## Research Article

# Hidden Periodicity and Chaos in the Sequence of Prime Numbers

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Logarithmic gaps have been used in order to find a periodic component of the sequence of prime numbers, hidden by a random noise (stochastic or chaotic). It is shown that multiplicative nature of the noise is the main reason for the successful application of the logarithmic gaps transforming the multiplicative noise into an additive one. A relation of this phenomenon to spontaneous neuron activity and to chaotic brain computations has been discussed.

## 1. Introduction

The prime number distribution is apparently random. The apparent randomness can be stochastic or chaotic (deterministic). It is well known that the Riemann zeros obey the chaotic GUE statistics (if the primes are interpreted as the classical periodic orbits of a chaotic system, see for a recent review [1]), whereas the primes themselves are believed to be stochastically distributed (Poissonian-like, etc., [2]). However, recent investigations suggest that primes themselves "...could be eigenvalues of a quantum system whose classical counterpart is chaotic at low energies but increasingly regular at higher energies." [3]. Therefore, the problem of chaotic (deterministic) behavior of the moderate and small prime numbers is still open. Moreover, there is also an indication of periodic patterns in the prime numbers distribution [3–7]. These patterns, however, have been observed in *probability* distribution of the gaps between neighboring primes and not in the prime numbers sequence itself (see also [8]). The intrinsic randomness (stochastic or chaotic) of the primes distribution makes the problem of finding the periodic patterns in the prime numbers themselves (if they exist after all) a very difficult one.

A physicists may ask: why should one be interested in finding these patterns? The answer is: comparison. If one can recognize patterns in an apparently random system, then

one can compare these patterns with the patterns known for some other system of interest. We have already mentioned the comparison with certain quantum systems. Another intrinsic comparison can be made with the computational properties of brains, where the natural numbers certainly should play a crucial role. The neuron signals are also apparently random. Can one compare patterns observed in these signals with the patterns in the prime numbers sequence in a constructive way, in order to shed a light on the computational apparatus of the brains?

## 2. Logarithmic Gaps

It is believed that properties of the gaps between consecutive primes can provide a lot of information about the primes distribution in the natural sequence. The so-called prime number theorem states that the "average length" of the gap between a prime $p$ and the next prime number is proportional (asymptotically) to $\ln p$ (see, e.g., [2]). This implies, in particular, statistical nonstationarity of the prime numbers sequence. That is a serious obstacle for practical applications of the statistical methods to this sequence. For relatively large prime numbers one can try to overcome this obstacle by using relatively short intervals [2]. Another (additional) way to overcome this obstacle is to use logarithmic gaps (logarithms of the gaps). The nonstationarity in the sequence of the logarithmic gaps is considerably "slower" than that in the original sequence of the gaps themselves. The logarithmic gaps have also another crucial advantage. If the noise corrupting the gaps sequence has a multiplicative nature, then for the logarithmic gaps this noise will be transformed into an additive one. It is well known that the additive noises can be much readily separated from the signal than the multiplicative ones (see below).

Before starting the analysis let us recall two rather trivial properties of the gaps, which will be used below. The sequence can be restored from the gaps by taking cumulative sum. Pure periodicity in a sequence corresponds to a constant value of the gaps (the period).

Let us take, as a first step, logarithms of the gaps for the sequence of prime numbers. Then, let us compute cumulative sum of these logarithms as a second step. Then, we will multiply each value in the cumulative sum by 10 and will replace each of the obtained values by a natural number which is nearest to it. As a result we will obtain a sequence of natural numbers: $7, 14, 28, 35, 49, 55, \ldots$ which will be called ln-sequence.

Let us now define a binary function $v(n)$ of natural numbers $n = 2, 3, 4, \ldots$, which takes two values +1 or −1 and changes its sign passing any number from the ln-sequence. This function contains full information about the ln-sequence numbers distribution in the sequence of natural numbers.

