

Research Article

Resonances for Perturbed Periodic Schrödinger Operator

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Received 29 September 2011; Accepted 27 November 2011

Academic Editor: Ali Mostafazadeh

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In the semiclassical regime, we obtain a lower bound for the counting function of resonances corresponding to the perturbed periodic Schrödinger operator $P(h) = -\Delta + V(x) + W(hx)$. Here V is a periodic potential, W a decreasing perturbation and h a small positive constant.

1. Introduction

The quantum dynamics of a Bloch electron in a crystal subject to external electric field, which varies slowly on the scale of the crystal lattice, is governed by the Schrödinger equation

$$P(h) = -\Delta + V(x) + W(hx). \quad (1.1)$$

Here V is periodic with respect to the crystal lattice $\Gamma \subset \mathbb{R}^n$, and it models the electric potential generated by the lattice of atoms in the crystal. The potential W is a decreasing perturbation and h a small positive constant.

There has been a growing interest in the rigorous study of the spectral properties of Bloch electrons in the presence of slowly varying external perturbations (see [1–11]).

Since the work of Peierls [10] and Slater [11], it is well known that, if h is sufficiently small, then solutions of $P(h)$ are governed by the “semiclassical” Hamiltonian

$$H(\mathbf{y}, \boldsymbol{\eta}) = \lambda(\boldsymbol{\eta} + A(\mathbf{y})) + V(\mathbf{y}). \quad (1.2)$$

Here $\lambda(k)$ is one of the “band functions” describing the Floquet spectrum of the unperturbed Hamiltonian

$$P_0 = -\Delta_x + V(x). \quad (1.3)$$

One argues that for suitable wave packets, which are spread over many lattice spacings, the main effect of a periodic potential on the electron dynamics consists in changing the dispersion relation from the free kinetic energy $E_{\text{free}}(k) = |k|^2$ to the modified kinetic energy $\lambda(k)$ given by the Bloch band.

The problem of resonances has been examined in [12] for the one-dimensional case and in [13] for the general case. In particular, a similar reduction to (1.2) for resonances has been obtained in [13].

This paper continues our previous works [13, 14] on the resonances and the eigenvalues counting function for $P(h)$. In [14], Dimassi and Zerzeri obtained a local trace formula for resonances. As a consequence, they obtained an upper bound for the number of resonances of $P(h)$ in any h -independent complex neighborhood of some energy E . The purpose of this paper is to give a lower bound for the number of resonances of $P(h)$.

In the case where $V = 0$, it is known that, for $0 < E$ in the analytic singular support (from now on sing supp_a for short) of the distribution $d\rho_0 * \mu$, then the operator $P(h) = -\Delta + W(hx)$ has at least $C_\Omega h^{-n}$ resonances in any h -independent complex neighborhood Ω of E (see, e.g., [15]). Here

$$\begin{aligned} \mu(t) &= \int_{\{x \in \mathbb{R}^n, W(x) > t\}} dx, \\ \rho_0(t) &= (2\pi)^{-n} \text{vol}(B(0, 1)) (\max(t, 0))^{n/2}. \end{aligned} \quad (1.4)$$

Using the explicit formula of ρ_0 we see that the analytic singular support of the distributions μ and $d\rho_0 * \mu$ coincide.

In the case where $V \neq 0$ the situation is different. Following Theorem 1.6 in [14] and Lemma 2.1 of the next section, we have to change ρ_0 by

$$\rho(\lambda) := \frac{1}{(2\pi)^n} \sum_{j \geq 1} \int_{\{k \in E^*; \lambda_j(k) \leq \lambda\}} dk, \quad (1.5)$$

which is the integrated density of states corresponding to the nonperturbed Hamiltonian P_0 (see Section 2).

If $\lambda_j(k)$ is a simple eigenvalue near some point e_0 , then $\lambda_j(k)$ is a smooth function, and if $e_0 = \lambda_j(k)$ is a critical value, we expect in general that e_0 will belong to the analytic singular support of $\rho(\lambda)$. In particular, we expect that near every point $e \in e_0 + \text{sing supp}_a(\mu)$ there exists at least Ch^{-n} , $C > 0$, resonances.

