Research Article

A Weighted Average Finite Difference Method for the Fractional Convection-Diffusion Equation

Lijuan Su and Pei Cheng

School of Mathematical Sciences, Anhui University, Hefei, Anhui 230601, China

Correspondence should be addressed to Lijuan Su; sulijuan8362@163.com

Received 18 January 2013; Revised 8 April 2013; Accepted 3 June 2013

Academic Editor: R. de la Llave

Copyright © 2013 L. Su and P. Cheng. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

A weighted average finite difference method for solving the two-sided space-fractional convection-diffusion equation is given, which is an extension of the weighted average method for ordinary convection-diffusion equations. Stability, consistency, and convergence of the new method are analyzed. A simple and accurate stability criterion valid for this method, arbitrary weighted factor, and arbitrary fractional derivative is given. Some numerical examples with known exact solutions are provided.

1. Introduction

The history of the fractional derivatives and integrals can date back to the 17th century. However, only after 124 years later, Lacroix first put forward a result of the simplest fractional calculus. Nowadays, the fractional derivatives and integrals have many important applications in various fields of physics [1–3], finance [4, 5], hydrology [6], engineering [7], mathematics [8], science, and so forth.

Anomalous diffusion is perhaps the most frequently studied complex problem. Classical (integer-order) partial differential equation of diffusion and wave has been extended to the equation with fractional time and/or space by means of fractional operator [9]. Furthermore, it has been extended to the problems of every kind of nonlinear fractional differential equations, and to present the solutions to the problems of initial and boundary values subject to above equations is another rapidly developing field of fractional operator applications. In general, all of these equations have important background of practice applications, such as dispersion in fractals and porous media [10], semiconductor, turbulence, and condensed matter physics.

As a special case of anomalous diffusion, the two-sided space-fractional convection-diffusion equation for the force-free case is usually written in the following way [11]:

$$\frac{\partial u(x, t)}{\partial t} = -V(x, t) \frac{\partial u(x, t)}{\partial x} + D_+(x, t) \frac{\partial^\alpha u(x, t)}{\partial_+ x^\alpha}$$

$$+ D_-(x, t) \frac{\partial^\alpha u(x, t)}{\partial_- x^\alpha} + s(x, t),$$

$$L \leq x \leq R, \quad 0 < t \leq T,$$

$$u(x, t = 0) = u_0(x), \quad L \leq x \leq R,$$

$$u(L, t) = u(R, t) = 0, \quad 0 \leq t \leq T,$$

(1)

where $V(x, t) > 0$ is the drift of the process, that is, the mean advective velocity, α is the order of fractional differentiation, $D_+(x, t) = (1 + \beta)D(x, t)/2$, $D_-(x, t) = (1 - \beta)D(x, t)/2$, $D(x, t) > 0$ is the coefficient of dispersion, and $-1 \leq \beta \leq 1$ indicates the relative weight of forward versus backward transition probability. The function $u_0(x)$ is the initial condition, the boundary conditions are zero Dirichlet boundary conditions, and the function $s(x, t)$ is a source/sink term. The $\partial^\alpha u(x, t)/\partial_+ x^\alpha$ and $\partial^\alpha u(x, t)/\partial_- x^\alpha$ in (1) are the Riemann-Liouville fractional derivatives. Equation (1) is a special case of the space-fractional Fokker-Planck equation, which more adequately describes the movement of solute in an aquifer than the traditional second-order Fokker-Planck equation.

The left-sided (+) and the right-sided (−) fractional derivatives in (1) are the Riemann-Liouville fractional derivatives of order α of a function $f(x)$ for $x \in [L, R]$ defined by [12]:

$$\frac{\partial^\alpha f(x)}{\partial_+ x^\alpha} = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dx^n} \int_L^x \frac{f(\xi) d\xi}{(x - \xi)^{\alpha - n + 1}},$$

$$\frac{\partial^\alpha f(x)}{\partial_- x^\alpha} = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^R \frac{f(\xi) d\xi}{(\xi-x)^{\alpha-n+1}}, \quad (2)$$

where $n-1 < \alpha \leq n$ (n is an integer) and $\Gamma(\cdot)$ is the Gamma function.

