Research Article

Maximum Norm Error Estimates of ADI Methods for a Two-Dimensional Fractional Subdiffusion Equation

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This paper is concerned with two alternating direction implicit (ADI) finite difference methods for solving a two-dimensional fractional subdiffusion equation. An explicit error estimate for each of the two methods is provided in the discrete maximum norm. It is shown that the methods have the same order as their truncation errors with respect to the discrete maximum norm. Numerical results are given to confirm the theoretical analysis results.

1. Introduction

Fractional differential equations and fractional calculus arise in various application problems in science and engineering [1–16]. Various numerical methods have been developed for the computation of fractional differential equations [17–34]. Fractional subdiffusion equations describe a special type of anomalous diffusion [35], and it is a more difficult task to solve this kind of equation numerically.

Numerical works for fractional subdiffusion equations are mostly focused on one-dimensional problems due to the memory effect in fractional derivatives; see, for example, [19, 20, 20–26, 31–33, 36–42]. A two-dimensional anomalous subdiffusion equation was numerically treated in [43, 44], where explicit and implicit finite difference schemes were proposed. Chen et al. [28] extended their work in [43] to a variable-order subdiffusion equation. Liu et al. [45] developed an implicit meshless approach based on the radial basis function for the numerical simulation of a two-dimensional subdiffusion problem. Chen and Liu [18] considered an implicit difference scheme for a three-dimensional fractional advection-diffusion equation, and a Richardson extrapolation was applied to improve the accuracy.

The complexity of the fractional differential equations comes from the involving fractional derivatives that are

nonlocal and have the character of history dependence and universal mutuality. This means that the computations would be costly if the implicit schemes were applied, especially for solving multidimensional problems [43, 44]. Some researchers have explored some techniques for reducing this cost. These techniques include the adaptive technique [46] and the matrix transfer technique [47, 48]. It is well known that alternating direction implicit (ADI) methods are unconditionally stable as the traditional implicit methods. On the other hand, they reduce a multidimensional problem to a series of independent one-dimensional problems, and thus the computational complexities and the computational cost can be greatly reduced. Therefore, ADI methods for fractional differential equations have the potential to significantly reduce the computational cost, while maintaining the stability of the numerical methods. The works in [29, 49–53] treated ADI finite difference methods for space fractional diffusion equations, and the work in [54] discussed ADI finite difference methods for fractional diffusion wave equations. Recently, Cui [55] derived an ADI compact finite difference scheme for a two-dimensional fractional subdiffusion equation, where the Grünwald formula is used to approximate the temporal Riemann-Liouville fractional derivative, and the spatial derivatives are approximated by a compact finite difference scheme. Another way to treat fractional subdiffusion

problem is to transform the original subdiffusion equation into an equivalent equation by replacing the temporal Riemann-Liouville fractional derivative by the temporal Caputo fractional derivative; see, for example, [40, 42, 56]. An advantage of this approach is that the L_1 approximation (see [1, 24, 25, 40, 57, 58]) can be used to deal with the temporal fractional derivative, and so the resulting scheme has the better temporal accuracy than the first order without the Crank-Nicolson technique (see [25, 40–42]). Based on the previous approach, Zhang and Sun [59] constructed two ADI finite difference schemes, called L_1 -ADI and BD-ADI schemes, for the following two-dimensional problem of subdiffusion equation with the temporal Caputo fractional derivative:

$$\begin{aligned} {}^C_0\mathcal{D}_t^\alpha u(x, y, t) &= \Delta u(x, y, t) + f(x, y, t), \\ (x, y) &\in \Omega, \quad t \in (0, T], \\ u(x, y, t) &= \varphi(x, y, t), \quad (x, y) \in \partial\Omega, \quad t \in (0, T], \\ u(x, y, 0) &= \psi(x, y), \quad (x, y) \in \bar{\Omega}, \end{aligned} \quad (1)$$

where $\Omega = (0, l_1) \times (0, l_2)$, $\bar{\Omega} = \Omega \cup \partial\Omega$, $\partial\Omega$ is the boundary of Ω , T is a positive constant, Δ is the two-dimensional Laplacian, and ${}^C_0\mathcal{D}_t^\alpha$ denotes the temporal Caputo fractional derivative operator defined as

$$\begin{aligned} {}^C_0\mathcal{D}_t^\alpha u(x, y, t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x, y, s)}{\partial s} (t-s)^{-\alpha} ds, \quad 0 < \alpha < 1. \end{aligned} \quad (2)$$

The main concern in that paper is the construction of the schemes and error estimates in the discrete H^1 -norm. Since H^1 -norm error estimates do not provide immediate insight on the phase error occurring during the time evolution, it is more preferable to give error estimates in the discrete maximum norm when we measure computation errors in practice. In this paper, we continue the investigation of the paper [59], by establishing a maximum norm error estimate for the ADI discretizations. It is known that an H^1 -norm error estimate does not imply a maximum norm error estimate for two-dimensional problems. We here present a technique of H^2 discrete energy analysis in order to obtain an explicit maximum norm error estimate.

The outline of the paper is as follows. In Section 2, we derive ADI finite difference schemes for (1) and present our main results of the maximum norm error estimates. The proof of the main error results is given in Section 3. In Section 4, we give some numerical results demonstrating the accuracy of the schemes in the discrete maximum norm. Section 5 contains some concluding remarks.

2. ADI Schemes and Maximum Norm Error Estimates

We partition Ω with nonisotropic uniform mesh sizes h_1 and h_2 in the x and y directions, respectively. The integers

$M_1 = l_1/h_1$ and $M_2 = l_2/h_2$. The mesh points $(x_i, y_j) = (ih_1, jh_2)$ ($0 \leq i \leq M_1, 0 \leq j \leq M_2$). Let Ω_h and $\partial\Omega_h$ be the sets of mesh points lying in Ω and on $\partial\Omega$, respectively, and let $\bar{\Omega}_h = \Omega_h \cup \partial\Omega_h$. For any grid function $v = \{v_{i,j} \mid 0 \leq i \leq M_1, 0 \leq j \leq M_2\}$, we denote

$$\begin{aligned} \delta_x v_{i,j} &= \frac{(v_{i+1,j} - v_{i,j})}{h_1}, & \delta_y v_{i,j} &= \frac{(v_{i,j+1} - v_{i,j})}{h_2}, \\ \delta_x^2 v_{i,j} &= \frac{(\delta_x v_{i,j} - \delta_x v_{i-1,j})}{h_1}, & \delta_y^2 v_{i,j} &= \frac{(\delta_y v_{i,j} - \delta_y v_{i,j-1})}{h_2}, \\ \Delta_h v_{i,j} &= \delta_x^2 v_{i,j} + \delta_y^2 v_{i,j}. \end{aligned} \quad (3)$$

For a positive integer N , we let $\tau = T/N$ be the time step. Define $t_n = n\tau$ and

$$\begin{aligned} U_{i,j}^n &= u(x_i, y_j, t_n), & f_{i,j}^n &= f(x_i, y_j, t_n), \\ \varphi_{i,j}^n &= \varphi(x_i, y_j, t_n), & \psi_{i,j} &= \psi(x_i, y_j). \end{aligned} \quad (4)$$

For the temporal approximation, we introduce the operators

$$\begin{aligned} \delta_t U_{i,j}^n &= \frac{1}{\tau} (U_{i,j}^{n+1} - U_{i,j}^n), \\ D_t^\alpha U_{i,j}^n &= \frac{1}{\mu} \left(U_{i,j}^n - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) U_{i,j}^k - a_{n-1} U_{i,j}^0 \right), \end{aligned} \quad (5)$$

where $\mu = \tau^\alpha \Gamma(2-\alpha)$ and $a_k = (k+1)^{1-\alpha} - k^{1-\alpha}$.