Multiple eigenvalues ($\lambda_j(k_0) = \lambda_{j+1}(k_0) = e_0$) can also give rise to singularities of $\rho(\lambda)$ and then lead to the existence of resonances near $e_0 + \text{sing supp}_a(\mu)$.

The purpose of this paper is to describe all these situations. Some results of this paper are announced without proofs in [16].

The paper is organized as follows: in the next section, we introduce some notations and state some technical lemmas. In Section 3 we give an upper bound for resonances near

singularities of the density of states measure ρ generated by a band crossing. In Section 4 we give an upper bound for resonances near the edge of bands.

2. Preliminaries

Let $\Gamma = \oplus_{i=1}^n \mathbb{Z}a_i$ be the lattice generated by the basis $a_1, a_2, \dots, a_n, a_i \in \mathbb{R}^n$. The dual lattice Γ^* is defined as the lattice generated by the dual basis $\{a_1^*, a_2^*, \dots, a_n^*\}$ determined by $a_i \cdot a_j^* = 2\pi\delta_{ij}$, $i, j = 1, 2, \dots, n$. Let E be a fundamental domain for Γ , and let E^* be a fundamental domain for Γ^* . If we identify opposite edges of E (resp., E^*), then it becomes a flat torus denoted by $\mathbb{T} = \mathbb{R}^n/\Gamma$ (resp., $\mathbb{T}^* = \mathbb{R}^n/\Gamma^*$).

Let V be a real valued potential, C^∞ and Γ -periodic. For k in \mathbb{R}^n , we define

$$P_0(k) = (D_x + k)^2 + V(x) \quad (2.1)$$

as an unbounded operator on $L^2(\mathbb{T})$ with domain $H^2(\mathbb{T})$. The Hamiltonian $P_0(k)$ is semi-bounded and self-adjoint. Since the resolvent of $(D_x + k)^2$ is compact, the resolvent of $P_0(k)$ is also compact, and therefore $P_0(k)$ has a complete set of (normalized) eigenfunctions $\Phi_n(\cdot, k) \in H^2(\mathbb{T}^*)$, $n \in \mathbb{N}$, called Bloch functions. The corresponding eigenvalues accumulate at infinity, and we enumerate them according to their multiplicities:

$$\lambda_1(k) \leq \lambda_2(k) \leq \dots \quad (2.2)$$

Since $e^{-ixy^*} H_0(k) e^{ixy^*} = H_0(\gamma^* + k)$, the band function $\lambda_n(k)$ is periodic with respect to Γ^* . The function $\lambda_n(k)$ is called a band function, and the closed intervals $\Lambda_n := \lambda_n(\mathbb{T}^*)$ are called bands.

Standard perturbation theory shows that $\lambda_n(k)$ is a continuous function of k and is real analytic in a neighborhood of any k such that

$$\lambda_{n-1}(k) < \lambda_n(k) < \lambda_{n+1}(k). \quad (2.3)$$

We fix λ in the spectrum of the unperturbed operator P_0 . We make the following hypothesis on the spectrum of the unperturbed Schrödinger operator.

(H1) For all k_0 with $\lambda_i(k_0) = \lambda$, the eigenvalue $\lambda_i(k_0)$ is simple and $d_k \lambda_i(k_0) \neq 0$.

Now, let us recall some well-known facts about the density of states associated with P_0 . The density of states measure ρ is defined as follows:

$$\rho(\lambda) := \frac{1}{(2\pi)^n} \sum_{j \geq 1} \int_{\{k \in E^*; \lambda_j(k) \leq \lambda\}} dk, \quad (2.4)$$

where E^* is a fundamental domain of \mathbb{R}^n/Γ^* . Since the spectrum of P_0 is absolutely continuous, the measure ρ is absolutely continuous with respect to the Lebesgue measure $d\lambda$. Thus, the density of states of P_0 , $\partial\rho/\partial\lambda$ is locally integrable.

We now consider the perturbed periodic Schrödinger operator:

$$P(h) := P_0 + W(hx), \quad (2.5)$$

where $W \in C^\infty(\mathbb{R}^n; \mathbb{R})$. We assume that there exist positive constants a and C such that W extends analytically to $\Gamma(a) := \{z \in \mathbb{C}^n; |\Im(z)| \leq a\Re(z)\}$ and

$$|W(z)| \leq C\langle z \rangle^{-\tilde{n}}, \quad \text{uniformly on } z \in \Gamma(a), \tilde{n} > n, \quad (2.6)$$

where $\langle z \rangle = (1 + |z|^2)^{1/2}$. Here $\Re(z), \Im(z)$ denote, respectively, the real part and the imaginary part of z .