In some cases, there are some methods to solve fractional partial differential equations and get the analytical solutions [12], such as Fourier transform methods, Laplace transform methods, Mellin transform methods, the method of images, and the method of separation of variables. In this paper, the exact solution of (1) can be obtained by Fourier transform methods. However, as in the cases of integer-order differential equations, there are only very few cases of fractional partial differential equations in which the closed-form analytical solutions are available. Therefore, numerical means have to be used in general.

Many of researches on the numerical methods for solving fractional partial differential equations have been proposed, for example, L2 or L2C methods [13], standard or shifted Grünwald-Letnikov formulae [14], convolution formulae [15], homotopy perturbation method, and so forth. For example, Langlands and Henry [16] use L1 scheme form [8] to discretize the Riemann-Liouville fractional time derivative of order between 1 and 2. Yuste [17] considered a Grünwald-Letnikov approximation for the Riemann-Liouville time fractional derivative and used a weighted average for the second-order space derivative. Lin and Xu [18] proposed the method based on a finite difference scheme in time, Legendre's spectral method in space, and so on.

In this paper, based on shifted Grünwald-Letnikov formula, we consider a fractional weighted average (FWA) finite difference method, which is very close to the classical WA methods for ordinary (nonfractional) partial differential equations. The FWA method has some better properties than the fractional explicit and full implicit methods [19], such as higher-order accuracy in time step when weighting coefficient $\lambda = 1/2$.

The rest of this paper is organized as follows. In Section 2, the FWA finite difference method is developed. The stability and convergence of the method are proved in Section 3. Some numerical examples are given in Section 4. Finally, we draw our conclusions in Section 5.

2. Fractional Weighted Average Methods

To present the new finite difference method, we give some notations: Δt is the time step, Δx is the spatial step, the coordinates of the mesh points are $x_j = L + j\Delta x$, $j = 0, 1, 2, \dots, N$, $N = (R-L)/\Delta x$, and $t_m = m\Delta t$, $m = 0, 1, 2, \dots, M$, $M = T/\Delta t$, and the values of the solution $u(x, t)$ at these grid points are $u(x_j, t_m) \equiv u_j^m \simeq U_j^m$, where we denote by U_j^m the numerical estimate of the exact value of $u(x, t)$ at the point (x_j, t_m) . Define $V_j^{m+1/2} = V(x_j, t_{m+1/2})$, $D_j^{m+1/2} = D(x_j, t_{m+1/2})$ and $s_j^{m+1/2} = s(x_j, t_{m+1/2})$.

The centered time difference scheme is [20]

$$\frac{\partial u}{\partial t} \Big|_{(x_j, t_{m+1/2})} = \frac{u_j^{m+1} - u_j^m}{\Delta t} + O(\Delta t)^2, \quad (3)$$

and the backward space difference scheme is

$$\frac{\partial u}{\partial x} \Big|_{(x_j, t_m)} = \frac{u_j^m - u_{j-1}^m}{\Delta x} + O(\Delta x). \quad (4)$$

According to the shifted Grünwald-Letnikov definition [8], the definition (2) can be written as

$$\frac{\partial^\alpha u(x)}{\partial_+ x^\alpha} \Big|_{(x_j, t_m)} = \frac{1}{h^\alpha} \sum_{k=0}^{j+1} g_k^{(\alpha)} u_{j-k+1}^m + O(h), \quad (5)$$

$$\frac{\partial^\alpha u(x)}{\partial_- x^\alpha} \Big|_{(x_j, t_m)} = \frac{1}{h^\alpha} \sum_{k=0}^{N-j+1} g_k^{(\alpha)} u_{j+k-1}^m + O(h).$$