Using the Taylor expansion and the L_1 approximation of ${}^C_0\mathcal{D}_t^\alpha u(x, y, t)$ (see [25, 58, 59]), we have

$$\begin{aligned} {}^C_0\mathcal{D}_t^\alpha u(x_i, y_j, t_n) &= D_t^\alpha U_{i,j}^n + (R_t)_{i,j}^n, \\ \Delta u(x_i, y_j, t_n) &= \Delta_h U_{i,j}^n - (R_x)_{i,j}^n - (R_y)_{i,j}^n, \\ (x_i, y_j) &\in \Omega_h, \quad 1 \leq n \leq N, \end{aligned} \quad (6)$$

where

$$\begin{aligned} (R_x)_{i,j}^n &= \frac{h_1^2}{6} \int_0^1 \left(\frac{\partial^4 u}{\partial x^4}(x_i + sh_1, y_j, t_n) \right. \\ &\quad \left. + \frac{\partial^4 u}{\partial x^4}(x_i - sh_1, y_j, t_n) \right) (1-s)^3 ds, \\ (R_y)_{i,j}^n &= \frac{h_2^2}{6} \int_0^1 \left(\frac{\partial^4 u}{\partial y^4}(x_i, y_j + sh_2, t_n) \right. \\ &\quad \left. + \frac{\partial^4 u}{\partial y^4}(x_i, y_j - sh_2, t_n) \right) (1-s)^3 ds. \end{aligned} \quad (7)$$

There exists a positive constant $C(\alpha)$ independent of τ, h_1, h_2 and the time level n such that

$$\left| (R_t)_{i,j}^n \right| \leq C(\alpha) \max_{0 \leq t \leq t_n} \left| \frac{\partial^2 u}{\partial t^2} (x_i, y_j, t) \right| \tau^{2-\alpha}. \quad (8)$$

Substituting (6) into (1), we obtain

$$\begin{aligned} D_t^\alpha U_{i,j}^n &= \Delta_h U_{i,j}^n + f_{i,j}^n - (R_t)_{i,j}^n - (R_x)_{i,j}^n - (R_y)_{i,j}^n, \\ (x_i, y_j) &\in \Omega_h, \quad 1 \leq n \leq N. \end{aligned} \quad (9)$$

2.1. Construction of L_1 -ADI and BD-ADI Schemes. In order to construct an ADI scheme, we add the term $\mu^2 D_t^\alpha \delta_x^2 \delta_y^2 U_{i,j}^n$ to (9). This yields

$$\begin{aligned} D_t^\alpha (U_{i,j}^n + \mu^2 \delta_x^2 \delta_y^2 U_{i,j}^n) &= \Delta_h U_{i,j}^n + f_{i,j}^n + (R_1)_{i,j}^n, \\ (x_i, y_j) &\in \Omega_h, \quad 1 \leq n \leq N, \end{aligned} \quad (10)$$

where

$$(R_1)_{i,j}^n = \mu^2 D_t^\alpha \delta_x^2 \delta_y^2 U_{i,j}^n - (R_t)_{i,j}^n - (R_x)_{i,j}^n - (R_y)_{i,j}^n. \quad (11)$$

It was shown in [59] that $|\mu^2 D_t^\alpha \delta_x^2 \delta_y^2 U_{i,j}^n| \leq C_L(\alpha) \tau^{2\alpha}$ for a positive constant $C_L(\alpha)$ independent of τ, h_1, h_2 and the time level n . Thus, there exists a positive constant C_1 independent of τ, h_1, h_2 and the time level n such that

$$\left| (R_1)_{i,j}^n \right| \leq C_1 (\tau^{\min\{2\alpha, 2-\alpha\}} + h_1^2 + h_2^2). \quad (12)$$

Denote by $u_{i,j}^n$ the finite difference approximation to $U_{i,j}^n$, and let $b_{n,k} = a_{n-k-1} - a_{n-k}$. After multiplying (10) by μ and then dropping the term $\mu(R_1)_{i,j}^n$, we derive a finite difference scheme as follows:

$$\begin{aligned} (I - \mu \delta_x^2) (I - \mu \delta_y^2) u_{i,j}^n &= \sum_{k=1}^{n-1} b_{n,k} (u_{i,j}^k + \mu^2 \delta_x^2 \delta_y^2 u_{i,j}^k) \\ &\quad + a_{n-1} (u_{i,j}^0 + \mu^2 \delta_x^2 \delta_y^2 u_{i,j}^0) + \mu f_{i,j}^n, \\ (x_i, y_j) &\in \Omega_h, \quad 1 \leq n \leq N, \\ u_{i,j}^n &= \varphi_{i,j}^n, \quad (x_i, y_j) \in \partial \Omega_h, \quad 1 \leq n \leq N, \\ u_{i,j}^0 &= \psi_{i,j}, \quad (x_i, y_j) \in \bar{\Omega}_h. \end{aligned} \quad (13)$$

By introducing the intermediate variable $v_{i,j}^n$, we obtain the following L_1 -ADI scheme (see [59]):

$$\begin{aligned} (I - \mu \delta_x^2) v_{i,j}^n &= \sum_{k=1}^{n-1} b_{n,k} (u_{i,j}^k + \mu^2 \delta_x^2 \delta_y^2 u_{i,j}^k) \\ &\quad + a_{n-1} (u_{i,j}^0 + \mu^2 \delta_x^2 \delta_y^2 u_{i,j}^0) + \mu f_{i,j}^n, \\ (x_i, y_j) &\in \Omega_h, \quad 1 \leq n \leq N, \\ v_{0,j}^n &= (I - \mu \delta_y^2) \varphi_{0,j}^n, \quad v_{M_1,j}^n = (I - \mu \delta_y^2) \varphi_{M_1,j}^n, \\ 0 \leq j &\leq M_2, \quad 1 \leq n \leq N, \end{aligned} \quad (14)$$

$$\begin{aligned} u_{i,j}^0 &= \psi_{i,j}, \quad (x_i, y_j) \in \bar{\Omega}_h, \\ (I - \mu \delta_y^2) u_{i,j}^n &= v_{i,j}^n, \quad (x_i, y_j) \in \Omega_h, \quad 1 \leq n \leq N, \\ u_{i,0}^n &= \varphi_{i,0}^n, \quad u_{i,M_2}^n = \varphi_{i,M_2}^n, \\ 0 \leq i &\leq M_1, \quad 1 \leq n \leq N. \end{aligned}$$

