Low-Dosed X-Ray Computed Tomography Imaging by Regularized Fully Spatial Fractional-Order Perona-Malik Diffusion

Zhiwu Liao

School of Computer Science, Sichuan Normal University, Chengdu, Sichuan 61000, China

Correspondence should be addressed to Zhiwu Liao; liaozhiwu@163.com

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Existing fractional-order Perona-Malik Diffusion (FOPMD) algorithms used in noise suppressing suffer from undesired artifacts and speckle effect, which hamper FOPMD used in low-dosed X-ray computed tomography (LDCT) imaging. In this paper, we propose a new FOPMD method for low-dose computed tomography (LDCT) imaging, which is called regularized fully spatial FOPMD (RFS-FOPMD), whose numerical scheme is also given based on Grünwald-Letnikov derivative (G-L derivative). Here, fully spatial FOPMD represents all the integer-order derivatives (IODs) in the right hand of Perona-Malik Diffusion (PMD) which are replaced by fractional-order derivatives (FODs). Since the new scheme has advantages of both regularization and FOPMD, it has good abilities in singularities preserving while suppressing noise. Some real sinogram of LDCT are used to compare the different performances not only for some classical but also for some state-of-art diffusion schemes. These schemes include PMD, regularized PMD (RPMD), and FOPMD in (Hu et al. 2012). Experimental results show that besides good ability in edge preserving, the new scheme also has good stability for iteration number and can avoid artifacts and speckle effect with suitable parameters.

1. Introduction

Perona-Malik diffusion (PMD) proposed in 1990 is a popular technique in image denoising and it is defined as [1]

\[
\frac{\partial u(x, y, t)}{\partial t} = \text{div} [c \left( \| \nabla u(x, y, t) \| \right) \nabla u(x, y, t)] ,
\]

(1)

where \(u(x, y, 0)\) is the initial gray scale image, \(u(x, y, t)\) is the smoothed gray scale image at time \(t\), \(\nabla\) denotes the gradient, \(\text{div} (\cdot)\) is the divergence operator, and \(c(\cdot)\) is the diffusion coefficient.

In 1992, Catté et al. indicated that PMD is ill-posed and they propose a new well-posed method named regularized Perona-Malik diffusion (RPMD), by replacing the gradient \(\nabla U\) in diffusion coefficients by the smoothed version \(G_{\alpha} \cdot \nabla U\) [2]. Thus, the RPMD can be represented as

\[
\frac{\partial u(x, y, t)}{\partial t} = \text{div} \left[ c \left( \| G_{\alpha} \cdot \nabla u(x, y, t) \| \right) \nabla u(x, y, t) \right] .
\]

(2)

Here \(G_{\alpha}\) is defined as:

\[
G_{\alpha} = \frac{1}{C} e^{-((x^2+y^2)/\sigma_1^2)}
\]

(3)

which is a Gaussian function and \(C\) is a constant.

In order to eliminate undesired “staircase” of PMD and RPMD, high-order PDEs (typically fourth-order PDEs) for image restoration have been introduced in [3, 4]. Though these methods can eliminate the staircase effect efficiently, they often leave the image with isolated black and white speckles (so-called speckle effect) [5].

Recently, fractional-order PMD (FOPMD) has been studied in image denoising [5–14], whose fractional order is \(\alpha\), \(0 \leq \alpha \leq 2\), which is a “natural interpolation” between PMD and fourth-order PDEs. Therefore, it has the benefits of both of PMD and high order PDEs.

Bai and Feng proposed a FSFOD method for image denoising with Euler-Lagrange equations of a cost functional and using Fourier-domain to compute the fractional
2. Regularized Fully Spatial Fractional-Order Perona-Malik Diffusion

In this paper, we use G-L definition defined as follows [29, 30]:

$$D^\alpha g(x) = \lim_{h \to 0^+} \sum_{k=0}^{\infty} (-1)^k C_k^\alpha g(x-kh)/h^\alpha, \quad \alpha > 0,$$

where $g(x)$ is a real function, $\alpha > 0$ is a real number, $C_k^\alpha = \Gamma(\alpha + 1)/[\Gamma(k+1)\Gamma(\alpha - k + 1)]$ is the generalized binomial coefficient, and $\Gamma(\cdot)$ denotes the gamma function.