Here, $g_k^{(\alpha)} = (-1)^k \binom{\alpha}{k}$ can be evaluated recursively:

$$g_0^{(\alpha)} = 1, \quad g_k^{(\alpha)} = \left(1 - \frac{\alpha+1}{k}\right) g_{k-1}^{(\alpha)}. \quad (6)$$

In the weighted average method, (1) can be evaluated at the intermediate point of the grid $(x_j, t_{m+1/2})$ by the following formula:

$$\begin{aligned} & \frac{\partial u(x, t)}{\partial t} \Big|_{(x_j, t_{m+1/2})} \\ &= -V(x_j, t_{m+1/2}) \left[\lambda \frac{\partial u(x, t)}{\partial x} \Big|_{(x_j, t_m)} + (1-\lambda) \frac{\partial u(x, t)}{\partial x} \Big|_{(x_j, t_{m+1})} \right] \\ &+ D_+(x_j, t_{m+1/2}) \left[\lambda \frac{\partial^\alpha u(x, t)}{\partial_+ x^\alpha} \Big|_{(x_j, t_m)} + (1-\lambda) \frac{\partial^\alpha u(x, t)}{\partial_+ x^\alpha} \Big|_{(x_j, t_{m+1})} \right] \\ &+ D_-(x_j, t_{m+1/2}) \left[\lambda \frac{\partial^\alpha u(x, t)}{\partial_+ x^\alpha} \Big|_{(x_j, t_m)} + (1-\lambda) \frac{\partial^\alpha u(x, t)}{\partial_+ x^\alpha} \Big|_{(x_j, t_{m+1})} \right] \\ &+ s(x, t) \Big|_{(x_j, t_{m+1/2})}, \end{aligned} \quad (7)$$

where $0 \leq \lambda \leq 1$ is the weighting coefficient.

Applying (3)~(5) to (7), letting $h = \Delta x$, and neglecting the truncation error, we get the FWA difference scheme

$$\begin{aligned}
 & U_j^{m+1} + (1 - \lambda) \left[-r_j^{m+1/2} U_{j-1}^{m+1} + r_j^{m+1/2} U_j^{m+1} \right. \\
 & \quad \left. - \xi_j^{m+1/2} \sum_{k=0}^{j+1} g_k^{(\alpha)} U_{j-k+1}^{m+1} \right. \\
 & \quad \left. - \eta_j^{m+1/2} \sum_{k=0}^{N-j+1} g_k^{(\alpha)} U_{j+k-1}^{m+1} \right] \\
 & = U_j^m - \lambda \left[-r_j^{m+1/2} U_{j-1}^m + r_j^{m+1/2} U_j^m \right. \\
 & \quad \left. - \xi_j^{m+1/2} \sum_{k=0}^{j+1} g_k^{(\alpha)} U_{j-k+1}^m \right. \\
 & \quad \left. - \eta_j^{m+1/2} \sum_{k=0}^{N-j+1} g_k^{(\alpha)} U_{j+k-1}^m \right] + \Delta t s_j^{m+1/2}, \\
 & j = 1, 2, \dots, N-1, \quad m = 0, 1, 2, \dots, M-1,
 \end{aligned} \tag{8}$$

where $r_j^{m+1/2} = V_j^{m+1/2} \Delta t / h$, $\xi_j^{m+1/2} = D_{+,j}^{m+1/2} \Delta t / h^\alpha$, $\eta_j^{m+1/2} = D_{-,j}^{m+1/2} \Delta t / h^\alpha$, and the initial values are calculated by $U_j^{(0)} = u_0(x_j)$, $j = 1, 2, \dots, N-1$. Generally, the quantity $r_j^{m+1/2}$ is called the Courant (or CFL) number, the $\xi_j^{m+1/2}$ and $\eta_j^{m+1/2}$ are associated with the diffusion coefficients.

