Research Article

Approximation Solutions for Local Fractional Schrödinger Equation in the One-Dimensional Cantorian System

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1. Introduction

As it is known, in classical mechanics, the equations of motions are described as Newton’s second law, and the equivalent formulations become the Euler-Lagrange equations and Hamilton's equations. In quantum mechanics, Schrödinger’s equation for a dynamic system like Newton's law plays an important role in Newton’s mechanics and conservation of energy. Mathematically, it is a partial differential equation, which is applied to describe how the quantum state of a physical system changes in time [1, 2]. In this work, the solutions of Schrödinger equations were investigated within the various methods [3–12] and other references therein.

Recently, the fractional calculus [13–30], which is different from the classical calculus, is now applied to practical techniques in many branches of applied sciences and engineering. Fractional Schrödinger’s equation was proposed by Laskin [31] via the space fractional quantum mechanics, which is based on the Feynman path integrals, and some properties of fractional Schrödinger’s equation are investigated by Naber [32]. In present works, the solutions of fractional Schrödinger equations were considered in [33–38].

Classical and fractional calculus cannot deal with nondifferentiable functions. However, the local fractional calculus (also called fractal calculus) [39–56] is best candidate and has been applied to model the practical problems in engineering, which are nondifferentiable functions. For example, the systems of Navier-Stokes equations on Cantor sets with local fractional derivative were discussed in [42]. The local fractional Fokker-Planck equation was investigated in [43]. The basic theory of elastic problems was considered in [44]. The anomalous diffusion with local fractional derivative was researched in [48–50]. Newtonian mechanics with local fractional derivative was proposed in [51]. The fractal heat transfer in silk cocoon hierarchy and heat conduction in a semi-infinite fractal bar were presented in [53–55] and other references therein.

More recently, the local fractional Schrödinger equation in three-dimensional Cantorian system was considered in [56] as

\[
i^\alpha h^a \partial^a \psi_a(x, y, z, t) = -\frac{h^a}{2m} \partial^{2a} \psi_a(x, y, z, t) + V_{a}(x, y, z) \psi_a(x, y, z, t),
\]

where the local fractional Laplace operator is [39, 40, 42]

\[
\psi^{2a} = \frac{\partial^{2a}}{\partial x^{2a}} + \frac{\partial^{2a}}{\partial y^{2a}} + \frac{\partial^{2a}}{\partial z^{2a}}.
\]
the wave function $\psi_\alpha(x, y, z, t)$ is a local fractional continuous function $[39, 40]$, and the local fractional differential operator is given by $[39, 40]

\[ f^{(\alpha)}(x_0) = \frac{D^\alpha f(x)}{dx^\alpha}\bigg|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha}, \]

with $\Delta^\alpha (f(x) - f(x_0)) \equiv (1 + \alpha)\Delta(f(x) - f(x_0))$.

The local fractional Schrödinger equation in two-dimensional Cantorian system can be written as

\[ L_\alpha \phi(x, y, t) = i^\alpha h_\alpha \frac{\partial^\alpha \psi(x, y)}{\partial t^\alpha} + V_\alpha(x, y) \psi(x, y, t), \]

where the local fractional Laplace operator is given by

\[ V^{2\alpha} = \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}}{\partial y^{2\alpha}}. \]

The local fractional Schrödinger equation in one-dimensional Cantorian system is presented as

\[ L_\alpha \phi(x, t) = i^\alpha h_\alpha \frac{\partial^\alpha \psi(x, t)}{\partial t^\alpha} + V_\alpha(x) \psi(x, t), \]

where the wave function $\psi_\alpha(x, t)$ is local fractional continuous function.