Adding the term $\tau \mu \delta_x^2 \delta_y^2 \delta_t U_{i,j}^n$ to (9), we have

$$D_t^\alpha U_{i,j}^n + \tau \mu \delta_x^2 \delta_y^2 \delta_t U_{i,j}^n = \Delta_h U_{i,j}^n + f_{i,j}^n + (R_2)_{i,j}^n, \quad (15)$$

where

$$(R_2)_{i,j}^n = \tau \mu \delta_x^2 \delta_y^2 \delta_t U_{i,j}^n - (R_t)_{i,j}^n - (R_x)_{i,j}^n - (R_y)_{i,j}^n. \quad (16)$$

Since $|\tau \mu \delta_x^2 \delta_y^2 \delta_t U_{i,j}^n| \leq C_{BD}(\alpha) \tau^{1+\alpha}$ for a positive constant $C_{BD}(\alpha)$ independent of τ, h_1, h_2 and the time level n (see [59]), there exists a positive constant C_2 independent of τ, h_1, h_2 and the time level n such that

$$\left| (R_2)_{i,j}^n \right| \leq C_2 (\tau^{\min\{1+\alpha, 2-\alpha\}} + h_1^2 + h_2^2). \quad (17)$$

By (15), we obtain the following finite difference scheme:

$$\begin{aligned} (I - \mu \delta_x^2) (I - \mu \delta_y^2) u_{i,j}^n &= \mu^2 \delta_x^2 \delta_y^2 u_{i,j}^{n-1} + \sum_{k=1}^{n-1} b_{n,k} u_{i,j}^k \\ &\quad + a_{n-1} u_{i,j}^0 + \mu f_{i,j}^n, \\ (x_i, y_j) &\in \Omega_h, \quad 1 \leq n \leq N, \\ u_{i,j}^n &= \varphi_{i,j}^n, \quad (x_i, y_j) \in \partial \Omega_h, \quad 1 \leq n \leq N, \\ u_{i,j}^0 &= \psi_{i,j}, \quad (x_i, y_j) \in \bar{\Omega}_h. \end{aligned} \quad (18)$$

It is equivalent to the following BD-ADI scheme (see [59]):

$$\begin{aligned}
(I - \mu\delta_x^2)v_{i,j}^n &= \mu^2\delta_x^2\delta_y^2u_{i,j}^{n-1} + \sum_{k=1}^{n-1} b_{n,k}u_{i,j}^k \\
&\quad + a_{n-1}u_{i,j}^0 + \mu f_{i,j}^n, \\
(x_i, y_j) &\in \Omega_h, \quad 1 \leq n \leq N, \\
v_{0,j}^n &= (I - \mu\delta_y^2)\varphi_{0,j}^n, \quad v_{M_1,j}^n = (I - \mu\delta_y^2)\varphi_{M_1,j}^n, \\
0 \leq j \leq M_2, \quad 1 \leq n \leq N, \\
u_{i,j}^0 &= \psi_{i,j}, \quad (x_i, y_j) \in \bar{\Omega}_h, \\
(I - \mu\delta_y^2)u_{i,j}^n &= v_{i,j}^n, \quad (x_i, y_j) \in \Omega_h, \quad 1 \leq n \leq N, \\
u_{i,0}^n &= \varphi_{i,0}^n, \quad u_{i,M_2}^n = \varphi_{i,M_2}^n, \\
0 \leq i \leq M_1, \quad 1 \leq n \leq N.
\end{aligned} \tag{19}$$

2.2. Maximum Norm Error Estimates. For any grid function $v = \{v_{i,j} \mid 0 \leq i \leq M_1, 0 \leq j \leq M_2\}$, we define its maximum norm $\|v\|_\infty$ by

$$\|v\|_\infty = \max_{0 \leq i \leq M_1, 0 \leq j \leq M_2} |v_{i,j}|. \tag{20}$$

Let $U_{i,j}^n$ be the value of the solution of (1) at the mesh point (x_i, y_j, t_n) , and let $u_{i,j}^n$ be the solution of the L_1 -ADI scheme (14) or the BD-ADI scheme (19). We now present our main results of the maximum norm estimate for the error $e_{i,j}^n = U_{i,j}^n - u_{i,j}^n$ in the following two theorems. Their proofs will be given in the next section.

Theorem 1. Assume that the solution $u(x, y, t)$ of (1) is sufficiently smooth, and let $u_{i,j}^n$ be the solution of the L_1 -ADI scheme (14). Then

$$\begin{aligned}
\left(\tau \sum_{l=1}^n \|U^l - u^l\|_\infty \right)^{1/2} &\leq C_1^* \left(\tau^{\min\{2\alpha, 2-\alpha\}} + h_1^2 + h_2^2 \right), \\
1 \leq n \leq N,
\end{aligned} \tag{21}$$

where $C_1^* = (l_0^2/2)\sqrt{\tau}C_1$ and $l_0 = \max\{l_1, l_2\}$.

Theorem 2. Assume that the solution $u(x, y, t)$ of (1) is sufficiently smooth, and let $u_{i,j}^n$ be the solution of the BD-ADI scheme (19). Then

$$\begin{aligned}
\left(\tau \sum_{l=1}^n \|U^l - u^l\|_\infty \right)^{1/2} &\leq C_2^* \left(\tau^{\min\{1+\alpha, 2-\alpha\}} + h_1^2 + h_2^2 \right), \\
1 \leq n \leq N,
\end{aligned} \tag{22}$$

where $C_2^* = (l_0^2/2)\sqrt{\tau}C_2$ and $l_0 = \max\{l_1, l_2\}$.

Theorems 1 and 2 show that the ADI difference solution $u_{i,j}^n$ from (14) or (19) converges to the analytical solution $U_{i,j}^n$

of (1) in the discrete maximum norm. We also see from the estimates (12), (17), (21), and (22) that the L_1 -ADI scheme (14) and the BD-ADI scheme (19) have the same order as their truncation error with respect to the discrete maximum norm.