Obviously, the scheme is explicit when $\lambda = 1$ and the scheme is fully implicit when $\lambda = 0$. particularly, when $\lambda = 1/2$, the FWA scheme is called the fractional Crank-Nicholson (FCN) scheme.

3. Stability and Accuracy Analysis

In this section, we study the stability of the FWA method and discuss the truncating error. According to our analysis, we can get a conclusion which is similar to the result of classical WA methods. In fact, the following theorem can be viewed as a generalization of these stability conditions for classical WA methods [20].

Lemma 1. The coefficients $g_k^{(\alpha)}$ given in (6) with $1 < \alpha \leq 2$ satisfy the following properties:

$$\begin{aligned}
 & g_0^{(\alpha)} = 1, \quad g_1^{(\alpha)} = -\alpha < 0, \\
 & 1 \geq g_2^{(\alpha)} \geq g_3^{(\alpha)} \geq \dots \geq 0, \\
 & \sum_{k=0}^{\infty} g_k^{(\alpha)} = 0, \\
 & \sum_{k=0}^m g_k^{(\alpha)} \leq 0 \quad (m \geq 1).
 \end{aligned} \tag{9}$$

Theorem 2. When $0 \leq \lambda \leq 1/2$, the FWA (8) is unconditionally stable, based on the shifted Grünwald approximation (5) to the fractional equation (1) with $1 < \alpha \leq 2$. When $1/2 < \lambda \leq 1$, the FWA (8) is conditionally stable if

$$\frac{\alpha \Delta t D_{\max}}{h^\alpha} + \frac{\Delta t V_{\max}}{h} \leq \frac{1}{2\lambda - 1}, \tag{10}$$

where $V_{\max} = \max_{L \leq x \leq R, 0 \leq t \leq T} V(x, t)$ and $D_{\max} = \max_{L \leq x \leq R, 0 \leq t \leq T} D(x, t)$.

Proof. The FWA scheme (8) can be rewritten as $[I + (1 - \lambda)A]U^{m+1} = (I - \lambda A)U^m$, $m = 0, 1, 2, \dots, M-1$; here, $U^m = [U_0^m, U_1^m, U_2^m, \dots, U_N^m]^T$, $A = (a_{ij})$, $i, j = 0, 1, 2, \dots, N$.

The matrix entries $a_{i,j}$ for $i = 1, 2, \dots, N-1$ and $j = 0, 1, \dots, N$ are defined by

$$a_{i,j} = \begin{cases} r_i^{m+1/2} - (\xi_i^{m+1/2} + \eta_i^{m+1/2}) g_1^{(\alpha)}, & j = i \\ -r_i^{m+1/2} - \xi_i^{m+1/2} g_2^{(\alpha)} - \eta_i^{m+1/2} g_0^{(\alpha)}, & j = i-1 \\ -\xi_i^{m+1/2} g_0^{(\alpha)} - \eta_i^{m+1/2} g_2^{(\alpha)}, & j = i+1 \\ -\xi_i^{m+1/2} g_{i-j+1}^{(\alpha)}, & j < i-1 \\ -\eta_i^{m+1/2} g_{j-i+1}^{(\alpha)}, & j > i+1 \end{cases} \tag{11}$$

while $a_{0,j} = a_{N,j} = 0$, for $j = 0, 1, \dots, N$.

According to Lemma 1 and the Gerschgorin theorem, the eigenvalues of the matrix A (noted ω_i) are in the disks centered at $a_{i,i} = r_i^{m+1/2} - (\xi_i^{m+1/2} + \eta_i^{m+1/2}) g_1^{(\alpha)}$, with radius

$$\begin{aligned}
 R_i & = \sum_{j=0, j \neq i}^N |a_{i,j}| \\
 & = r_i^{m+1/2} + \xi_i^{m+1/2} \sum_{k=0, k \neq 1}^{i+1} g_k^{(\alpha)} + \eta_i^{m+1/2} \sum_{k=0, k \neq 1}^{N-i+1} g_k^{(\alpha)} \\
 & \leq r_i^{m+1/2} - (\xi_i^{m+1/2} + \eta_i^{m+1/2}) g_1^{(\alpha)}.
 \end{aligned} \tag{12}$$