Isotropic diffusion will damage the image features such as edges, lines, and textures. To avoid the damage, the smoothing has to be adaptively controlled by the amount of smoothing or the direction of smoothing. A classic example of adaptive smoothing is the anisotropic diffusion scheme proposed by Perona and Malik [1], in which the smoothing process is formulated by a partial differential equation (PDE). PMD is formulated in (1).

However, PMD methods suffer from their “staircase” effects. Therefore, FOPMD is proposed to suppress the staircase of PMD.

The fractional-order gradient vector with $\alpha$ order is defined as

$$\nabla^\alpha u(x, y, t) = \left[ \nabla^\alpha u(x, y, t), \nabla_{\alpha}^\alpha u(x, y, t) \right],$$

where $\alpha$ is a positive real, $\nabla^\alpha u(x, y, t)$ represents the partial fractional-order derivative of $u(x, y, t)$ with respect to the variable $x$ whose order is $\alpha$, and $\nabla_{\alpha}^\alpha u(x, y, t)$ represents the partial fractional-order derivative of $u(x, y, t)$ with respect to the variable $y$ whose order is $\alpha$.

According to [8], FOPMD is defined as

$$\frac{\partial u(x, y, t)}{\partial t} = \text{div}^\alpha \left[ c \left( \nabla^\alpha u(x, y, t) \right) \right] \nabla^\alpha u(x, y, t),$$

where $\text{div}^\alpha$ denotes the $\alpha$-order divergence. For the vector $\nabla^\beta V = [v^\beta_x, v^\beta_y]$ where $v^\beta_x, v^\beta_y$ represent the partial fractional-order derivative of $u(x, y, t)$ with whose order is $\beta$ with respect to the variable $x$ and $y$ respectively, its $\alpha$-order divergence is defined as:

$$\text{div}^\alpha \nabla^\beta V = v^\beta_x + v^\beta_y.$$

However, FOPMD defined by (3) will produce some artifacts for sinogram restoration of LDCT, which increases the probability of error diagnosis. In order to avoid artifacts produced in sinogram restoration of LDCT using FOPMD, we propose a new diffusion model, named regularized fully spatial fractional-order PMD (RFS-FOPMD), where “fully spatial” indicates all derivatives of the right-hand side of (6). That is, the FOD in diffusion coefficient is replaced by its smoothed version.

Therefore, the RFS-FOPMD is given by

$$\frac{\partial u(x, y, t)}{\partial t} = \text{div}^\alpha \left[ c \left( \nabla_{\alpha}^\beta V \right) \right] \nabla^\alpha u(x, y, t)$$
with the observed image as the initial condition and $G_{\sigma_i}$ is defined in (3).

When $\alpha = 1$, (6) is precisely the PMD and (8) is precisely the RPMD; when $\alpha = 2$, (6) is precisely the fourth-order anisotropic diffusion equation. In this paper, we are interested in $0.5 \leq \alpha \leq 1.5$ since Bai and Feng in [5] suggest that $\alpha = 1.5$ in their model has the best performance.

### 3. The Numerical Scheme

An image $U$ will be a 2-dimensional matrix of size $N \times N$. In order to get the aim of anisotropic diffusion along different directions and because the discrete $\alpha$-order gradient $\nabla^\alpha u$ is an 8-dimensional vector

\[
\nabla^\alpha u(i,j) = (\nabla_0^\alpha u(i,j), \nabla_1^\alpha u(i,j), \nabla_2^\alpha u(i,j), \nabla_3^\alpha u(i,j), \nabla_4^\alpha u(i,j), \nabla_5^\alpha u(i,j), \nabla_6^\alpha u(i,j), \nabla_7^\alpha u(i,j))^T,
\]