This assumption allows us to define the resonances of $P(h)$ by the spectral deformation method (see [17]). We follow essentially the presentation of [13].

Let $v \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ be Γ^* -periodic. For $t \in \mathbb{R}$, we introduce the spectral deformation family \mathcal{U}_t defined by for all $u \in \mathcal{S}$:

$$\mathcal{U}_t u(r) := \mathcal{F}_h^{-1} \left\{ \left(J_t^{1/2} (\mathcal{F}_h u)(v_t(k)) \right) \right\}(r), \quad \forall x \in \mathbb{R}^n, \quad (2.7)$$

where $v_t(k) = k - tv(k)$ and $J_t(k)$ its Jacobian. Here \mathcal{F}_h is the semiclassical Fourier transform:

$$[\mathcal{F}_h u](\xi) := \int_{\mathbb{R}^n} e^{-i/h x \xi} u(x) dx, \quad \forall u \in \mathcal{S}(\mathbb{R}^n). \quad (2.8)$$

Consider, for $t \in \mathbb{R}$, the family of unitarily equivalent operators

$$P_1(t, h) := \mathcal{U}_t P_1(h) \mathcal{U}_t^{-1}. \quad (2.9)$$

It was established in [13, Proposition 2.8] that $P_1(t, h)$ extends to an analytic type- \mathcal{A} family of operators on $D(t_0) := \{t \in \mathbb{C}; |t| < t_0\}$ with domain $H^2(\mathbb{R}^n)$. Moreover, under the assumptions (H1) and (2.6), there exists a neighborhood $\tilde{\Omega}$ of z_0 and a small positive constant η such that, for $t \in D(t_0)$ with $\Im t > 0$, the spectrum of $P_1(t, h)$ in $\Omega_t := \{z \in \tilde{\Omega}; \Im z > -\eta \Im t\}$ consists of discrete eigenvalues of finite multiplicities that lie in the lower half plane (see [13, formula (4.9)]). These eigenvalues are t -independent under small variations of $\Im t > 0$ and are called resonances. We will denote the set of resonances by $\text{Res}(P(h))$.

For $f \in C_0^\infty(\mathbb{R})$, we set

$$\langle \mu, f \rangle = \int [f(W(x)) - f(0)] dx, \quad (2.10)$$

$$\langle \omega, f \rangle = \frac{1}{(2\pi)^n} \sum_{j \geq 1} \int_{E^*} \int_{\mathbb{R}_x^n} [f(W(x) + \lambda_j(k)) - f(\lambda_j(k))] dk dx, \quad (2.11)$$

For $E > 0$, let

$$\nu_+(E) := \int_{\{x \in \mathbb{R}^n; W(x) \geq E\}} dx. \quad (2.12)$$

Similarly, for $E < 0$, we set

$$\nu_-(E) := \int_{\{x \in \mathbb{R}^n; W(x) \leq E\}} dx. \quad (2.13)$$

Clearly, $\nu_+(E)$ (resp., $\nu_-(E)$) is a decreasing function of E (resp., an increasing function of E) and

$$\mu_{|\mathbb{R}^\pm} = -\frac{d}{dE} \nu_\pm(E). \quad (2.14)$$

Lemma 2.1. *The distributions ω and μ are real valued of order ≤ 1 . Moreover, in $\mathfrak{D}'(\mathbb{R})$, one has*

$$\omega = d\rho * \mu. \quad (2.15)$$

Proof. Applying Taylor's formula to the right-hand side of (2.10), we obtain

$$|\langle \mu, f \rangle| \leq \sup |f'| \int |W(x)| dx, \quad (2.16)$$

which together with (2.6) imply that μ is a distribution of order ≤ 1 , with

$$\text{supp } \mu \subset [\inf W(x), \sup W(x)]. \quad (2.17)$$

Consequently, $d\rho * \mu$ is well defined in $\mathfrak{D}'(\mathbb{R})$ and for all $f \in C_0^\infty(\mathbf{R})$, we have