Therefore, we have

$$\begin{aligned}
 0 \leq \omega_i & \leq 2 \left[r_i^{m+1/2} - (\xi_i^{m+1/2} + \eta_i^{m+1/2}) g_1^{(\alpha)} \right] \\
 & = 2 \left[\frac{D_i^{m+1/2} \alpha \Delta t}{h^\alpha} + \frac{V_i^{m+1/2} \Delta t}{h} \right].
 \end{aligned} \tag{13}$$

Next, note that ω_i is an eigenvalue of A if and only if $(1 - \lambda \omega_i)/(1 + (1 - \lambda) \omega_i)$ is an eigenvalue of the matrix $(I - \lambda A)[I + (1 - \lambda)A]^{-1}$. Because of $0 \leq \omega_i$ and $0 \leq \lambda \leq 1$, we get

$$\frac{1 - \lambda \omega_i}{1 + (1 - \lambda) \omega_i} = 1 - \frac{\omega_i}{1 + (1 - \lambda) \omega_i} \leq 1. \tag{14}$$

In addition, $(1 - \lambda \omega_i)/(1 + (1 - \lambda) \omega_i) \geq -1$ as long as $(2\lambda - 1)\omega_i \leq 2$.

Hence, when $0 \leq \lambda \leq 1/2$, we can find that $-1 \leq (1 - \lambda \omega_i)/(1 + (1 - \lambda) \omega_i) \leq 1$ always holds; that is, $|(1 - \lambda \omega_i)/(1 + (1 - \lambda) \omega_i)| \leq 1$. Then, the FWA (8) is unconditionally stable. On the other hand, when $1/2 < \lambda \leq 1$, from (13) and $\omega_i \leq 2/(2\lambda - 1)$, we get the limited condition $(\alpha \Delta t D_{\max}/h^\alpha) + (\Delta t V_{\max}/h) \leq 1/(2\lambda - 1)$ where $V_{\max} = \max_{L \leq x \leq R, 0 \leq t \leq T} V(x, t)$ and $D_{\max} = \max_{L \leq x \leq R, 0 \leq t \leq T} D(x, t)$. Therefore, the FWA (8) is conditionally stable. $\square$

FIGURE 1: Numerical solutions of (1) by means of the FWA method for $\lambda = 0.9$, $\Delta x = 1/40$, and $S = 1.2$. The numerical solutions are shown after 100 (dots), 1000 (stars), and 4000 (circles) time steps. The lines correspond to the exact solutions.

Let

$$S = \frac{\alpha \Delta t D_{\max}}{h^\alpha} + \frac{\Delta t V_{\max}}{h}, \quad (15)$$

the stability limit $S_\times$ is $S_\times = 1/(2\lambda - 1)$.

In addition, taking into account (3)~(5), for arbitrary Δx and Δt , we derive that this method is consistent with a local truncation error $O(\Delta x + \Delta t)$, except for the FCN method, whose accuracy is of $(\Delta t)^2$ with respect to the time step [21]. Therefore, according to Lax's equivalence theorem, the FWA method converges at the same rate, too.

Remark 3. Instead of (4), if forward space difference scheme is used, Theorem 2 still holds, and its proof does not change basically. However, if centered space difference scheme is used, we cannot obtain the same conclusion as Theorem 2.