where $T$ represents the transpose of the vector and $\nabla^\alpha u_k(i,j)$, $k = 0, \ldots, 7$ are defined as

\[
\nabla_0^\alpha u(i,j) = \sum_{k=0}^{K-1} (-1)^k C^\alpha_k u(i, j + k - (K - 1)),
\]

\[
\nabla_1^\alpha u(i,j) = \sum_{k=0}^{K-1} (-1)^k C^\alpha_k u(i, j + k) + (K - 1),
\]

\[
\nabla_2^\alpha u(i,j) = \sum_{k=0}^{K-1} (-1)^k C^\alpha_k u(i, j + k - (K - 1)),
\]

\[
\nabla_3^\alpha u(i,j) = \sum_{k=0}^{K-1} (-1)^k C^\alpha_k u(i, j - k + (K - 1)),
\]

\[
\nabla_4^\alpha u(i,j) = \sum_{k=0}^{K-1} (-1)^k C^\alpha_k u(i, j + k - (K - 1)),
\]

\[
\nabla_5^\alpha u(i,j) = \sum_{k=0}^{K-1} (-1)^k C^\alpha_k u(i, j + k - (K - 1)),
\]

\[
\nabla_6^\alpha u(i,j) = \sum_{k=0}^{K-1} (-1)^k C^\alpha_k u(i, j - k + (K - 1)),
\]

\[
\nabla_7^\alpha u(i,j) = \sum_{k=0}^{K-1} (-1)^k C^\alpha_k u(i, j - k + (K - 1)).
\]

Thus,

\[
\nabla^{2\alpha} u(i,j) = (\nabla_0^{2\alpha} u(i,j), \nabla_1^{2\alpha} u(i,j), \nabla_2^{2\alpha} u(i,j), \nabla_3^{2\alpha} u(i,j), \nabla_4^{2\alpha} u(i,j), \nabla_5^{2\alpha} u(i,j), \nabla_6^{2\alpha} u(i,j), \nabla_7^{2\alpha} u(i,j))^T,
\]

where $T$ represents the transpose of the vector. From (4), we have

\[
\nabla^{2\alpha} u_0(i,j) = \sum_{k=0}^{K-1} (-1)^k C^{2\alpha}_k u(i, j + k - (K - 1)),
\]

\[
\nabla^{2\alpha} u_1(i,j) = \sum_{k=0}^{K-1} (-1)^k C^{2\alpha}_k u(i, j + k + (K - 1), j + k - (K - 1)),
\]

\[
\nabla^{2\alpha} u_2(i,j) = \sum_{k=0}^{K-1} (-1)^k C^{2\alpha}_k u(i, j + k + (K - 1), j - k + (K - 1)),
\]

\[
\nabla^{2\alpha} u_3(i,j) = \sum_{k=0}^{K-1} (-1)^k C^{2\alpha}_k u(i, j + k + (K - 1), j - k + (K - 1)),
\]

\[
\nabla^{2\alpha} u_4(i,j) = \sum_{k=0}^{K-1} (-1)^k C^{2\alpha}_k u(i, j - k + (K - 1)),
\]

\[
\nabla^{2\alpha} u_5(i,j) = \sum_{k=0}^{K-1} (-1)^k C^{2\alpha}_k u(i, j - k + (K - 1)),
\]

\[
\nabla^{2\alpha} u_6(i,j) = \sum_{k=0}^{K-1} (-1)^k C^{2\alpha}_k u(i, j - k + (K - 1)),
\]

\[
\nabla^{2\alpha} u_7(i,j) = \sum_{k=0}^{K-1} (-1)^k C^{2\alpha}_k u(i, j - k + (K - 1)),
\]