$$\begin{aligned} \langle d\rho * \mu, f \rangle &= \langle d\rho(t), \langle \mu, f(\cdot + t) \rangle \rangle \\ &= -\left\langle \rho(t), \int [f'(W(x) + t) - f'(t)] dx \right\rangle \\ &= -\frac{1}{(2\pi)^n} \sum_j \int_{E^*} \int_{\lambda_j(k)}^\infty \int_{\mathbf{R}_x^n} [f'(W(x) + t) - f'(t)] dx dt dk \\ &= \frac{1}{(2\pi)^n} \sum_j \int_{E^*} \int_{\mathbf{R}_x^n} [f(W(x) + \lambda_j(k)) - f(\lambda_j(k))] dx dk \\ &= \langle \omega, f \rangle. \end{aligned} \quad (2.18)$$

This ends the proof of the lemma. \square

Let Ω be an open-bounded set in \mathbb{R}^n , and let $\tilde{\Omega}$ be a complex neighborhood of Ω . Let $x \rightarrow \varphi(x)$ be analytic on $\tilde{\Omega}$ and real valued for all x in Ω . Let us introduce the real function

$$I(e) := \int_{\{x \in \Omega; \varphi(x) \leq e\}} dx. \quad (2.19)$$

For $e \in \varphi(\Omega)$, we set

$$\Sigma(e) := \{x \in \Omega; \varphi(x) = e\}. \quad (2.20)$$

Lemma 2.2. *Let $e_0 \in \varphi(\Omega)$, and let $\Sigma(e)$, $I(e)$ be as above. One assumes that*

- (i) $\nabla \varphi(x) \neq 0$ for all $x \in \Sigma(e_0)$,
- (ii) $\partial\Omega \cap \Sigma(e_0) = \emptyset$.

Then the function

$$I(e) := \int_{\{x \in \Omega; \varphi(x) \leq e\}} dx \quad (2.21)$$

is analytic near e_0 .

Proof. Let ϵ be a small positive constant such that $\nabla \varphi(x) \neq 0$ when $x \in \Sigma_\epsilon(e_0) := \varphi^{-1}(]e_0 - \epsilon, e_0 + \epsilon[)$. Without any loss of generality we may assume that $\partial_{x_1} \varphi \neq 0$ for all $x \in \Sigma_\epsilon(e_0)$. By the change of variable $H : x \rightarrow (\varphi(x), x_2, \dots, x_n)$ we have

$$\int_{\{x \in \Sigma_\epsilon(e_0); \varphi(x) \leq e\}} dx = \int_{\{x \in H(\Sigma_\epsilon(e_0)); x_1 \leq e\}} \text{Jac}(H^{-1}(x)) dx. \quad (2.22)$$

Clearly the right-hand side of the above equality is analytic. Combining this with the fact that $\int_{\{x \in \Omega \setminus \Sigma_\epsilon(e_0); \varphi(x) \leq e\}} dx$ is constant for e near e_0 we get the lemma. \square

Lemma 2.3. *If φ has a nondegenerate extremum at x_0 with $\varphi(x_0) = e_0$ and if $\nabla \varphi(x) \neq 0$ for all $x \in \Sigma_{e_0} \setminus \{x_0\}$, then*

$$I(e) = f(e - e_0) + H(\pm(e - e_0))g\left(\sqrt{\pm(e - e_0)}\right), \quad (2.23)$$

where f and g are analytic near zero and

$$g(t) \sim_{t \rightarrow 0} \frac{\text{vol}(S^{n-1})}{n\sqrt{\det \varphi''(x_0)}} 2^{n/2} t^n. \quad (2.24)$$

Here $H(t)$ is the Heaviside function and $+$ ($-$) corresponds to a minimum (maximum, resp.).