Let us consider a relatively short interval: $9 \times 10^4 < n < 10^5$. For this interval the autocorrelation function

$$C(n, \tau) = \langle v(n)v(n + \tau) \rangle - \langle v(n) \rangle \langle v(n + \tau) \rangle \tag{2.1}$$

computed for the $v(n)$ function of the ln-sequence will be approximately independent on $n$. Figure 1 shows the autocorrelation function (circles) computed for the $v(n)$ function of the ln-sequence for the "short interval": $9 \times 10^4 < n < 10^5$. Figure 2 shows an autocorrelation function (circles) computed for the $v(n)$ function of the ln-sequence for interval of much smaller natural numbers: $1000 < n < 2 \times 10^4$ (here we rely on the much slower nonstationarity in the logarithmic gaps sequence in comparison with the original gaps sequence).
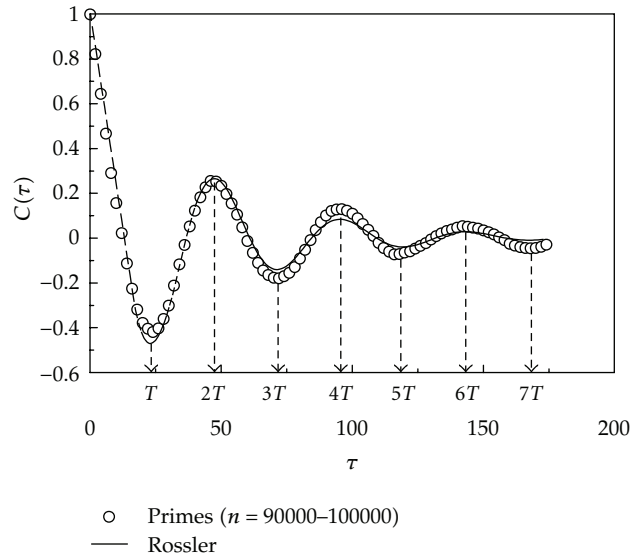
**Figure 1:** Autocorrelation of the $v(n)$ functions for the ln-sequence (circles) in the interval $9 \times 10^4 < n < 10^5$. The solid line correspond to autocorrelation function computed for the telegraph signal generated by the Rössler attractor fluctuations overcoming the threshold $x = 7$. In order to make the autocorrelation functions comparable a rescaling has been made for the Rössler autocorrelation function.
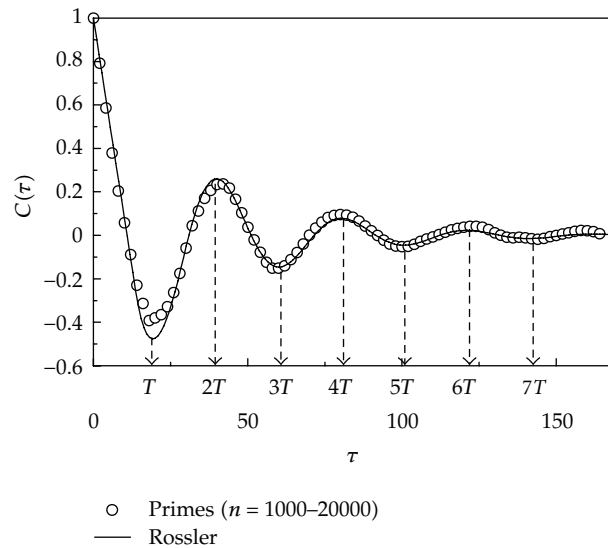


**Figure 2:** As in Figure 1 but for interval $1000 < n < 2 \times 10^4$.

## 3. Models

In order to understand origin of the oscillating autocorrelation function shown in Figures 1 and 2 let us consider a very simple telegraph signal, which allows analytic calculation of its
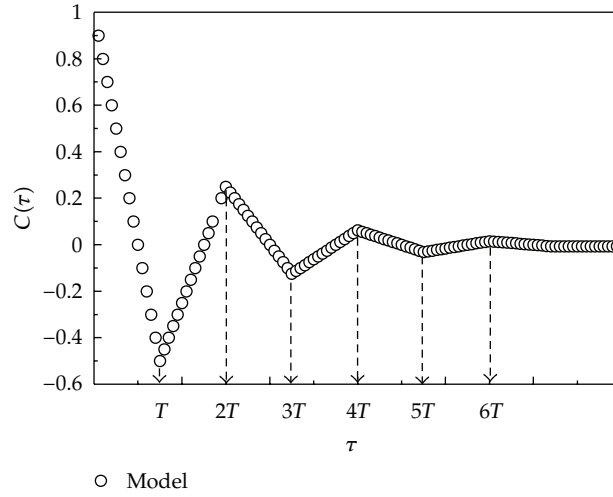
**Figure 3:** Autocorrelation function for the simple model telegraph signal: (3.2) with $q = 0.25$.