3. Proof of the Main Results

Let $\mathcal{V}_h$ be the set of all grid functions $v = \{v_{i,j} \mid 0 \leq i \leq M_1, 0 \leq j \leq M_2\}$ defined in $\bar{\Omega}_h$ and vanishing on $\partial\Omega_h$. For arbitrary $v, w \in \mathcal{V}_h$, we define the following inner products:

$$\begin{aligned}
(v, w) &= h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} v_{i,j} w_{i,j}, \\
(\delta_x v, \delta_x w)_x &= h_1 h_2 \sum_{i=0}^{M_1-1} \sum_{j=1}^{M_2-1} (\delta_x v_{i,j}) (\delta_x w_{i,j}), \\
(\delta_y v, \delta_y w)_y &= h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=0}^{M_2-1} (\delta_y v_{i,j}) (\delta_y w_{i,j}).
\end{aligned} \tag{23}$$

For any $v \in \mathcal{V}_h$, we introduce the following norms:

$$\begin{aligned}
\|v\| &= (v, v)^{1/2}, \quad \|\delta_y v\|_y = (\delta_y v, \delta_y v)_y^{1/2} \quad (v = x, y), \\
|v|_1 &= \left(\|\delta_x v\|_x^2 + \|\delta_y v\|_y^2 \right)^{1/2}, \\
\|\delta_x \delta_y^2 v\|_x &= (\delta_x \delta_y^2 v, \delta_x \delta_y^2 v)_x^{1/2}, \\
\|\delta_y \delta_x^2 v\|_y &= (\delta_y \delta_x^2 v, \delta_y \delta_x^2 v)_y^{1/2}.
\end{aligned} \tag{24}$$

Using a simple calculation, we have that for arbitrary $v, w \in \mathcal{V}_h$,

$$(\delta_y^2 v, w) = -(\delta_y v, \delta_y w)_y, \quad v = x, y. \tag{25}$$

Before proving Theorems 1 and 2, we first introduce the following embedding theorem from [60, page 281].

Lemma 3. For any $v \in \mathcal{V}_h$, one has

$$\|v\|_\infty \leq l_0^2 \left(2\sqrt{l_1 l_2} \right)^{-1} \|\Delta_h v\|_\infty, \quad l_0 = \max\{l_1, l_2\}. \tag{26}$$

Proof of Theorem 1. Let $e_{i,j}^n = U_{i,j}^n - u_{i,j}^n$. Then by (10) and (13),

$$\begin{aligned}
e_{i,j}^n - \mu \Delta_h e_{i,j}^n + \mu^2 \delta_x^2 \delta_y^2 e_{i,j}^n &= w_{i,j}^{n-1} + \mu^2 \delta_x^2 \delta_y^2 w_{i,j}^{n-1} + \mu (R_1)_{i,j}^n, \\
(x_i, y_j) &\in \Omega_h, \quad 1 \leq n \leq N, \\
e_{i,j}^n &= 0, \quad (x_i, y_j) \in \partial\Omega_h, \quad 1 \leq n \leq N, \\
e_{i,j}^0 &= 0, \quad (x_i, y_j) \in \bar{\Omega}_h,
\end{aligned} \tag{27}$$

where

$$w_{i,j}^{n-1} = \sum_{k=1}^{n-1} b_{n,k} e_{i,j}^k + a_{n-1} e_{i,j}^0. \tag{28}$$

This implies that $e^n = \{e_{i,j}^n \mid 0 \leq i \leq M_1, 0 \leq j \leq M_2\} \in \mathcal{V}_h$ for each n . Taking the inner product of the first equation in (27) and $-\Delta_h e^n$, we get

$$\begin{aligned} & (e^n + \mu^2 \delta_x^2 \delta_y^2 e^n, -\Delta_h e^n) + (-\mu \Delta_h e^n, -\Delta_h e^n) \\ &= (w^{n-1} + \mu^2 \delta_x^2 \delta_y^2 w^{n-1}, -\Delta_h e^n) + (\mu(R_1)^n, -\Delta_h e^n), \\ & \quad 1 \leq n \leq N. \end{aligned} \quad (29)$$

It follows from (25) that

$$(e^n + \mu^2 \delta_x^2 \delta_y^2 e^n, -\Delta_h e^n) = |e^n|_1^2 + \mu^2 \left(\|\delta_y \delta_x^2 e^n\|_y^2 + \|\delta_x \delta_y^2 e^n\|_x^2 \right). \quad (30)$$

Similarly,

$$\begin{aligned} & (w^{n-1} + \mu^2 \delta_x^2 \delta_y^2 w^{n-1}, -\Delta_h e^n) \\ &= \sum_{k=1}^{n-1} b_{n,k} \left((\delta_x e^k, \delta_x e^n)_x + (\delta_y e^k, \delta_y e^n)_y \right. \\ & \quad \left. + \mu^2 (\delta_y \delta_x^2 e^k, \delta_y \delta_x^2 e^n)_y + \mu^2 (\delta_x \delta_y^2 e^k, \delta_x \delta_y^2 e^n)_x \right). \end{aligned} \quad (31)$$

Since

$$b_{n,k} = a_{n-k-1} - a_{n-k} > 0, \quad \sum_{k=1}^{n-1} b_{n,k} \leq 1, \quad (32)$$

we have from Cauchy-Schwarz inequality that

$$\begin{aligned} & (w^{n-1} + \mu^2 \delta_x^2 \delta_y^2 w^{n-1}, -\Delta_h e^n) \\ & \leq \frac{1}{2} \sum_{k=1}^{n-1} b_{n,k} \left(|e^k|_1^2 + \mu^2 \left(\|\delta_y \delta_x^2 e^k\|_y^2 + \|\delta_x \delta_y^2 e^k\|_x^2 \right) \right) \\ & \quad + \frac{1}{2} \left(|e^n|_1^2 + \mu^2 \left(\|\delta_y \delta_x^2 e^n\|_y^2 + \|\delta_x \delta_y^2 e^n\|_x^2 \right) \right), \\ & (\mu(R_1)^n, -\Delta_h e^n) \leq \frac{\mu}{2} \left(\|(R_1)^n\|^2 + \|\Delta_h e^n\|^2 \right). \end{aligned} \quad (33)$$

Substituting (30) and (33) into (29) gives the following:

$$\mu \|\Delta_h e^n\|^2 + E^n \leq E^{n-1} + \mu \|(R_1)^n\|^2, \quad 1 \leq n \leq N, \quad (34)$$

where

$$E^n = \sum_{k=1}^n a_{n-k} \left(|e^k|_1^2 + \mu^2 \left(\|\delta_y \delta_x^2 e^k\|_y^2 + \|\delta_x \delta_y^2 e^k\|_x^2 \right) \right), \quad (35)$$

$$E^0 = 0.$$

This implies that

$$\mu \sum_{l=1}^n \|\Delta_h e^l\|^2 + E^n \leq \mu \sum_{l=1}^n \|(R_1)^l\|^2, \quad 1 \leq n \leq N. \quad (36)$$

Since $E^n \geq 0$ and by (12), $\|(R_1)^l\|^2 \leq l_1 l_2 C_1^2 (\tau^{\min\{2\alpha, 2-\alpha\}} + h_1^2 + h_2^2)^2$, the estimate (21) follows from (36) and (26) immediately. $\square$