Proof. Here we only give a sketch of the proof. For the details we refer to [18]. Without any loss of generality, we only consider the case of minimum. By Morse lemma there exist a neighborhood U of x_0 , $\epsilon > 0$ and a local analytic diffeomorphism $D : \Omega \rightarrow B(0, \epsilon)$ such that

$$\int_{\{x \in U; \varphi(x) \leq \epsilon\}} dx = \int_{\{x \in B(0, \epsilon); |x|^2 \leq \epsilon - e_0\}} \text{Jac}(D^{-1}(x)) dx. \quad (2.25)$$

By a simple calculus we show, using polar coordinates, that the integral of the r.h.s. is equal to $H(\epsilon - e_0)g(\sqrt{\epsilon - e_0})$. On the other hand, since $\nabla\varphi(x) \neq 0$ for $x \in \Sigma_{e_0} \setminus \{x_0\}$, it follows from Lemma 2.2 that

$$\int_{\{x \in O \setminus U; \varphi(x) \leq \epsilon\}} dx \quad (2.26)$$

is analytic near e_0 . This ends the proof of the lemma. \square

3. Lower-Bound Near Singularities due to Band Crossing

Here we are interested in the C^∞ singular support (which will be denoted by sing supp). Recall that $x_0 \notin \text{sing supp } \mu$ if and only if μ is C^∞ near x_0 . The case of analytic singular support can be treated similarly.

In this section we study resonances near singularities of $\rho(\lambda)$ generated by a band crossing. We will only consider the two-dimensional case. With similar assumptions, one can treat the case $n \geq 2$.

We assume that $\lambda_j(k)$ is double eigenvalues $\lambda_{j-1}(k_0) < \lambda_j(k_0) = e_0 = \lambda_{j+1}(k_0) < \lambda_{j+2}(k_0)$ and that for all $k \neq k_0$ such that $\lambda_i(k) = e_0$, $\lambda_i(k)$ is simple and $\nabla\lambda_i(k) \neq 0$.

Since $P_0(k)$ is analytic in k , this implies that, for $|k - k_0| \leq \delta$ (with δ small enough), the span $V(k)$, of the eigenvectors of $P_0(k)$ corresponding to eigenvalues in the set $\{e; |e - e_0| \leq \delta\}$, has a basis $\varphi_j(x, k)$, $\varphi_{j+1}(x, k)$, which is orthonormal and real analytic in k . The restriction of $P_0(k)$ to $V(k)$ has the matrix

$$\begin{pmatrix} \alpha(k) & \overline{b(k)} \\ b(k) & \beta(k) \end{pmatrix}, \quad (3.1)$$

which can be written

$$\begin{pmatrix} a(k) + c(k) & b_1(k) - ib_2(k) \\ b_1(k) + ib_2(k) & a(k) - c(k) \end{pmatrix}, \quad (3.2)$$

where $a(k) = \alpha(k) + \beta(k)/2$, $c(k) = \alpha(k) - \beta(k)/2$, $b_1(k)$ and $b_2(k)$ are real valued. Next the periodic potential is assumed to have the symmetry $V(x) = V(-x)$. This symmetry is typical of metals. This symmetry forces $b(k)$ to be real valued (i.e., $b_2(k) = 0$), (see [19]). Consequently, near k_0 we have

$$\lambda_j(k) = a(k) - \sqrt{c^2(k) + b_1^2(k)}, \quad \lambda_{j+1}(k) = a(k) + \sqrt{c^2(k) + b_1^2(k)}. \quad (3.3)$$

We assume that $\nabla b_1(k_0), \nabla c(k_0)$ are independent. Since $n = 2$, $(\nabla b_1(k_0), \nabla c(k_0))$ is a basis in \mathbb{R}^2 . Set $\nabla a(k_0) = \alpha_1 \nabla b_1(k_0) + \alpha_2 \nabla c(k_0)$.

Lemma 3.1. *Let $\nabla a(k_0) = \alpha_1 \nabla b_1(k_0) + \alpha_2 \nabla c(k_0)$ be as above. One assumes that*

$$\alpha_1^2 + \alpha_2^2 < 1. \quad (3.4)$$

Then there exist an open connected neighborhood J of e_0 and analytic functions f and g such that

$$\rho(e) = f(e) + (H(e - e_0) - H(e_0 - e))g(e), \quad (3.5)$$

with

$$g''(e_0) \neq 0, \quad \forall e \in J. \quad (3.6)$$

Proof. To simplify the notation we assume that $k_0 = 0$ and $e_0 = 0$.