autocorrelation function. The telegraph signal takes two values $+1$ or $-1$, and it changes its sign at discrete moments:

$$t_n = nT + \zeta, \tag{3.1}$$

where $\zeta$ is a uniformly distributed over the interval $[0, T]$ random variable, $n = 1, 2, 3 \ldots$, and $T$ is a fixed period. If $q$ is a probability of a sign change at a current moment ($0 \leq q < 1$), then the autocorrelation function of such telegraph signal is

$$C(\tau) = \left(\frac{n - \tau}{T}\right)(2q - 1)^{n-1} + \left(\frac{\tau}{T} - (n - 1)\right)(2q - 1)^n \tag{3.2}$$

in the interval $(n - 1)T \leq \tau < nT$. Figure 3 shows the autocorrelation function (3.2) calculated for $q = 0.25$, as an example (cf. with Figures 1 and 2). In model (3.1) $\zeta$ was taken as a pure stochastic variable. This variable, however, can be also a chaotic (deterministic) one. Let us consider a chaotic solution of the Rössler system [9]

$$\frac{dx}{dt} = -(y + z), \qquad \frac{dy}{dt} = x + ay, \qquad \frac{dz}{dt} = b + xz - cz, \tag{3.3}$$

where $a$, $b$, and $c$ are parameters (a reason for this system relevance will be given in Section 4). Figure 4 shows the $X$-component fluctuations of a chaotic solution of the Rössler system.

Let us consider a telegraph signal generated by the variable $x$ crossing certain threshold from below (Figure 4). This telegraph signal takes values $+1$ or $-1$ and changes its sign at the threshold-crossing points. Then, the fundamental period of the Rössler chaotic attractor provides the period $T$ in (3.1), whereas the chaotic fluctuations of the variable $x$ of the attractor provide chaotic variable $\zeta$ in (3.1) for the considered telegraph
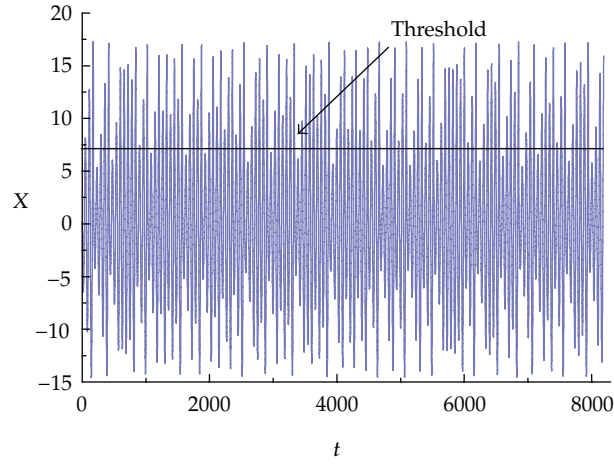
**Figure 4:** $X$-component fluctuations of a chaotic solution of the Rössler system (3.3) ($a = 0.15$, $b = 0.20$, $c = 10.0$).



(a)

(b)
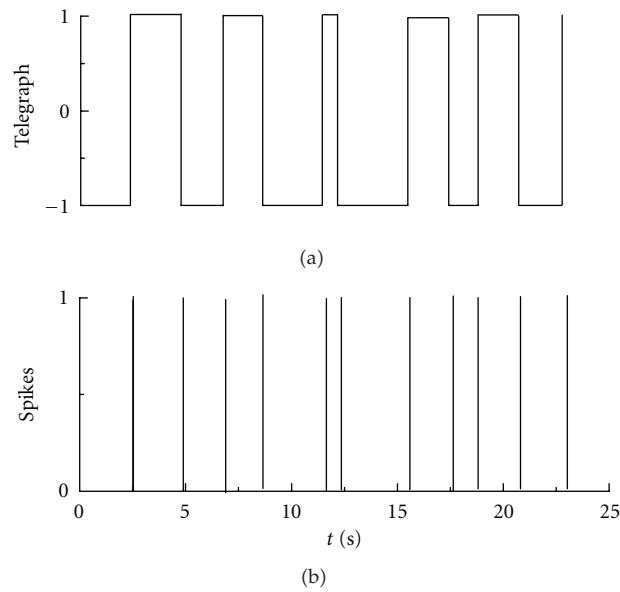
**Figure 5:** Mapping of a spike train (b) into a telegraph signal (a).