Proof of Theorem 2. The proof follows from the similar argument as that in the proof of Theorem 1 and we give a sketch. Let $e_{i,j}^n = U_{i,j}^n - u_{i,j}^n$. By (15) and (18),

$$\begin{aligned} e_{i,j}^n - \mu \Delta_h e_{i,j}^n + \mu^2 \delta_x^2 \delta_y^2 e_{i,j}^n &= \mu^2 \delta_x^2 \delta_y^2 e_{i,j}^{n-1} + w_{i,j}^{n-1} + \mu(R_2)_{i,j}^n, \\ (x_i, y_j) &\in \Omega_h, \quad 1 \leq n \leq N, \end{aligned}$$

$$e_{i,j}^n = 0, \quad (x_i, y_j) \in \partial\Omega_h, \quad 1 \leq n \leq N,$$

$$e_{i,j}^0 = 0, \quad (x_i, y_j) \in \overline{\Omega}_h, \quad (37)$$

where $w_{i,j}^{n-1}$ is defined by (28). Taking the inner product of the first equation in (37) and $-\Delta_h e^n$, we get

$$\begin{aligned} & (e^n, -\Delta_h e^n) + (-\mu \Delta_h e^n, -\Delta_h e^n) \\ & \quad + (\mu^2 \delta_x^2 \delta_y^2 (e^n - e^{n-1}), -\Delta_h e^n) \\ &= (w^{n-1}, -\Delta_h e^n) + (\mu(R_2)^n, -\Delta_h e^n), \quad 1 \leq n \leq N. \end{aligned} \quad (38)$$

Since

$$\begin{aligned} (e^n, -\Delta_h e^n) &= |e^n|_1^2, \quad (-\mu \Delta_h e^n, -\Delta_h e^n) = \mu \|\Delta_h e^n\|^2, \\ & (\mu^2 \delta_x^2 \delta_y^2 (e^n - e^{n-1}), -\Delta_h e^n) \\ & \geq \frac{\mu^2}{2} \left(\|\delta_y \delta_x^2 e^n\|_y^2 + \|\delta_x \delta_y^2 e^n\|_x^2 \right. \\ & \quad \left. - \|\delta_y \delta_x^2 e^{n-1}\|_y^2 - \|\delta_x \delta_y^2 e^{n-1}\|_x^2 \right), \\ (w^{n-1}, -\Delta_h e^n) &\leq \frac{1}{2} \left(\sum_{k=1}^{n-1} b_{n,k} |e^k|_1^2 + |e^n|_1^2 \right), \\ (\mu(R_2)^n, -\Delta_h e^n) &\leq \frac{\mu}{2} \left(\|(R_2)^n\|^2 + \|\Delta_h e^n\|^2 \right), \end{aligned} \quad (39)$$

we obtain that

$$\mu \|\Delta_h e^n\|^2 + F^n \leq F^{n-1} + \mu \|(R_2)^n\|^2, \quad 1 \leq n \leq N, \quad (40)$$

where

$$F^n = \sum_{k=1}^n a_{n-k} |e^k|_1^2 + \mu^2 \left(\|\delta_y \delta_x^2 e^n\|_y^2 + \|\delta_x \delta_y^2 e^n\|_x^2 \right), \quad (41)$$

$$F^0 = 0.$$

TABLE 1: The maximum norm error and the temporal accuracy of $u_{i,j}^n$ for Example 4.

τ	L_1 -ADI scheme ($h = \pi/200$)			BD-ADI scheme ($h = \pi/200$)		
	α	error(τ, h)	order(τ)	α	error(τ, h)	order(τ)
1/20	1/2	1.1393833280e-03	0.860979	1/3	1.5093973350e-03	1.266265
1/40		6.2732035469e-04	0.882911		6.2750871863e-04	1.285881
1/80		3.4017847773e-04	0.910189		2.5735409495e-04	1.301046
1/160		1.8101416127e-04	0.936057		1.0444249965e-04	1.319085
1/320		9.4608749632e-05	0.959731		4.1859417673e-05	1.351071
1/640		4.8643355059e-05			1.6408936533e-05	
1/20	2/3	1.2585768427e-03	1.260363	1/2	3.6622305737e-04	1.463622
1/40		5.2537885652e-04	1.288661		1.3278581143e-04	1.474131
1/80		2.1505367805e-04	1.303399		4.7796268443e-05	1.488745
1/160		8.7133397922e-05	1.307319		1.7030874694e-05	1.514438
1/320		3.5208078966e-05	1.297645		5.9613630191e-06	1.535827
1/640		1.4322284746e-05			2.0559641134e-06	
1/20	3/4	2.3004703569e-03	1.174836	2/3	1.4653603051e-03	1.207510
1/40		1.0189566568e-03	1.197125		6.3452327488e-04	1.242740
1/80		4.4441136189e-04	1.210140		2.6813016026e-04	1.264770
1/160		1.9208646006e-04	1.216775		1.1158658090e-04	1.276115
1/320		8.2643912130e-05	1.217031		4.6074754492e-05	1.274583
1/640		3.5550691871e-05			1.9044750580e-05	

Therefore

$$\mu \sum_{l=1}^n \|\Delta_h e^l\|^2 + F^n \leq \mu \sum_{l=1}^n \|(R_2)^l\|^2, \quad 1 \leq n \leq N. \quad (42)$$

The estimate (22) follows from the previous inequality, (17), and (26) immediately. $\square$

4. Numerical Results

In this section, we give some numerical results to demonstrate the accuracy of the L_1 -ADI scheme (14) and the BD-ADI scheme (19) with respect to the discrete maximum norm. Some detailed numerical comparisons of these two ADI schemes with the implicit scheme proposed in [43] can be found in [59]. The dependence of the solution of (1) on the anomalous diffusion exponent α was also exhibited in [59] through some numerical results.

In our numerical computations, we take an equal mesh size in each of the space directions; that is, $h_1 = h_2 = h$. We compute the discrete maximum norm error of the numerical solution $u_{i,j}^n$ by

$$\text{error}(\tau, h) = \left(\tau \sum_{l=1}^N \|U^l - u^l\|_{\infty} \right)^{1/2}, \quad (43)$$

and its convergence orders by

$$\begin{aligned} \text{order}(\tau) &= \log_2 \left(\frac{\text{error}(2\tau, h)}{\text{error}(\tau, h)} \right), \\ \text{order}(h) &= \log_2 \left(\frac{\text{error}(\tau, 2h)}{\text{error}(\tau, h)} \right), \end{aligned} \quad (44)$$

where $U_{i,j}^n$ represents the value of the exact analytic solution of (1) at (x_i, y_j, t_n) .

Example 4. We consider the problem (1) in the domain $\Omega = (0, \pi) \times (0, \pi)$, and let $T = 1/2$. Assume that the solution $u(x, y, t)$ of this problem is $u(x, y, t) = t^2 \sin(x) \sin(y)$. It can be checked that the corresponding known functions are given by

$$f(x, y, t) = 2 \sin(x) \sin(y) \left(\frac{t^{2-\alpha}}{\Gamma(3-\alpha)} + t^2 \right), \quad (45)$$

$$\varphi(x, y, t) = t^2 \sin(x) \sin(y), \quad \psi(x, y) = 0.$$

In Table 1, we present the maximum norm error $\text{error}(\tau, h)$ and the temporal convergence order $\text{order}(\tau)$ of the numerical solution $u_{i,j}^n$ by the L_1 -ADI scheme (14) and the BD-ADI scheme (19). We see that the L_1 -ADI scheme has the temporal accuracy of $\min\{2\alpha, 2 - \alpha\}$ in the discrete maximum norm, and the best temporal accuracy is attained at $\alpha = 2/3$. It is also seen that the BD-ADI scheme generates the temporal accuracy of $\min\{1 + \alpha, 2 - \alpha\}$ in the discrete maximum norm, and it gets the best temporal accuracy when $\alpha = 1/2$. These observations coincide well with the theoretical analysis.

