of Physics*, vol. 42, p. 1146, 1997.
[150] V. S. Olkhovsky and N. L. Doroshko, "Cross-sections and durations of the proton-nucleus scattering near a resonance distorted by the nonresonance background and their phase-shift analysis," *Europhysics Letters*, vol. 18, no. 6, pp. 483–486, 1992.
[151] A. D'Arrigo, N. L. Doroshko, N. V. Eremin, et al., "Bremsstrahlung study of nuclear-reaction dynamics: the 16_O + p reaction," *Nuclear Physics A*, vol. 549, no. 3, pp. 375–386, 1992.
[152] A. D'Arrigo, N. L. Doroshko, N. V. Eremin, et al., "Delay-advance phenomenon observed by bremsstrahlung spectrum of the 12_C + p collision," *Nuclear Physics A*, vol. 564, no. 2, pp. 217–226, 1993.

[153] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory*, Butterworth-Heinemann, Oxford, UK, 1981.
[154] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory*, Butterworth-Heinemann, Oxford, UK, 1998.
[155] A. J. Kálnay, "Lorentz-covariant localized states and the extended-type position operator," *Boletin del IMAF*, vol. 2, p. 11, 1966.
[156] A. J. Kálnay and B. P. Toledo, "A reinterpretation of the notion of localization," *Il Nuovo Cimento A*, vol. 48, no. 4, pp. 997–1007, 1967.
[157] J. A. Gallardo, A. J. Kálnay, B. A. Stec, and B. P. Toledo, "The punctual approximations to the extended-type position," *Il Nuovo Cimento A*, vol. 48, no. 4, pp. 1008–1013, 1967.
[158] J. A. Gallardo, A. J. Kálnay, B. A. Stec, and B. P. Toledo, "Philips' Lorentz-covariant localized states and the extended-type position operator," *Il Nuovo Cimento A*, vol. 49, no. 3, pp. 393–398, 1967.
[159] J. C. Gallardo, A. J. Kálnay, and S. H. Risemberg, "Lorentz-invariant localization for elementary systems," *Physical Review*, vol. 158, no. 5, pp. 1484–1490, 1967.
[160] E. Recami and P. Caldirola, "Descrizione spazio-temporale di processi elementary, tavhioni e caysalita," *Atti della Accademia Nazionale dei Lincei*, vol. 49, p. 77, 1970.
[161] M. Baldo and E. Recami, "Comments about recent letters on spacelike states," *Letters to II Nuovo Cimento*, vol. 2, p. 613, 1969.
[162] J. H. Fermor, "Quantum-mechanical tunneling," *American Journal of Physics*, vol. 34, no. 12, pp. 1168–1170, 1966.
[163] K. W. McVoy, L. Heller, and M. Bolsterli, "Optical analysis of potential well resonances," *Reviews of Modern Physics*, vol. 39, no. 1, pp. 245–258, 1967.
[164] A. Anderson, "Multiple scattering approach to one-dimensional potential problems," *American Journal of Physics*, vol. 57, no. 3, pp. 230–235, 1989.
[165] F. Cardone, S. P. Maidanyuk, R. Mignani, and V. S. Olkhovsky, "Multiple internal reflections during particle and photon tunneling," *Foundations of Physics Letters*, vol. 19, no. 5, pp. 441–457, 2006.
[166] S. P. Maidanyuk, V. S. Olkhovsky, and A. K. Zaichenko, "Multiple internal reflections method in the description of tunneling evolution of nonrelativistic particles and photons," *Journal of Physical Studies*, vol. 6, no. 1, pp. 24–39, 2002 (Ukrainian).
[167] F. Cardone, R. Mignani, and V. S. Olkhovsky, "Are particle and photon tunneling and filling in barriers local or non-local phenomena?" *Physics Letters A*, vol. 289, no. 6, pp. 279–286, 2001.
[168] A. J. Leggett, "Macroscopic quantum systems and the quantum theory of measurement," *Progress of Theoretical Physics. Supplement*, no. 69, pp. 80–100, 1980.
[169] D. Mugnai and A. Ranfagni, "Instanton bounces as real physical processes," *Europhysics Letters*, vol. 6, no. 1, pp. 1–5, 1988.
[170] A. Enders and G. Nimtz, "On superluminal barrier traversal," *Journal de Physique I*, vol. 2, no. 9, pp. 1693–1698, 1992.
[171] A. Enders and G. Nimtz, "Zero-time tunneling of evanescent mode packets," *Journal de Physique I*, vol. 3, no. 5, pp. 1089–1092, 1993.
[172] A. Enders and G. Nimtz, "Photonic-tunneling experiments," *Physical Review B*, vol. 47, no. 15, pp. 9605–9609, 1993.
[173] A. Enders and G. Nimtz, "Evanescent-mode propagation and quantum tunneling," *Physical Review E*, vol. 48, no. 1, pp. 632–634, 1993.
[174] A. E. Bernardini, S. De Leo, and P. P. Rotelli, "Above barrier potential diffusion," *Modern Physics Letters A*, vol. 19, no. 36, pp. 2717–2725, 2004.
[175] S. De Leo and P. P. Rotelli, "Above barrier Dirac multiple scattering and resonances," *European Physical Journal C*, vol. 46, no. 2, pp. 551–558, 2006.
[176] S. De Leo and P. P. Rotelli, "Tunnelling through two barriers," *Physics Letters A*, vol. 342, no. 4, pp. 294–298, 2005.
[177] N. S. Krylov and V. A. Fock, "On two main interpretations of energy-time uncertainty," *Zhurnal Eksperimental noi i Teoreticheskoi Fiziki*, vol. 17, p. 93, 1947.
[178] V. A. Fock, "On the energy-time uncertainty and on an attempt to refute it," *Zhurnal Eksperimental noi i Teoreticheskoi Fiziki*, vol. 42, p. 1135, 1962.
[179] Y. Aharonov and D. Bohm, "Time in the quantum theory and the uncertainty relation for time and energy," *Physical Review*, vol. 122, no. 5, pp. 1649–1658, 1961.
[180] Y. Aharonov and D. Bohm, "Answer to Fock concerning the time energy indeterminacy relation," *Physical Review*, vol. 134, no. 6B, pp. B1417–B1418, 1964.

