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Research Article

Power Spectrum of Generalized Fractional Gaussian Noise

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Recently, we introduced a type of autocorrelation function (ACF) to describe a long-range dependent (LRD) process indexed with two parameters, which takes standard fractional Gaussian noise (fGn for short) as a special case. For simplicity, we call it the generalized fGn (GfGn). This short paper gives the power spectrum density function (PSD) of GfGn.

1. Introduction

LRD time series increasingly gains applications to many fields of science and technologies; see, for example, Mandelbrot [1] and references therein. In this regard, standard fGn introduced by Mandelbrot and van Ness is a widely used tool for modeling LRD time series; see, for example, Beran [2], Abuzeid et al. [3, 4], and Liao et al. [5]. Following [1, H11], [2], its ACF is given by

$$\begin{aligned}\rho(\tau) &= r(\tau; H)\\ &= \frac{\sigma^2}{2}\left[(|\tau|+1)^{2H} - 2|\tau|^{2H} + ||\tau|-1|^{2H}\right],\end{aligned} \tag{1}$$

where H is the Hurst parameter and $\sigma^2 = \Gamma(1-2H)\cos(H\pi)/H\pi$. It implies three families of time series. In the case of $H \in (0.5, 1)$, ρ is nonintegrable, and a corresponding series is LRD. For $H \in (0, 0.5)$, r is integrable, and a corresponding series is short-range dependent (SRD). The case of $H = 0.5$ corresponds to white noise. Note that statistics of LRD series substantially differ from SRD ones. From a practice view, SRD fGn may be less interesting in applications as can be seen from [1, 2]. This paper only considers LRD series unless otherwise stated.

Li [6] recently introduced an ACF form that is a generalization of ACF of fGn. Since ACF is an even function, we write ACF of GfGn by

$$\begin{aligned}C(\tau) &= C(\tau; H, \alpha)\\ &= 0.5\sigma^2\left((|\tau|^\alpha+1)^{2H} - 2(|\tau|^\alpha)^{2H} + \left||\tau|^\alpha - 1\right|^{2H}\right),\end{aligned} \tag{2}$$

where $H \in (0.5, 1)$ and $\alpha \in (0, 1]$. We call a process whose ACF follows (2) GfGn for simplicity because it takes fGn as a special case of $C(\tau; H, 1) = \rho(\tau; H)$. Without loss of generality, the following considers the normalized ACF by letting $r(\tau) = C(\tau)/\sigma^2$. This paper aims at giving PSD of GfGn. The Fourier transform (FT) of $r(\tau)$ is treated as a generalized function over Schwartz space of test functions since $r(\tau)$ is nonintegrable.

2. PSD of GfGn

Denote

$$r(\tau) = 0.5\left[r_1(\tau) - 2r_2(\tau) + r_3(\tau)\right], \tag{3}$$

where $r_1 = (|\tau|^a+1)^{2H}$, and $r_2 = (|\tau|^\alpha)^{2H}$, $r_3 = ||\tau|^a - 1|^{2H}$. Denote $S_m(\omega) = F(r_m)$, where F means FT and $m = 1, 2, 3$. Then, FT of $r(\tau)$ is given by

$$S(\omega) = 0.5\left[S_1(\omega) - 2S_2(\omega) + S_3(\omega)\right]. \tag{4}$$

Lemma 1 (see [7] or Gelfand and Vilenkin [8, Chapter 2]). *FT of $|t|^\lambda$ is expressed by*

$$F\left[|t|^\lambda\right] = -\sin\left(\frac{\lambda\pi}{2}\right)\Gamma(\lambda+1)|\omega|^{-\lambda-1}, \tag{5}$$

where $\lambda \neq -1, -3, \ldots$.

Corollary 2. *$S_2(\omega)$ equals* $-\sin(H\alpha\pi)\Gamma(2H\alpha+1)|\omega|^{-2H\alpha-1}$.

Proof. Note $2H\alpha \neq -1, -3, \ldots$. Thus, doing $F(|\tau|^{2H\alpha})$ with (5) yields Corollary 2. □

Lemma 3 (binomial series). *$(1+x)^\nu$ and $(1-x)^\nu$ can be expanded as*

$$(1+x)^\nu = \sum_{k=0}^{\infty}\binom{\nu}{k}x^k = \sum_{k=0}^{\infty}\frac{\Gamma(\nu+k)}{\Gamma(\nu)\Gamma(1+k)}x^k \quad \text{for } |x|<1, \tag{6a}$$

$$(1-x)^\nu = \sum_{k=0}^{\infty}\binom{\nu}{k}(-x)^k = \sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(\nu+k)}{\Gamma(\nu)\Gamma(1+k)}x^k \quad \text{for } |x|<1, \tag{6b}$$

where x and ν are real number, and $\binom{\nu}{k}$ is binomial coefficient [9].