Table 2 gives the maximum norm error $\text{error}(\tau, h)$ and the spatial convergence order $\text{order}(h)$ of the numerical solution $u_{i,j}^n$ by the L_1 -ADI scheme (14) and the BD-ADI scheme (19). As expected from the theoretical analysis, these two schemes have the second-order spatial accuracy.

TABLE 2: The maximum norm error and the spatial accuracy of $u_{i,j}^n$ for Example 4.

h	L_1 -ADI scheme ($\alpha = 2/3, \tau = 1/4000$)		BD-ADI scheme ($\alpha = 1/2, \tau = 1/4000$)	
	error(τ, h)	order(h)	error(τ, h)	order(h)
$\pi/4$	$1.4977394377e - 03$	1.994343	$1.8149573747e - 03$	2.001730
$\pi/8$	$3.7590600110e - 04$	1.989441	$4.5319556650e - 04$	2.000950
$\pi/16$	$9.4666844874e - 05$	1.960585	$1.1322433260e - 04$	2.002088
$\pi/32$	$2.4322216030e - 05$	1.844870	$2.8265146732e - 05$	2.007753
$\pi/64$	$6.7708311670e - 06$		$7.0284169450e - 06$	

TABLE 3: The maximum norm error and the temporal accuracy of $u_{i,j}^n$ for Example 5.

τ	L_1 -ADI scheme ($h = 1/200$)			BD-ADI scheme ($h = 1/200$)		
	α	error(τ, h)	order(τ)	α	error(τ, h)	order(τ)
1/20	1/2	$4.6386801902e - 03$	0.835782	1/3	$3.0682418814e - 03$	1.247710
1/40		$2.5989567998e - 03$	0.906370		$1.2920863692e - 03$	1.295329
1/80		$1.3866102907e - 03$	0.947443		$5.2645185538e - 04$	1.315075
1/160		$7.1902789341e - 04$	0.970534		$2.1158337290e - 04$	1.325682
1/320		$3.6693233527e - 04$	0.983962		$8.4413446753e - 05$	1.335982
1/640		$1.8551714096e - 04$			$3.3438043901e - 05$	
1/20	2/3	$1.5621553331e - 03$	1.192172	1/2	$1.5491678500e - 03$	1.423830
1/40		$6.8366714097e - 04$	1.262779		$5.7740819917e - 04$	1.463914
1/80		$2.8491177407e - 04$	1.297640		$2.0931522211e - 04$	1.481554
1/160		$1.1589959988e - 04$	1.317329		$7.4956359178e - 05$	1.496663
1/320		$4.6507867889e - 05$	1.333601		$2.6562452368e - 05$	1.524750
1/640		$1.8453234155e - 05$			$9.2315064167e - 06$	
1/20	3/4	$1.7962917770e - 03$	1.063599	2/3	$1.1129679333e - 03$	1.112230
1/40		$8.5941268352e - 04$	1.128142		$5.1483494465e - 04$	1.171085
1/80		$3.9318529135e - 04$	1.161548		$2.2863166690e - 04$	1.211600
1/160		$1.7576660520e - 04$	1.180856		$9.8720772742e - 05$	1.239224
1/320		$7.7528818903e - 05$	1.191836		$4.1818156321e - 05$	1.255314
1/640		$3.3937881480e - 05$			$1.7517727276e - 05$	

Example 5. We consider the subdiffusion equation

$$\begin{aligned}
 &u_t(x, y, t) \\
 &= {}_0\mathcal{D}_t^{1-\alpha}(\Delta u(x, y, t)) + e^{x+y} \left((1+\alpha)t^\alpha - \frac{2\Gamma(2+\alpha)}{\Gamma(1+2\alpha)}t^{2\alpha} \right), \\
 &\quad (x, y) \in \Omega, \quad t \in [0, 1],
 \end{aligned} \tag{46}$$

with boundary and initial conditions

$$\begin{aligned}
 &u(x, y, t) = e^{x+y}t^{1+\alpha}, \quad (x, y) \in \partial\Omega, \quad t \in (0, 1], \\
 &u(x, y, 0) = 0, \quad (x, y) \in \overline{\Omega},
 \end{aligned} \tag{47}$$

where $\Omega = (0, 1) \times (0, 1)$ and ${}_0\mathcal{D}_t^{1-\alpha}$ denotes the Riemann-Liouville fractional derivative operator defined as

$${}_0\mathcal{D}_t^{1-\alpha}y(t) = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_0^t \frac{y(s)}{(t-s)^{1-\alpha}} ds, \quad 0 < \alpha < 1. \tag{48}$$

Operating Riemann-Liouville fractional derivative operator ${}_0\mathcal{D}_t^{\alpha-1}$ on both sides of (46), we obtain the equivalent problem of the subdiffusion equation with the temporal Caputo fractional derivative [40, 42, 59] as follows:

$$\begin{aligned}
 &{}_0^C\mathcal{D}_t^\alpha u(x, y, t) = \Delta u(x, y, t) + e^{x+y} (\Gamma(2+\alpha)t - 2t^{1+\alpha}), \\
 &\quad (x, y) \in \Omega, \quad t \in [0, 1],
 \end{aligned}$$

$$u(x, y, t) = e^{x+y}t^{1+\alpha}, \quad (x, y) \in \partial\Omega, \quad t \in (0, 1],$$

$$u(x, y, 0) = 0, \quad (x, y) \in \overline{\Omega}. \tag{49}$$

We now solve the aforementioned problem by the L_1 -ADI scheme (14) and the BD-ADI scheme (19). Tables 3 and 4 give the maximum norm error $\text{error}(\tau, h)$ and the convergence orders $\text{order}(\tau)$ and $\text{order}(h)$ of the numerical solution $u_{i,j}^n$. It is easily seen that the numerical results confirm the theoretical analysis results.

TABLE 4: The maximum norm error and the spatial accuracy of $u_{i,j}^n$ for Example 5.

h	L_1 -ADI scheme ($\alpha = 2/3, \tau = 1/4000$)		BD-ADI scheme ($\alpha = 1/2, \tau = 1/4000$)	
	error(τ, h)	order(h)	error(τ, h)	order(h)
1/4	$8.9083790169e - 04$	1.901804	$9.4164046481e - 04$	1.901896
1/8	$2.3839587645e - 04$	1.985610	$2.5197489313e - 04$	1.974325
1/16	$6.0196414976e - 05$	2.072000	$6.4124816806e - 05$	2.032967
1/32	$1.4316490746e - 05$	2.263270	$1.5669032476e - 05$	2.158045
1/64	$2.9821150680e - 06$		$3.5107982244e - 06$	

5. Conclusions

We have studied two ADI finite difference methods for a two-dimensional fractional subdiffusion equation. An explicit error estimate for each of the two methods has been provided in the discrete maximum norm. It has been shown that the methods have the same order as their truncation errors with respect to the discrete maximum norm. The maximum norm error estimates presented here are more preferable for measuring computation errors in practice, compared to the H^1 -norm error estimates in [59]. Numerical results have confirmed the theoretical analysis results.