Let

\[
\mathbf{g} = (g_0, g_1, g_2, g_3, g_4, g_5, g_6, g_7)^T,
\]

where $T$ represents the transpose of the vector and $g_k, k = 0, \ldots, 7$ are defined as

\[
g_k = \frac{g(\|G_{\sigma_i} \cdot \nabla^\alpha u_k(i,j)\|)}{\sum_{n=0}^{7} g(\|G_{\sigma_i} \cdot \nabla^\alpha u_k(i,j)\|)}, \quad k = 0, 1, \ldots, 7,
\]

where $\nabla^\alpha u_k(i, j), k = 0, \ldots, 7$ defined in (9) are the components of vector $\nabla^\alpha u(i,j)$ and $\sum_{n=0}^{7} g(\|G_{\sigma_i} \cdot \nabla^\alpha u_k(i, j)\|)$ is the normalized constant, $g$ is the decreasing function of absolute value of $\nabla^\alpha u_k(i,j), k = 0, \ldots, 7$, and $g(\|\nabla^\alpha u(x, y, t)\|)$ is defined as

\[
g(\|\nabla^\alpha u(x, y, t)\|) = e^{-\|\nabla^\alpha u(x, y, t)\| / \sigma^2}, \quad k = 0, \ldots, 7
\]

or

\[
g(\|\nabla^\alpha u(x, y, t)\|) = \frac{1}{1 + (\|\nabla^\alpha u(x, y, t)\| / \sigma)^2}, \quad k = 0, \ldots, 7
\]

where $\| \cdot \|$ is the module of the fractional-order vector and the constant $\sigma$ controls the sensitivity to edges.
The new FOPMD based on G-L fractional-order derivative is defined as
\[
\frac{\partial u(i,j,t)}{\partial t} = \text{div}^{\alpha} \begin{pmatrix}
g_0 V_0^\alpha u(i,j,t) \\
g_1 V_1^\alpha u(i,j,t) \\
g_2 V_2^\alpha u(i,j,t) \\
g_3 V_3^\alpha u(i,j,t) \\
g_4 V_4^\alpha u(i,j,t) \\
g_5 V_5^\alpha u(i,j,t) \\
g_6 V_6^\alpha u(i,j,t) \\
g_7 V_7^\alpha u(i,j,t)
\end{pmatrix},
\]
where \(g_k\), \(k = 0, \ldots, 7\) defined in (14) are the components of \(\mathbf{g}\) in (13).

The above equation can be represented as
\[
\frac{\partial u(i,j,t)}{\partial t} = \sum_{k=0}^{7} g_k V_k^{2\alpha} u(i,j,t),
\]
where \(\sum_{k=0}^{7} g_k = 1\) and \(V_k^{2\alpha} u(i,j,t)\) can be computed according to (8).

Thus, the explicit form for solving (18) is
\[
u(i,j,t+1) = u(i,j,t) + \lambda \sum_{k=0}^{7} g_k V_k^{2\alpha} u(i,j,t),
\]
where \(u(i,j,t+1)\) is the gray level of \((i,j)\) at time \(t+1\) and \(\lambda\) is the integration constant \((0 \leq \lambda \leq 1/7)\).

To summarize, our sinogram restoration approach is done in the following steps:

1. let the input sinogram be \(U\) and set \(t = 1, U_1 = U\), input iterative numbers \(n\), Gaussian deviations \(\sigma_1\) of regularized Gaussian kernel in (3), fractional order \(\alpha\), integration constant \(\lambda\) in (19), gradient modulus threshold \(\sigma\) that controls the conduction in (15) or (16), and choose (15) or (16) as diffusion coefficients;
2. compute \(\alpha\)-order gradient vector \(V^\alpha \mathbf{u}\) using (9);
3. compute diffusion coefficients vector \(\mathbf{g}\) using (13)–(16);
4. compute \(2\alpha\)-order gradient vector \(V^{2\alpha} \mathbf{u}\) using (11);
5. compute \(U_{t+1}\) using (19), and set \(t = t + 1\), if \(t = n\), output sinogram \(U_t\); else goto step 2;
6. return back-project sinogram \(U_t\) into the image \(I\).

4. Experiments and Discussion

The main objective for LDCT imaging is to delete the noise and avoid artifacts while preserving anatomy details for the back-projection images.