Let Ω be a neighborhood of $k_0 = 0$. We introduce

$$(2\pi)^n \rho_1(e) = \int_{\{k \in \Omega; \lambda_n(k) \leq e\}} dk + \int_{\{k \in \Omega; \lambda_{n+1}(k) \leq e\}} dk, \quad (3.7)$$

so that

$$(2\pi)^n (\rho(e) - \rho_1(e)) = \sum_{j \notin \{n, n+1\}} \int_{\{k \in E^*; \lambda_j(k) \leq e\}} dk + \int_{\{k \in E^* \setminus \Omega; \lambda_n(k) \leq e\}} dk + \int_{\{k \in E^* \setminus \Omega; \lambda_{n+1}(k) \leq e\}} dk. \quad (3.8)$$

Due to Lemma 2.2, the right-hand side of the above equalities is analytic near 0.

Since $\nabla b_1(k_0), \nabla c(k_0)$ are independent, there exist a neighborhood Ω of $k_0 = 0$, $\epsilon > 0$ and a local analytic diffeomorphism $\kappa : \Omega \rightarrow B(0, \epsilon)$ such that, with the change of variable $k \rightarrow \kappa(k)$, we obtain

$$(2\pi)^n \rho_1(e) = \int_{\{|k| \leq \epsilon; G(k) + |k| \leq e\}} F(k) dk + \int_{\{|k| \leq \epsilon; G(k) - |k| \leq e\}} F(k) dk, \quad (3.9)$$

where $G(k) = a(\kappa^{-1}(k))$ and $F(k) = \text{Jac}(\kappa(k))$ are analytic near $k = 0$ and $\nabla G(0) = (\alpha_1, \alpha_2)$.

Using polar coordinates and making the change $r \rightarrow -r, \omega \rightarrow -\omega$ in the second integral, we get

$$(2\pi)^n \rho_1(e) = \int_{S^1} \int_{\{0 \leq r \leq \delta; G(r\omega) + r \leq e\}} F(r\omega) r dr d\omega - \int_{S^1} \int_{\{-\delta \leq r \leq 0; G(r\omega) + r \leq e\}} F(r\omega) r dr d\omega, \quad (3.10)$$

which can be written

$$(2\pi)^n \rho_1(e) = \int_{S^1} \int_{\{0 \leq r \leq \delta; G(r\omega) + r \leq e\}} F(r\omega) r \, dr \, d\omega + \int_{S^1} \int_{\{-\delta \leq r \leq 0; G(r\omega) + r \geq e\}} F(r\omega) r \, dr \, d\omega - c_0, \quad (3.11)$$

where $c_0 = \int_{S^1} \int_{\{-\delta \leq r \leq 0\}} F(r\omega) r \, dr \, d\omega$. Since

$$\partial_r (G(r\omega) + r)|_{r=0} = \langle \nabla G(0), \omega \rangle + 1 \geq \eta > 0, \quad (3.12)$$

uniformly on $\omega \in S^1$, there exist $\delta_1, \delta_2 > 0$ (independent on $\omega \in S^1$) such that $Y : r \rightarrow Y(r) = G(r\omega) + r$ from $] -\delta_1, \delta_1[$ into $] -\delta_2, \delta_2[$ is an analytic diffeomorphism. Hence, for $|e|$ small enough

$$\begin{aligned} (2\pi)^n \rho_1(e) + c_0 &= \int_{S^1} \int_{\{t \geq 0; t \leq e\}} F(Y^{-1}(t)\omega) \frac{Y^{-1}(t)}{Y'(t)} \, dt \, d\omega \\ &\quad + \int_{S^1} \int_{\{t \leq 0; t \geq e\}} F(Y^{-1}(t)\omega) \frac{Y^{-1}(t)}{Y'(t)} \, dt \, d\omega \\ &= (H(e) - H(-e))g(e), \end{aligned} \quad (3.13)$$

where

$$g(e) = \int_0^e \int_{S^1} F(Y^{-1}(t)\omega) \frac{Y'(t)}{Y^{-1}(t)} \, dt \, d\omega. \quad (3.14)$$

Using that

$$Y^{-1}(0) = 0 \quad (3.15)$$

we deduce $g''(0) = F(0) \int_{S^1} (\langle \nabla G(0), \omega \rangle + 1)^{-2} \, d\omega \neq 0$. \square

We denote by $\#A$ the number of elements of A , counted with their multiplicity. The main result of this section is the following.