4. Numerical Simulations

In this section, we apply the FWA scheme (8) to solve the two-sided space-fractional convection-diffusion equation (1) with $\beta = 0$, $V(x, t) = V$, $D(x, t) = D$, and $s(x, t) = 0$; the initial condition is

$$u_0(x) = \frac{20}{\pi} \int_0^{+\infty} \cos[(x - 0.1V)\xi] e^{0.1D \cos(\pi\alpha/2)\xi^\alpha} d\xi. \quad (16)$$

In this case, the analytical solution of (1) solved by the Fourier transform methods is [12]

$$\begin{aligned} u(x, t) &= \frac{20}{\pi} \int_0^{+\infty} \cos[(x - V(t + 0.1))\xi] e^{D(t+0.1) \cos(\pi\alpha/2)\xi^\alpha} d\xi. \end{aligned} \quad (17)$$

In the following numerical experiments, the data are chosen as follow: $\alpha = 1.9$, $D = 2$, $V = 2$, $T = 2.5$, $L = -5$, and $R = 15$.

FIGURE 2: The same as Figure 1 but for $S = 1.3$. The errors between numerical solution and exact solution after 1000 time steps are shown by line.

FIGURE 3: Numerical solutions of (1) by means of the FCN method for $\Delta x = 1/40$ and $S = 100$. The numerical solutions are shown after 10 (dots), 50 (stars), and 100 (circles) time steps. The lines correspond to the exact solutions.

The numerical solutions are obtained from the FWA scheme (8) discussed above, with different λ , Δt , t , S , and h . From (15), the values of Δt for fixed S and $\Delta x = h$ are

$$\Delta t = \frac{S}{((\alpha D_{\max}/h^\alpha) + (V_{\max}/h))} = \frac{S}{((\alpha D/h^\alpha) + (V/h))}. \quad (18)$$

The computational results are shown in Figures 1, 2, and 3. Figures 1 and 2 show two different cases where the FWA method is stable and unstable according to the theoretical predictions of Theorem 2. Figure 1 shows numerical solutions obtained by the FWA method (8) with $\lambda = 0.9$, $\Delta x = 1/40$, and small $S = 1.2$ after 100, 1000, and 4000 time steps. The numerical solutions compare well to the exact solutions,

which proves that the FWA method is stable. At the moment, we gain the very small time step $\Delta t = 2.8 \times 10^{-4}$ calculated from (18). Figure 2 has the same assumptions as Figure 1 but for $S = 1.3$ after 1000 time steps, and the large errors between numerical solutions and exact solutions obviously prove that the FWA method is unstable. In the both figures, because of $\lambda = 0.9$, the stability limit is $S_x = 1/(2\lambda - 1) = 1.25$.

Next, we consider the special case of $\lambda = 1/2$, under the assumption that the FWA method becomes the FCN method. Figure 3 shows numerical solutions obtained by the FCN method with $\Delta x = 1/40$ and large $S = 100$ after 10, 50, and 100 time steps. Meanwhile, we can gain the large time step $\Delta t = 2.3 \times 10^{-2}$ calculated from (18), which is much larger than $\Delta t = 2.8 \times 10^{-4}$ in Figure 1. The numerical solutions approximate well to the exact solutions, and the FCN method is always stable, so it allows the large time steps to be used.

5. Conclusions

Based on the shifted Grünwald approximation to the fractional derivative, we propose the FWA method in this paper, which can be viewed as a generalization of the classical WA methods for ordinary diffusion equations [17]. The stability of the FWA method depends on weighting parameter λ , and its accuracy is of order $O(\Delta x + \Delta t)$, except for the FCN method, whose accuracy with respect to the time step is of $(\Delta t)^2$ (see [21]).

Obviously, the FCN method is much better and more convenient than the fractional explicit and fully implicit methods because it is not only unconditionally stable, but also of second-order accuracy in time.

Acknowledgments

This research was supported by the National Natural Science Foundations of China (Grants nos. 11126179 and 11226247), the 211 Project of Anhui University (nos. 02303319 and 12333010266), the Scientific Research Award for Excellent Middle-Aged and Young Scientists of Shandong Province (no. BS2010HZ012), and the Nature Science Foundation of Anhui Provincial (no. 1308085QA15). The authors acknowledge the anonymous reviewers for their helpful comments.