With the potential energy $V_\alpha = 0$, the local fractional Schrödinger equation in the one-dimensional Cantorian system is

\[ L_\alpha \phi(x, t) = i^\alpha h_\alpha \frac{\partial^\alpha \psi(x, t)}{\partial t^\alpha}. \]

In this paper our aim is to investigate the nondifferentiable solutions for local fractional Schrödinger equations in the one-dimensional Cantorian system by using the local fractional series expansion method $[49]$. The organization of the paper is organized as follows. In Section 2, we introduce the local fractional series expansion method. Section 3 is devoted to the solutions for local fractional Schrödinger equations. Finally, conclusions are given in Section 4.

2. The Local Fractional Series Expansion Method

According to local fractional series expansion method $[49]$, we consider the following local fractional differentiable equation:

\[ \phi^\alpha_i = L_\alpha \phi, \]

where $L_\alpha$ is the linear local fractional operator and $\phi$ is a local fractional continuous function.

In view of (8), the multitemr separated functions with respect to $x, t$ are expressed as follows:

\[ \phi(x, t) = \sum_{i=0}^{\infty} \phi_i(t) \psi_i(x), \]

where $\phi_i(t)$ and $\psi_i(x)$ are the local fractional continuous function.

There are nondifferentiable terms, which are written as

\[ \phi_i(t) = \frac{i^\alpha}{\Gamma(1 + i\alpha)} \psi_i(x). \]

where $\chi_i$ is a coefficient.

In view of (10), we get

\[ \phi(x, t) = \sum_{i=0}^{\infty} \frac{i^\alpha}{\Gamma(1 + i\alpha)} \psi_i(x). \]

Therefore,

\[ \phi^\alpha_i = \sum_{i=0}^{\infty} \frac{i^\alpha}{\Gamma(1 + i\alpha)} (L_\alpha \psi_i)(x). \]

Then, following (12), we have

\[ \sum_{i=0}^{\infty} \frac{i^\alpha}{\Gamma(1 + i\alpha)} \psi_i(x) = \sum_{i=0}^{\infty} \frac{i^\alpha}{\Gamma(1 + i\alpha)} (L_\alpha \psi_i)(x). \]

Let $\chi_i = \chi_i = 1$; then

\[ \sum_{i=0}^{\infty} \frac{i^\alpha}{\Gamma(1 + i\alpha)} \psi_i(x) = \sum_{i=0}^{\infty} \frac{i^\alpha}{\Gamma(1 + i\alpha)} (L_\alpha \psi_i)(x). \]

So, we have

\[ \psi_{i+1}(x) = L_\alpha \psi_i, \]

where $L_\alpha$ is a linear local fractional operator.

In $[49]$, the linear local fractional operators are considered as

\[ L_\alpha = \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} \]

\[ L_\alpha = \frac{\chi^{2\alpha}}{\Gamma(1 + 2\alpha)} \frac{\partial^{2\alpha}}{\partial x^{2\alpha}}, \]

\[ L_\alpha = \mu \frac{\partial^{2\alpha}}{\partial x^{2\alpha}}, \]

where $\mu$ is a constant.

Here, we consider the following operator:

\[ L_\alpha = \eta \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} + \gamma, \]

where $\eta$ and $\gamma$ are two constants.

Using the iterative formula (18), we obtain

\[ \phi(x, t) = \sum_{i=0}^{\infty} \frac{i^\alpha}{\Gamma(1 + i\alpha)} \psi_i(x), \]

which is the solution of (8).
3. Approximation Solutions

Let us change (6) into the following form:

\[
\frac{\partial^n \psi_a(x, t)}{\partial t^a} = -\frac{h_\alpha}{2^m m} \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} \psi_a(x, t) + \frac{V_a(x)}{i^m h_\alpha} \psi_a(x, t),
\]

(20)

where the linear local fractional operator is

\[
L_\alpha = -\frac{h_\alpha}{2^m m} \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} + \frac{V_a(x)}{i^m h_\alpha}.
\]

(21)

With \( V_a(x) = 1 \), we have

\[
L_\alpha = -\frac{h_\alpha}{2^m m} \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} + \frac{1}{i^m h_\alpha},
\]

(22)

so that

\[
\frac{\partial^n \psi_a(x, t)}{\partial t^a} = L_\alpha \psi_a(x, t),
\]