[181] C. K. Carniglia and L. Mandel, "Phase-shift measurement of evanescent electromagnetic waves," *Journal of the Optical Society of America*, vol. 61, no. 8, pp. 1035–1043, 1971.

[182] S. Zhu, A. W. Yu, D. Hawley, and R. Roy, "Frustrated total internal reflection: a demonstration and review," *American Journal of Physics*, vol. 54, no. 7, pp. 601–607, 1986.

[183] A. M. Steinberg and R. Y. Chiao, "Tunneling delay times in one and two dimensions," *Physical Review A*, vol. 49, no. 5, pp. 3283–3295, 1994.

[184] H. Chew, D.-S. Wang, and M. Kerker, "Elastic scattering of evanescent electromagnetic waves," *Applied Optics*, vol. 18, no. 15, pp. 2679–2687, 1979.

[185] K. L. Chandiramani, "Diffraction of evanescent waves, with applications to aerodynamically scattered sound and radiation from unbaffled plates," *Journal of the Acoustical Society of America*, vol. 55, no. 1, pp. 19–29, 1974.

[186] V. V. Davydovsky, A. K. Zaichenko, and V. S. Olkhovsky, "New possibilities of one-particle alpha-decay model," *Yaderna Fizika*, vol. 1, p. 28, 2005.

[187] V. V. Davydovsky, M. E. Dolinska, N. L. Doroshko, and V. S. Olkhovsky, "Recent developments in time analysis of nuclear processes and revealing of some new temporal phenomena," in *Proceedings of the International Conference on Current Problems in Nuclear Physics and Atomic Energy*, Kyiv, Ukraine, June 2008, (to be printed).

[188] A. D'Arrigo, G. Giardina, A. Taccone, G. Hassani, and V. S. Olkhovsky, "A way to evaluate the level density, total width and compound-nucleus decay functions in the range of unresolved resonances," *International Journal of Modern Physics E*, vol. 2, no. 4, pp. 809–820, 1993.

[189] V. S. Olkhovsky, "Recent developments in joint energy and time analysis of nucleon-nucleus reactions at the range of isolated and overlapped resonance," in *International Conference on Nuclear Data for Science and Technology*, vol. 59 of *Conference Proceedings, Part I*, pp. 280–282, Italian Physical Society, Trieste, Italy, May 1997.

[190] V. S. Olkhovsky, "Influence of excited radioactive nuclei for results in large-scale nuclear Chronometry," in *Proceedings of the International Conference on Nuclear Physics at Border Lines*, pp. 244–247, World Scientific, Lipari, Italy, May 2002.

[191] V. S. Olkhovsky, "Is it essential for nuclear chronometry of astrophysical processes to consider the decays of the excited radioactive nuclei," *Atti dell'Academia di Pericolanti, Classe di Scienze Fisiche, Matematiche e Naturali, Università di Messina*, vol. 76, p. 59, 1998.