Theorem 3.2. *Let $\lambda, e_0 \in \sigma(P_0)$ with $\lambda \in (e_0 + \text{sing supp}(\mu))$. One assumes the following.*

- (i) *The periodic potential V satisfies $V(x) = V(-x)$.*
- (ii) *There exists $k_0 \in \mathbb{R}^n / \Gamma^*$ such that $\lambda_{j-1}(k_0) < \lambda_j(k_0) = e_0 = \lambda_{j+1}(k_0) < \lambda_{j+2}(k_0)$.*
- (iii) *For all $k \notin k_0 + \Gamma^*$ such that $\lambda_i(k) = e_0$, the eigenvalue $\lambda_i(k)$ is simple and $\nabla \lambda_i(k) \neq 0$.*
- (iv) *The numbers (α_1, α_2) satisfy (3.4), and $(\lambda - \text{supp}(\mu)) \subset J$. Here J is the interval given by Lemma 3.1.*
- (v) *λ satisfies (H1).*

Then for all h -independent complex neighborhoods Ω of λ , there exist $h_0 = h(\Omega) > 0$ sufficiently small and $C = C(\Omega) > 0$ such that, for $h \in]0, h_0[$,

$$\#\{z \in \Omega; z \in \text{Res}(P(h))\} \geq C_\Omega h^{-n}. \quad (3.16)$$

Proof. Without any loss of generality we may assume that $e_0 = 0$. Set

$$K(\cdot) := (H(\cdot) - H(\cdot - \cdot))g(\cdot), \quad (3.17)$$

where $g(\cdot)$ is the function given in Lemma 3.1.

The assumption that $(\lambda - \text{supp}(\mu)) \subset J$ ensures that, in the study of $d\rho * \mu$ near λ , one only needs the value of ρ in J given by (3.4). More precisely, it implies that

$$\omega(t) = d\rho * \mu(t) = \rho * d\mu(t) = f * d\mu + K(\cdot) * d\mu = (1) + (2), \quad (3.18)$$

for t near λ .

Since f is smooth, the first term of the right-hand side of the above equation is also smooth.

Clearly, it follows from assumption (2.6) and Lemma 2.2 that the $\text{sing supp}(\mu)$ is a discrete set. Thus, the point λ is isolated in $\text{sing supp}(\mu)$. We recall that we have assumed that $e_0 = 0$.

Let $\chi \in C_0^\infty(B(0, 1))$ (resp., $\theta \in C_0^\infty(B(\lambda, 1))$) be equal to one near zero (resp., λ). Here $B(y, r)$ is the disc of center y and radius r . Set $\chi_\epsilon = \chi(\cdot/\epsilon)$ and $\theta_\epsilon = \theta(\cdot/\epsilon)$. We choose $\epsilon > 0$ small enough such that

$$\text{sing supp}(\mu) \cap \text{supp } \theta_\epsilon = \{\lambda\}. \quad (3.19)$$

To study the second term of the right-hand side of (3.18), we write it in the form

$$(2) = K(\cdot)(1 - \chi_\epsilon) * d\mu + K(\cdot)\chi_\epsilon * \theta_\epsilon d\mu + K(\cdot)\chi_\epsilon * (1 - \theta_\epsilon)d\mu = (3) + (4) + (5). \quad (3.20)$$

Since $K(\cdot)(1 - \chi_\epsilon)$ is smooth the term (3) is also smooth. Using (3.19) and the fact that the support of $K(\cdot)\chi_\epsilon$ is small for $\epsilon \ll 1$, we see that the term (5) is C^∞ near λ .

Now, we claim that

$$\text{sing supp}(4) = \{\lambda\}. \quad (3.21)$$

First, from a standard result on the singular support, we have

$$\text{sing supp}(4) \subset \text{sing supp}(K(\cdot)\chi_\epsilon) + \text{sing supp}(\theta_\epsilon d\mu) = \{\lambda\}. \quad (3.22)$$

Consequently, to prove the claim it suffices to show that (4) $\notin C_0^\infty(\mathbf{R})$. We recall that (4) has a compact support.