signal. Autocorrelation function computed for this telegraph signal has been shown in Figures 1 and 2 as the solid line. In order to make the autocorrelation functions comparable a rescaling has been made for the Rössler telegraph signal's autocorrelation function: the scaling coefficient is equal to 0.36 for Figure 1 and to 0.30 for Figure 2.
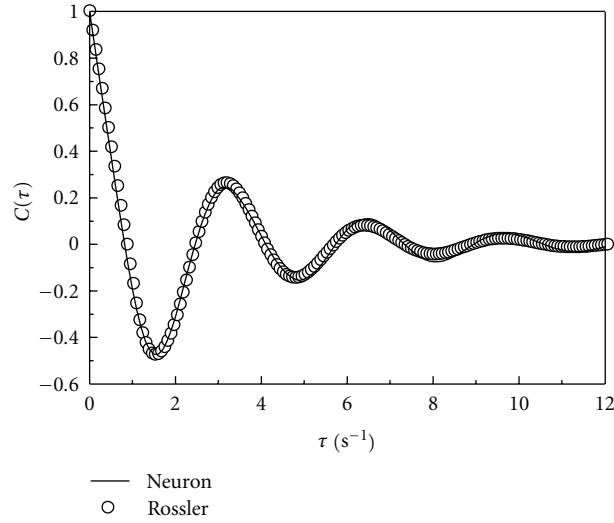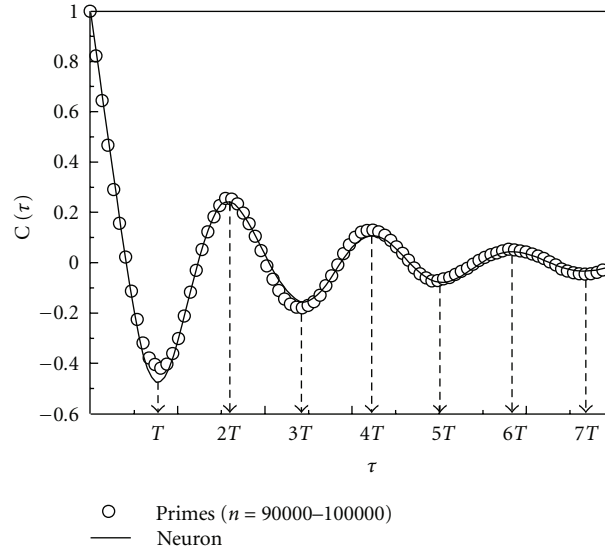
**Figure 6:** Autocorrelation functions for the telegraph signals corresponding to a spontaneous activity of a hippocampal (CA3) singular neuron (solid curve) and to the spike train generated by the Rössler attractor fluctuations overcoming a threshold (circles). In order to make the autocorrelation functions comparable a rescaling has been made for the Rössler autocorrelation function. The data has been taken from [10].

## 4. Prime Numbers and Spontaneous Neuron Activity

All types of information, which is received by sensory system, are encoded by nerve cells into sequences of pulses of similar shape (spikes) before they are transmitted to the brain. Brain neurons use such sequences as main instrument for intercells connection. The information is reflected in the time intervals between successive firings (interspike intervals of the action potential train, see Figure 5(b)). There need be no loss of information in principle when converting from dynamical amplitude information to spike trains, and the irregular spike sequences are the foundation of neural information processing. Although understanding of the origin of interspike intervals irregularity has important implications for elucidating the temporal components of the neuronal code the problem is still very far from its solution.