(23)

where

\[
\psi_a(x, 0) = \frac{x^{2\alpha}}{\Gamma(1 + 2\alpha)}.
\]

(24)

Using iteration relation (15), we set up

\[
\psi_{a,n+1}(x, t) = L_\alpha \psi_{a,n}(x, t),
\]

(25)

and an initial value is given by

\[
\psi_{a,0}(x, t) = \frac{x^{2\alpha}}{\Gamma(1 + 2\alpha)}.
\]

(26)

Therefore, following (25), we get

\[
\psi_{a,0}(x, t) = \frac{x^{2\alpha}}{\Gamma(1 + 2\alpha)}.
\]

(27)

\[
\psi_{a,1}(x, t) = \left( -\frac{h_\alpha}{2^m m} \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} + \frac{1}{i^m h_\alpha} \right) \psi_{a,0}(x, t)
= -\frac{h_\alpha}{2i^m m} + \frac{1}{i^m h_\alpha} \frac{x^{2\alpha}}{\Gamma(1 + 2\alpha)},
\]

(28)

\[
\psi_{a,2}(x, t) = \left( -\frac{h_\alpha}{2^m m} \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} + \frac{1}{i^m h_\alpha} \right) \psi_{a,1}(x, t)
= \left( -\frac{h_\alpha}{2^m m} \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} + \frac{1}{i^m h_\alpha} \right) \psi_{a,1}(x, t)
\times \left( -\frac{h_\alpha}{2i^m m} + \frac{1}{i^m h_\alpha} \frac{x^{2\alpha}}{\Gamma(1 + 2\alpha)} \right)
\times \left( -\frac{2}{2} \frac{h_\alpha}{2^m m} + \frac{1}{i^m h_\alpha} \frac{x^{2\alpha}}{\Gamma(1 + 2\alpha)} \right)
= -\frac{2}{2i^m m} + \frac{1}{i^m h_\alpha} \frac{x^{2\alpha}}{\Gamma(1 + 2\alpha)},
\]

(29)

Applying (17), we can write the iterative relations as follows:

\[
\psi_{a,n+1}(x, t) = L_\alpha \psi_{a,n}(x, t),
\]

(30)

\[
\psi_{a,0}(x, t) = \frac{x^{2\alpha}}{\Gamma(1 + 2\alpha)}.
\]

(31)
where
\[ L_\alpha = -\frac{\hbar^2}{2m} \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} \]
(37)

From (36)–(38), we give the local fractional series terms as follows:
\[ \psi_{\alpha,0}(x, t) = \psi_{\alpha,0}(x, t) = \frac{x^{2\alpha}}{\Gamma(1 + 2\alpha)}, \]
(38)
\[ \psi_{\alpha,1}(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} \psi_{\alpha,0}(x, t) = 0, \]
(39)
\[ \psi_{\alpha,2}(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} \psi_{\alpha,1}(x, t) = 0, \]
(40)
\[ \psi_{\alpha,3}(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} \psi_{\alpha,2}(x, t) = 0, \]
(41)
\[ \vdots \]
\[ \psi_{\alpha,n}(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} \psi_{\alpha,n-1}(x, t) = 0, \]
(42)
and so forth.

Hence, we have the nondifferentiable solution of (34) as follows:
\[ \psi_{\alpha}(x, t) = \sum_{n=0}^{\infty} \psi_{\alpha,n}(x, t) = \frac{x^{2\alpha}}{\Gamma(1 + 2\alpha)} + \frac{\hbar^2}{2m} \frac{x^{\alpha}}{\Gamma(1 + \alpha)}. \]
(43)

4. Conclusions

In the work, we have obtained the nondifferentiable solutions for the local fractional Schrödinger equations in the one-dimensional Cantorian system by using the local fractional series expansion method. The present method is shown that is an effective method to obtain the local fractional series solutions for the partial differential equations within local fractional differentiable operator.

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References