A simple calculus and Lemma 3.1 yield

$$c(1 + |\xi|^2)^{-1} \leq \left| \widehat{K(\cdot) \chi_\epsilon(\xi)} \right| \leq C. \quad (3.23)$$

Here $\widehat{f}(\xi)$ is the Fourier transform of f . Consequently, $\widehat{\theta_\epsilon d\mu} \in S(\mathbb{R})$ if and only if $(4) \in S(\mathbb{R})$, where $S(\mathbb{R})$ is the Schwartz space of C^∞ function of rapid decrease.

On the other hand, (3.19) implies that $\widehat{\theta_\epsilon \mu} \notin S(\mathbb{R})$. Combining this with the above remarks we get the claim.

Summing up, we have proved that $\lambda \in \text{sing supp}(\omega = d\rho * \mu)$.

Now, applying the following result of [14] we obtain Theorem 3.2.

Theorem 3.3 (see [14]). *Let $\lambda \in \text{sing supp}_a(\omega)$. Assume that λ satisfies (H1). Then for every h -independent complex neighborhood $\tilde{\Omega}$ of λ , there exists $h_0 = h(\tilde{\Omega})$ sufficiently small and $C = C(\tilde{\Omega})$ large enough such that, for $h \in]0, h_0[$,*

$$\#\{z \in \tilde{\Omega}; z \in \text{Res}(P(h))\} \geq C(\tilde{\Omega})h^{-n}. \quad (3.24)$$

□

Remark 3.4. Let e_0 be a singularity of the integrated density of states, generated by a band crossing. Theorem 3.2 shows that there is at least $\sim h^{-n}$ resonances near $e_0 + t$, where t is in the singular support of the distribution μ defined by

$$\mu(t) = \int_{\{x \in \mathbb{R}^n; W(x) > t\}} dx. \quad (3.25)$$

4. Lower Bound of the Counting Function near the Edges of Bands

In this section we study resonances generated by analytic singularities of ρ near the edge of bands. The following result is a consequence of Lemma 2.3.

Lemma 4.1. *Let $e_0 \in \sigma(P_0)$. One assumes the following.*

- (i) *If $\lambda_j(k) = e_0$, then $\lambda_j(k)$ is a simple eigenvalue of $H_0(k)$.*
- (ii) *There exist i_0 and k_0 such that $\lambda_{i_0}(k_0) = e_0$, $\nabla \lambda_{i_0}(k_0) = 0$, $\pm \partial^2 \lambda_{i_0}(k_0) > 0$ and $\nabla \lambda_{i_0}(k) \neq 0$, for all $k \in E^*$, $k \neq k_0$.*
- (iii) *For all $k \in \lambda_i^{-1}\{e_0\}$ with $i \neq i_0$, $\nabla \lambda_i(k) \neq 0$.*

Then there exists an open connected neighborhood J of e_0 such that

$$\rho(e) = f(e - e_0) + H(\pm(e - e_0))g\left(\sqrt{\pm(e - e_0)}\right), \quad \forall e \in J, \quad (4.1)$$

where f and g are analytic near zero and $g(0) = 0, \dots, g^{(n-1)}(0) = 0$, $g^{(n)}(0) \neq 0$. Here, $+$ ($-$) corresponds to a local minimum (maximum, resp.).

Now, repeating the arguments in the proof of Theorem 3.2 and using Lemma 4.1, we obtain the following.

Theorem 4.2. Let $e_0, \lambda \in \sigma(P_0)$ with $\lambda \in (e_0 + \text{sing supp}_a(\mu))$. One assumes the following.

- (i) λ satisfies (H1),
- (ii) e_0 satisfies the assumptions of Lemma 4.1,
- (iii) $(\lambda - \text{supp}(\mu)) \subset J$. Here J is the interval given by Lemma 4.1.

Then for all h -independent complex neighborhoods Ω of λ , there exist $h_0 = h(\Omega) > 0$ sufficiently small and $C = C(\Omega) > 0$ such that, for $h \in]0, h_0[$,

$$\#\{z \in \Omega; z \in \text{Res}(P(h))\} \geq C_\Omega h^{-n}. \quad (4.2)$$

Remark 4.3. Notice that the assumptions (iv) in Theorem 3.2 and (iii) in Theorem 4.2 are satisfied if $\|W\|_\infty$ is small.

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