Two abdominal CT images of a 58-year-old man and two abdominal CT images of a 62-year-old woman with different doses were scanned from a 16-multi detector-row CT unit (Somatom Sensation 16; Siemens Medical Solutions) using 120 kVp and 5 mm slice thickness. Other remaining scanning parameters are gantry rotation time, 0.5 second; detector configuration (number of detector rows section thickness), 16 x 1.5 mm; table feed per gantry rotation, 24 mm; pitch, 1:1, and reconstruction method, back projection (FBP) algorithm with the soft-tissue convolution kernel "B30F." Different CT doses were controlled by using two different fixed tube currents 60 mAs and 150 mAs (60 mA or 150 mAs) for LDCT and standard-dose CT (SDCT) protocols, resp.). The CT dose index volume (CTDIvol) for LDCT images and SDCT images is in positive linear correlation to the tube current and is calculated to be approximately ranging between 15.32 mGy and 3.16 mGy [28] (see Figures 1(a)–1(d)).

In order to compare our method with classical PM and other state-of-art FOPMD methods, three compared methods: PMD [1], regularized PMD (RPM) [2], and FOPMD are proposed in [8]. According to the numerical scheme of PMD and RPM, they used half-point central difference discretization scheme, while FOPMD in [8] and RFS-FOPMD use integer-point unilateral difference discretization scheme.

In order to ensure that the comparison is put on a fair level, the common used parameters are set to the same value. The common used parameters for four methods include gradient modulus threshold \(\sigma\) that controls the conduction, integration constant \(\lambda\), and iteration number \(t\). Due to numerical stability, \(\lambda\) is set to its maximum value \(1/100\) and \(\sigma\) is set to 30 to reduce iteration number.

The iteration number \(t\) is very important in all comparison methods. That is, big \(t\) will make smooth image while small \(t\) will still leave a lot of noise. In order to study the performance of four compared methods with different iteration numbers \(t\) and other fixed parameters, \(t\) is set to 20, 50, and 100, respectively.

The standard deviation of smoothed Gaussian kernel for the image \(\sigma_1\) used for RPMD is set to 1 since, in [2], the authors suggest that \(\sigma_1\) should be a small number.

On sinogram space, FOPMD with \(\alpha = 0.5, \alpha = 0.8\) and \(\alpha = 1.2\) is carried on two image collections.

Since bigger iteration number leads to smoother denoised results sometimes, it also leads to dark processed images by posing too big integration constant \(\lambda\). In order to observe the behaviors of big iteration number clearly, Figure 1 sets iteration number \(t = 100\). Comparing all the original SDCT images in Figures 1(a) and 1(c), LDCT images Figures 1(b) and 1(d) were severely degraded by nonstationary noise. All denoised images in Figure 1 can suppress most of noises. Particular, FOPMD and RFS-FOPMD can provide very satisfied images with little noise and preserving all useful anatomy structures. However, denoised images of PMD and RPMD are oversmooth, which lost a lot of details.

In order to test the consistency of the definitions of different integer order or fractional order, we set fractional-order \(\alpha = 1\), in which a two fractional-order PMD should have same forms and they also correspond to the order of PMD and RPMD (see Figures 1(e)–1(l)). Observing Figures
Figure 1: Original SDCT and LDCT images ((a)–(d)), LDCT images processed by PMD, RPMD, FOPMD, and RFS-FOPMD with fractional order \( \alpha = 1 \) and iteration number is set to 100.

We can find that the denoised images are identical, which demonstrate that the fractional-order definitions between [8] and RFS-FOPMD are identical when \( \alpha = 1 \).

However, the resulting images of PMD and RPMD are quite different to the images denoised by FOPMD and RFS-FOPMD. That is, the images processed by PMD and RPMD are smoother than the images processed by FOPMD and RFS-FOPMD. Just as introduced in the previous paragraph, different discretization schemes lead to this interesting result.

Since two FOMD schemes provide more satisfied results, we only compare two FOPMD methods with different fractional orders (see Figure 2) and different iteration numbers (see Figure 3).