References

- [1] M. de la Sen, "Positivity and stability of the solutions of Caputo fractional linear time-invariant systems of any order with internal point delays," *Abstract and Applied Analysis*, vol. 2011, Article ID 161246, 25 pages, 2011.
- [2] H. Yang, "Existence of mild solutions for a class of fractional evolution equations with compact analytic semigroup," *Abstract and Applied Analysis*, vol. 2012, Article ID 903518, 15 pages, 2012.
- [3] A. Ashyralyev, "A note on fractional derivatives and fractional powers of operators," *Journal of Mathematical Analysis and Applications*, vol. 357, no. 1, pp. 232–236, 2009.
- [4] M. Raberto, E. Scalas, and F. Mainardi, "Waiting-times and returns in high-frequency financial data: an empirical study," *Physica A*, vol. 314, no. 1–4, pp. 749–755, 2002.
- [5] L. Sabatelli, S. Keating, J. Dudley, and P. Richmond, "Waiting time distributions in financial markets," *The European Physical Journal B*, vol. 27, no. 2, pp. 273–275, 2002.
- [6] B. Baeumer, D. A. Benson, M. M. Meerschaert, and S. W. Wheatcraft, "Subordinated advection-dispersion equation for contaminant transport," *Water Resources Research*, vol. 37, no. 6, pp. 1543–1550, 2001.
- [7] R. L. Magin, *Fractional Calculus in Bioengineering*, Begell House Publishers, New York, NY, USA, 2006.
- [8] K. B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York, NY, USA, 1974.
- [9] F. Mainardi, "Fractional relaxation-oscillation and fractional diffusion-wave phenomena," *Chaos, Solitons and Fractals*, vol. 7, no. 9, pp. 1461–1477, 1996.
- [10] B. I. Henry and S. L. Wearne, "Fractional reaction-diffusion," *Physica A*, vol. 276, no. 3–4, pp. 448–455, 2000.
- [11] D. A. Benson, S. W. Wheatcraft, and M. M. Meerschaert, "The fractional-order governing equation of Levy motion," *Water Resources Research*, vol. 36, no. 6, pp. 1413–1423, 2000.
- [12] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, NY, USA, 1999.
- [13] F. Liu, V. Anh, and I. Turner, "Numerical solution of the space fractional Fokker-Planck equation," *Journal of Computational and Applied Mathematics*, vol. 166, no. 1, pp. 209–219, 2004.
- [14] M. M. Meerschaert and C. Tadjeran, "Finite difference approximations for fractional advection-dispersion flow equations," *Journal of Computational and Applied Mathematics*, vol. 172, no. 1, pp. 65–77, 2004.
- [15] Ch. Lubich, "Discretized fractional calculus," *SIAM Journal on Mathematical Analysis*, vol. 17, no. 3, pp. 704–719, 1986.
- [16] T. A. M. Langlands and B. I. Henry, "The accuracy and stability of an implicit solution method for the fractional diffusion equation," *Journal of Computational Physics*, vol. 205, no. 2, pp. 719–736, 2005.
- [17] S. B. Yuste, "Weighted average finite difference methods for fractional diffusion equations," *Journal of Computational Physics*, vol. 216, no. 1, pp. 264–274, 2006.
- [18] Y. Lin and C. Xu, "Finite difference/spectral approximations for the time-fractional diffusion equation," *Journal of Computational Physics*, vol. 225, no. 2, pp. 1533–1552, 2007.
- [19] M. M. Meerschaert and C. Tadjeran, "Finite difference approximations for two-sided space-fractional partial differential equations," *Applied Numerical Mathematics*, vol. 56, no. 1, pp. 80–90, 2006.
- [20] K. W. Morton and D. F. Mayers, *Numerical Solution of Partial Differential Equations*, Cambridge University Press, Cambridge, UK, 1994.
- [21] L. Su, W. Wang, and Z. Yang, "Finite difference approximations for the fractional advection-diffusion equation," *Physics Letters A*, vol. 373, pp. 4405–4408, 2009.