Corollary 4. *$r_1(\tau)$ and $r_3(\tau)$ for $|\tau|<1$ can be expanded as*

$$(1+|\tau|^\alpha)^{2H} = \sum_{k=0}^{\infty}\binom{2H}{k}|\tau|^{\alpha k} = \sum_{k=0}^{\infty}\frac{\Gamma(2H+k)}{\Gamma(2H)\Gamma(1+k)}|\tau|^{\alpha k}, \tag{7a}$$

$$(1-|\tau|^\alpha)^{2H} = \sum_{k=0}^{\infty}(-1)^k\binom{2H}{k}|\tau|^{\alpha k} = \sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(2H+k)}{\Gamma(2H)\Gamma(1+k)}|\tau|^{\alpha k}. \tag{7b}$$

Proof. This corollary is straightforward from Lemma 3. □

Corollary 5. *For $|\tau|^\alpha>1$, $r_1(\tau)$ and $r_3(\tau)$ can be expanded as*

$$(1+|\tau|^\alpha)^{2H} = |\tau|^{2H\alpha}\sum_{k=0}^{\infty}\binom{2H}{k}|\tau|^{-\alpha k} = \sum_{k=0}^{\infty}\frac{\Gamma(2H+k)}{\Gamma(2H)\Gamma(1+k)}|\tau|^{\alpha(2H-k)}, \tag{8a}$$

$$\left||\tau|^a-1\right|^{2H} = |\tau|^{2H\alpha}\sum_{k=0}^{\infty}(-1)^k\binom{2H}{k}|\tau|^{-\alpha k} = \sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(2H+k)}{\Gamma(2H)\Gamma(1+k)}|\tau|^{\alpha(2H-k)}. \tag{8b}$$

Proof. Since $(1+|\tau|^\alpha)^{2H} = |\tau|^{2H\alpha}(1+|\tau|^{-\alpha})^{2H}$, according to (7a), (8a) results. Similarly, (8b) follows due to $||\tau|^a-1|^{2H} = |\tau|^{2H\alpha}|1-|\tau|^{-\alpha}|^{2H}$ and (7b). □

Corollary 6. *For $|\tau|<1$, S_1 and S_3 are given by* (6), *respectively,*

$$\begin{aligned} S_1(\omega) &= \sum_{k=0}^{\infty}\frac{-\Gamma(2H+k)\Gamma(\alpha k+1)}{\Gamma(2H)\Gamma(1+k)} \times \sin\left(\frac{\alpha k\pi}{2}\right)|\omega|^{-\alpha k-1}, \\ S_3(\omega) &= \sum_{k=0}^{\infty}\frac{(-1)^{k+1}\Gamma(2H+k)\Gamma(\alpha k+1)}{\Gamma(2H)\Gamma(1+k)} \times \sin\left(\frac{\alpha k\pi}{2}\right)|\omega|^{-\alpha k-1}. \end{aligned} \tag{6}$$

Proof. Doing $F[|\tau|^{\alpha k}]$ term by term for (7a) and (7b) with Lemma 1 yields (6), respectively. □

Corollary 7. *For $|\tau|^\alpha>1$, S_1 and S_3 are given by* (7), *respectively,*

$$\begin{aligned} S_1(\omega) &= \sum_{k=0}^{\infty}\frac{-\Gamma(2H+k)\Gamma[(\alpha(2H-k)+1]}{\Gamma(2H)\Gamma(1+k)} \times \sin\left[\frac{\alpha(2H-k)\pi}{2}\right]|\omega|^{-\alpha(2H-k)-1}, \\ S_3(\omega) &= \sum_{k=0}^{\infty}\frac{(-1)^{k+1}\Gamma(2H+k)\Gamma[\alpha(2H-k)+1]}{\Gamma(2H)\Gamma(1+k)} \times \sin\left[\frac{\alpha(2H-k)\pi}{2}\right]|\omega|^{-\alpha(2H-k)-1}. \end{aligned} \tag{7}$$

Proof. Doing FTs for (8a) and (8b) based on Lemma 1 results in (7). □

The following proposition is a consequence of Corollaries 2, 6, and 7.

Proposition 8. *PSD of GfGn is given by*

$$S(\omega) = \begin{cases} \sin(H\alpha\pi)\Gamma(2H\alpha+1)|\omega|^{-2H\alpha-1} \\ +0.5\sum_{k=0}^{\infty}\frac{\left[(-1)^{k+1}-1\right]\Gamma(2H+k)\Gamma(\alpha k+1)}{\Gamma(2H)\Gamma(1+k)} \times \sin\left(\frac{\alpha k\pi}{2}\right)|\omega|^{-\alpha k-1}, \quad |\tau|<1, \\ \sin(H\alpha\pi)\Gamma(2H\alpha+1)|\omega|^{-2H\alpha-1} \\ +0.5\sum_{k=0}^{\infty}\frac{\left[(-1)^{k}-1\right]\Gamma(2H+k)\Gamma[\alpha(2H-k)+1]}{\Gamma(2H)\Gamma(1+k)} \times \sin\left[\frac{\alpha(2H-k)\pi}{2}\right]|\omega|^{-\alpha(2H-k)-1}, \\ |\tau|^\alpha>1. \end{cases} \tag{8}$$

Considering the leading term of (8) results in the following proposition.

Proposition 9. *PSD of GfGn has the following approximate value:*

$$S(\omega) \approx \sin(H\alpha\pi)\,\Gamma(2H\alpha+1)\,|\omega|^{-2H\alpha-1}. \tag{9}$$

From (9), we can easily get the two notes below.

Note 1. $S(\omega)$ is divergent at the origin for $0 < 2H\alpha + 1 < 1$, which is the LRD condition. This is the basic feature of LRD process.

Note 2. Recall $2H\alpha + 1 > 0$. Then, the cases of $2H\alpha + 1 < 1$ and $H \in (0.5, 1)$ imply $\alpha \in (0, 1]$. This explains the range of α for GfGn from a view in the frequency domain.

3. Conclusions

We have derived PSD of GfGn. Its approximate expression has been given. The range of α has been explained from a spectral view.

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