In order to compare denoised results of two FOPMD schemes with different fractional orders, two original LDCT images in Figures 1(b) and 1(d) are used with iteration number \( t = 100 \) and fractional-order \( \alpha = 0.5, \alpha = 0.8, \) and \( \alpha = 1.2 \). From the second and the fourth rows of Figure 2, we can conclude that the resulting images of RFS-FOPMD are very
satisfied and they become smoother when \( \alpha \) becomes bigger, which is coherent with our intuition, for example, bigger fractional-order smoother resulting images.

However, denoised images in Figures 2(a) and 2(g) with \( \alpha = 0.5 \) using FOPMD in [8] have many artifacts, which are small black circles in two images. Although big fractional-order FOPMD proposed in [8] will decrease the artifacts, its denoised images in Figures 2(c) and 2(i) are very dark comparing with the original LDCT images in Figures 1(b) and 1(d). Images in Figures 2(c) and 2(i) also have some isolated artificial white points, which are called speckle effect. It is obvious that resulting images in Figures 2(b) and 2(h) with \( \alpha = 0.8 \) processed by FOPMD in [8] have the best performance in three image series with different \( \alpha \).
Generally, artifacts in denoised images are oscillations near edges, caused by that the low-passed filtering is not processed correctly near the real edges. That is, some smooth regions near edges are regarded falsely as edges, which makes these error edges preserved. Therefore, improving accuracy of edge detection is a good choice for improving the performance of FOPMD in [8]. In this paper, we use regularization for FOPMD in [8] to locate edges correctly.

Intuitively, processed images with bigger iteration number correspond to smoother images. In order to check the influence of iteration numbers for FOPMD in [8] and RFS-FOPMD, processed image series of two original LDCT images, Figures I(b) and I(d) with different iteration numbers 20, 50, and 100 are shown in Figure 3. The resulting images in the first and the third rows of Figure 3 are the processed images using FOPMD. Figure 3: LDCT images (Figures 1(b) and 1(d)) processed by FOPMD and RFS-FOPMD with different iteration numbers and the fractional-order $\alpha = 1.2$. The first column: iteration number is 20; the second column: iteration number is 50 and the third column: iteration number is 100.
Comparing with original LDCT images in Figures 1(b) and 1(d), all resulting images with different iteration numbers have less noise. In addition, the smoother images can be obtained as the iteration number becomes bigger. However, the most undesired default for FOPMD in [8] is that resulting images become dark as the iteration number becomes big. Moreover, except for Figure 3(g), the resulting images in the third row have some isolated white points, which are the speckle effect.

The resulting images in the second and the fourth rows of Figure 3 show that RFS-FOPMD with different iteration numbers is very satisfied and it becomes smoother when the iteration number becomes bigger, which is coherent with our intuition. Another attractive nature for RFS-FOPMD about iteration is that the smoothing shown in these images is very slow. That is, the resulting images Figures 3(d) and 3(j) with iteration number 20 are slightly different to the images Figures 3(f) and 3(l) with iteration number 100. This nature shows that RFS-FOPMD has good stability. Therefore, it is not sensitive to iteration number.

All existing FOPMD methods at least suffer from speckle effect from the resulting images of these images. Fortunately, RFS-FOPMD can avoid artifacts, dark images, and speckle effect partly, which ensure its applications in sinogram restoration. More important for the new scheme is its stability, which makes it not sensitive to the iteration number.

5. Conclusions

In this paper, we propose a new FOPMD, RFS-FOPMD, for LDCT sinogram imaging based on G-L fractional-order derivative definition. RFS-FOPMD not only has good ability in preserving edges while denoising, but it also can avoid artifacts, dark images, and speckle effects of FOPMD in [8] and other existing FOPMD schemes partly by improving the performance of edges locating by regularization, which ensures that RFS-FOPMD can be used for sinogram restoration of LDCT. Of more importance, RFS-FOPMD has good stability for iteration numbers, which makes it not sensitive to the iteration number choice.

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