In a recent paper [10] a membrane threshold-crossing model of spontaneous (without external stimulus) neuron firing based on the chaotic Rössler attractor was suggested. Good agreement between the model and the spiking time series, obtained in vitro from a spontaneous activity of hippocampal (CA3) singular neurons (rats brain slice culture), was reported in this paper. The spiking time series were mapped into a telegraph signal as it is shown in Figure 5. The comparison between the experimental data and the signal generated by the Rössler attractor fluctuations overcoming a threshold (cf. Section 3) is shown in Figure 6.

The hippocampus is a significant part of a brain system responsible, in particular, for spatial memory, body orientation, and navigation. These functions of brain are related to the very sophisticated real-time computations, which the brains make with astonishing effectiveness. One should distinguish between the inner brain computations and our conscious arithmetic calculations, which are performed in the more specialized brain's departments such as the intraparietal sulcus. Therefore, if one wants to investigate how the inner brain computational abilities are related to neuron activity the hippocampal

**Figure 7:** Autocorrelation of the $v(n)$ functions for the ln-sequence (circles) in the interval $9 \times 10^4 < n < 10^5$. The solid curve corresponds to autocorrelation function computed for the telegraph signals corresponding to a spontaneous activity of a hippocampal (CA3) singular neuron (cf. Figure 3). In order to make the autocorrelation functions comparable a rescaling has been made for autocorrelation of the $v(n)$ function.

pyramidal cells are an appropriate subject for this investigation. Spontaneous activity in brain slice preparations purely reflects the intrinsic properties of local circuits and individual neurons. The comparison of the spontaneous activity of the hippocampal singular neurons with the random properties of the sequence of prime numbers can shed certain light on the long-standing problem: whether mathematics belongs completely to the outer world or it is a production of our brain activity (at least in the form of the natural numbers). For instance, in order to work together the brain neurons have to make an adjustment of their rhythms. The main problem for this adjustment is the very noisy environment of the brain neurons. For pure periodic inner clocks this adjustment would be impossible due to the noise. Nature, however, has another option. This option is a chaotic clock (see [10] and references therein). In chaotic attractors certain characteristic frequencies can be embedded by broad-band spectra, that makes them much more stable to the noise perturbations. The Riemann's hypothesis, for instance, which states that the apparent randomness in the distribution of prime numbers only comes from a noise (otherwise the prime numbers are as regularly distributed as possible), can reflect an optimization of such mechanism.

On the other hand, the compatibility of the statistical properties of the neuron spontaneous activity with the statistics of the prime numbers (see Figures 7 and 8) could be a key for understanding of the inner computational methods of the brains. If brains use chaotic dynamics (Figure 6) also for computational processing, then such compatibility should play a crucial role in the inner brain computations. The reduction to the additive random processes by means of the logarithmic gaps (see Section 3) could be a part of these inner brain computations. If the gaps can be considered as a result of a multiplicative random process, that is, each gap between consecutive primes can be represented as a product of independent random variables (the random variables could take their values from
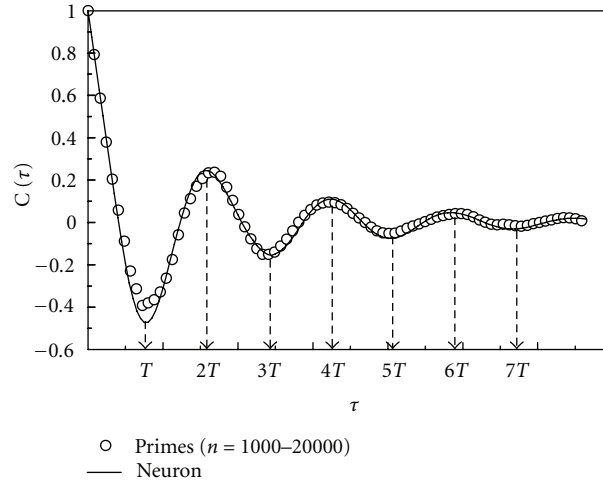
**Figure 8:** As in Figure 7 but for interval $1000 < n < 2 \times 10^4$.

the prime numbers set, including 1), then logarithmic gap represents the sum of random variables (i.e., an additive random process). It is known that unlike the additive random processes the multiplicative random processes are much more complex and difficult for analysis.

## Acknowledgment

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