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References

- [1] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, NY, USA, 1999.
- [2] R. Gorenflo, F. Mainardi, D. Moretti, G. Pagnini, and P. Paradisi, "Discrete random walk models for space-time fractional diffusion," *Chemical Physics*, vol. 284, pp. 521–541, 2002.
- [3] B. I. Henry and S. L. Wearne, "Fractional reaction-diffusion," *Physica A*, vol. 276, no. 3–4, pp. 448–455, 2000.
- [4] M. Giona and H. E. Roman, "Fractional diffusion equation for transport phenomena in random media," *Physica A*, vol. 185, pp. 87–97, 1992.
- [5] T. Kosztolowicz, "Subdiffusion in a system with a thick membrane," *Journal of Membrane Science*, vol. 320, pp. 492–499, 2008.
- [6] R. Metzler, E. Barkai, and J. Klafter, "Anomalous diffusion and relaxation close to thermal equilibrium: a fractional Fokker-Planck equation approach," *Physical Review Letters*, vol. 82, pp. 3563–3567, 1999.
- [7] R. Metzler and J. Klafter, "The random walk's guide to anomalous diffusion: a fractional dynamics approach," *Physics Reports*, vol. 339, no. 1, p. 77, 2000.
- [8] R. Metzler and J. Klafter, "Boundary value problems for fractional diffusion equations," *Physica A*, vol. 278, no. 1–2, pp. 107–125, 2000.
- [9] M. Li, "Approximating ideal filters by systems of fractional order," *Computational and Mathematical Methods in Medicine*, vol. 2012, Article ID 365054, 6 pages, 2012.
- [10] M. Li, S. C. Lim, and S. Chen, "Exact solution of impulse response to a class of fractional oscillators and its stability," *Mathematical Problems in Engineering*, vol. 2011, Article ID 657839, 9 pages, 2011.
- [11] M. Li and W. Zhao, "Essay on fractional Riemann-Liouville integral operator versus Mikusinski's," *Mathematical Problems in Engineering*, vol. 2013, Article ID 635412, 3 pages, 2013.
- [12] M. Li and W. Zhao, "Solving the Abel's type integral equation with the Mikusinski's operator of fractional order," *Advances in Mathematical Physics*, vol. 2013, Article ID 806984, 4 pages, 2013.
- [13] J. A. Tenreiro MacHado, M. F. Silva, R. S. Barbosa et al., "Some applications of fractional calculus in engineering," *Mathematical Problems in Engineering*, vol. 2010, Article ID 639801, 34 pages, 2010.
- [14] H. Sun, Y. Chen, and W. Chen, "Random-order fractional differential equation models," *Signal Processing*, vol. 91, no. 3, pp. 525–530, 2011.
- [15] C. Cattani, "Fractional calculus and Shannon wavelet," *Mathematical Problems in Engineering*, vol. 2012, Article ID 502812, 26 pages, 2012.
- [16] C. Cattani, G. Pierro, and G. Altieri, "Entropy and multifractality for the myeloma multiple TET 2 gene," *Mathematical Problems in Engineering*, vol. 2012, Article ID 193761, 14 pages, 2012.
- [17] B. Baeumer, M. Kovács, and M. M. Meerschaert, "Numerical solutions for fractional reaction-diffusion equations," *Computers & Mathematics with Applications*, vol. 55, no. 10, pp. 2212–2226, 2008.
- [18] C.-M. Chen and F. Liu, "A numerical approximation method for solving a three-dimensional space Galilei invariant fractional advection-diffusion equation," *Journal of Applied Mathematics and Computing*, vol. 30, no. 1–2, pp. 219–236, 2009.
- [19] C.-M. Chen, F. Liu, I. Turner, and V. Anh, "A Fourier method for the fractional diffusion equation describing sub-diffusion," *Journal of Computational Physics*, vol. 227, no. 2, pp. 886–897, 2007.
- [20] C.-M. Chen, F. Liu, V. Anh, and I. Turner, "Numerical schemes with high spatial accuracy for a variable-order anomalous subdiffusion equation," *SIAM Journal on Scientific Computing*, vol. 32, no. 4, pp. 1740–1760, 2010.
- [21] C.-m. Chen, F. Liu, and K. Burrage, "Finite difference methods and a Fourier analysis for the fractional reaction-subdiffusion equation," *Applied Mathematics and Computation*, vol. 198, no. 2, pp. 754–769, 2008.
- [22] A. Mohebbi, M. Abbaszadeh, and M. Dehghan, "A high-order and unconditionally stable scheme for the modified anomalous fractional sub-diffusion equation with a nonlinear source term," *Journal of Computational Physics*, vol. 240, pp. 36–48, 2013.

- [23] G.-H. Gao, Z.-Z. Sun, and Y.-N. Zhang, "A finite difference scheme for fractional sub-diffusion equations on an unbounded domain using artificial boundary conditions," *Journal of Computational Physics*, vol. 231, no. 7, pp. 2865–2879, 2012.
- [24] T. A. M. Langlands and B. I. Henry, "The accuracy and stability of an implicit solution method for the fractional diffusion equation," *Journal of Computational Physics*, vol. 205, no. 2, pp. 719–736, 2005.
- [25] Y. Lin and C. Xu, "Finite difference/spectral approximations for the time-fractional diffusion equation," *Journal of Computational Physics*, vol. 225, no. 2, pp. 1533–1552, 2007.
- [26] F. Liu, C. Yang, and K. Burrage, "Numerical method and analytic technique of the modified anomalous subdiffusion equation with a nonlinear source term," *Journal of Computational and Applied Mathematics*, vol. 231, no. 1, pp. 160–176, 2009.
- [27] F. Liu, P. Zhuang, V. Anh, I. Turner, and K. Burrage, "Stability and convergence of the difference methods for the space-time fractional advection-diffusion equation," *Applied Mathematics and Computation*, vol. 191, no. 1, pp. 12–20, 2007.
- [28] C.-M. Chen, F. Liu, V. Anh, and I. Turner, "Numerical methods for solving a two-dimensional variable-order anomalous subdiffusion equation," *Mathematics of Computation*, vol. 81, no. 277, pp. 345–366, 2012.
- [29] Q. Liu, F. Liu, I. Turner, and V. Anh, "Numerical simulation for the 3D seepage flow with fractional derivatives in porous media," *IMA Journal of Applied Mathematics*, vol. 74, no. 2, pp. 201–229, 2009.
- [30] C. Tadjeran, M. M. Meerschaert, and H.-P. Scheffler, "A second-order accurate numerical approximation for the fractional diffusion equation," *Journal of Computational Physics*, vol. 213, no. 1, pp. 205–213, 2006.
- [31] S. B. Yuste, "Weighted average finite difference methods for fractional diffusion equations," *Journal of Computational Physics*, vol. 216, no. 1, pp. 264–274, 2006.
- [32] S. B. Yuste and L. Acedo, "An explicit finite difference method and a new von Neumann-type stability analysis for fractional diffusion equations," *SIAM Journal on Numerical Analysis*, vol. 42, no. 5, pp. 1862–1874, 2005.
- [33] P. Zhuang, F. Liu, V. Anh, and I. Turner, "New solution and analytical techniques of the implicit numerical method for the anomalous subdiffusion equation," *SIAM Journal on Numerical Analysis*, vol. 46, no. 2, pp. 1079–1095, 2008.
- [34] P. Zhuang, F. Liu, V. Anh, and I. Turner, "Numerical methods for the variable-order fractional advection-diffusion equation with a nonlinear source term," *SIAM Journal on Numerical Analysis*, vol. 47, no. 3, pp. 1760–1781, 2009.
- [35] J.-P. Bouchaud and A. Georges, "Anomalous diffusion in disordered media: statistical mechanisms, models and physical applications," *Physics Reports*, vol. 195, no. 4-5, pp. 127–293, 1990.
- [36] P. Zhuang, F. Liu, V. Anh, and I. Turner, "Stability and convergence of an implicit numerical method for the non-linear fractional reaction-subdiffusion process," *IMA Journal of Applied Mathematics*, vol. 74, no. 5, pp. 645–667, 2009.
- [37] Y. T. Gu, P. Zhuang, and F. Liu, "An advanced implicit meshless approach for the non-linear anomalous subdiffusion equation," *Computer Modeling in Engineering & Sciences*, vol. 56, no. 3, pp. 303–333, 2010.
- [38] M. Cui, "Compact finite difference method for the fractional diffusion equation," *Journal of Computational Physics*, vol. 228, no. 20, pp. 7792–7804, 2009.
- [39] Y.-N. Zhang, Z.-Z. Sun, and H.-W. Wu, "Error estimates of Crank-Nicolson-type difference schemes for the subdiffusion equation," *SIAM Journal on Numerical Analysis*, vol. 49, no. 6, pp. 2302–2322, 2011.
- [40] G.-h. Gao and Z.-z. Sun, "A compact finite difference scheme for the fractional sub-diffusion equations," *Journal of Computational Physics*, vol. 230, no. 3, pp. 586–595, 2011.
- [41] X. Li and C. Xu, "A space-time spectral method for the time fractional diffusion equation," *SIAM Journal on Numerical Analysis*, vol. 47, no. 3, pp. 2108–2131, 2009.
- [42] X. Zhao and Z.-z. Sun, "A box-type scheme for fractional sub-diffusion equation with Neumann boundary conditions," *Journal of Computational Physics*, vol. 230, no. 15, pp. 6061–6074, 2011.
- [43] C.-M. Chen, F. Liu, I. Turner, and V. Anh, "Numerical schemes and multivariate extrapolation of a two-dimensional anomalous sub-diffusion equation," *Numerical Algorithms*, vol. 54, no. 1, pp. 1–21, 2010.
- [44] P. Zhuang and F. Liu, "Finite difference approximation for two-dimensional time fractional diffusion equation," *Journal of Algorithms & Computational Technology*, vol. 1, no. 1, pp. 1–15, 2007.
- [45] Q. Liu, Y. T. Gu, P. Zhuang, F. Liu, and Y. F. Nie, "An implicit RBF meshless approach for time fractional diffusion equations," *Computational Mechanics*, vol. 48, no. 1, pp. 1–12, 2011.
- [46] H. Brunner, L. Ling, and M. Yamamoto, "Numerical simulations of 2D fractional subdiffusion problems," *Journal of Computational Physics*, vol. 229, no. 18, pp. 6613–6622, 2010.
- [47] Q. Yang, T. Moroney, K. Burrage, I. Turner, and F. Liu, "Novel numerical methods for time-space fractional reaction diffusion equations in two dimensions," *ANZIAM Journal. Electronic Supplement*, vol. 52, pp. C395–C409, 2010.
- [48] Q. Yang, I. Turner, F. Liu, and M. Ilić, "Novel numerical methods for solving the time-space fractional diffusion equation in two dimensions," *SIAM Journal on Scientific Computing*, vol. 33, no. 3, pp. 1159–1180, 2011.
- [49] Q. X. Liu and F. W. Liu, "Modified alternating direction methods for solving a two-dimensional non-continuous seepage flow with fractional derivatives," *Mathematica Numerica Sinica*, vol. 31, no. 2, pp. 179–194, 2009.
- [50] S. Chen and F. Liu, "ADI-Euler and extrapolation methods for the two-dimensional fractional advection-dispersion equation," *Journal of Applied Mathematics and Computing*, vol. 26, no. 1-2, pp. 295–311, 2008.
- [51] M. M. Meerschaert, H.-P. Scheffler, and C. Tadjeran, "Finite difference methods for two-dimensional fractional dispersion equation," *Journal of Computational Physics*, vol. 211, no. 1, pp. 249–261, 2006.
- [52] H. Wang and K. Wang, "An $O(N \log^2 N)$ alternating-direction finite difference method for two-dimensional fractional diffusion equations," *Journal of Computational Physics*, vol. 230, no. 21, pp. 7830–7839, 2011.
- [53] C. Tadjeran and M. M. Meerschaert, "A second-order accurate numerical method for the two-dimensional fractional diffusion equation," *Journal of Computational Physics*, vol. 220, no. 2, pp. 813–823, 2007.
- [54] Y.-N. Zhang, Z.-Z. Sun, and X. Zhao, "Compact alternating direction implicit scheme for the two-dimensional fractional diffusion-wave equation," *SIAM Journal on Numerical Analysis*, vol. 50, no. 3, pp. 1535–1555, 2012.

- [55] M. Cui, "Compact alternating direction implicit method for two-dimensional time fractional diffusion equation," *Journal of Computational Physics*, vol. 231, no. 6, pp. 2621–2633, 2012.
- [56] S. Chen, F. Liu, P. Zhuang, and V. Anh, "Finite difference approximations for the fractional Fokker-Planck equation," *Applied Mathematical Modelling*, vol. 33, no. 1, pp. 256–273, 2009.
- [57] W. Deng, "Finite element method for the space and time fractional Fokker-Planck equation," *SIAM Journal on Numerical Analysis*, vol. 47, no. 1, pp. 204–226, 2008/09.
- [58] Z.-Z. Sun and X. Wu, "A fully discrete difference scheme for a diffusion-wave system," *Applied Numerical Mathematics*, vol. 56, no. 2, pp. 193–209, 2006.
- [59] Y.-N. Zhang and Z.-Z. Sun, "Alternating direction implicit schemes for the two-dimensional fractional sub-diffusion equation," *Journal of Computational Physics*, vol. 230, no. 24, pp. 8713–8728, 2011.
- [60] A. A. Samarskii, *The Theory of Difference Schemes*, Marcel Dekker, New York, NY, USA, 2001.